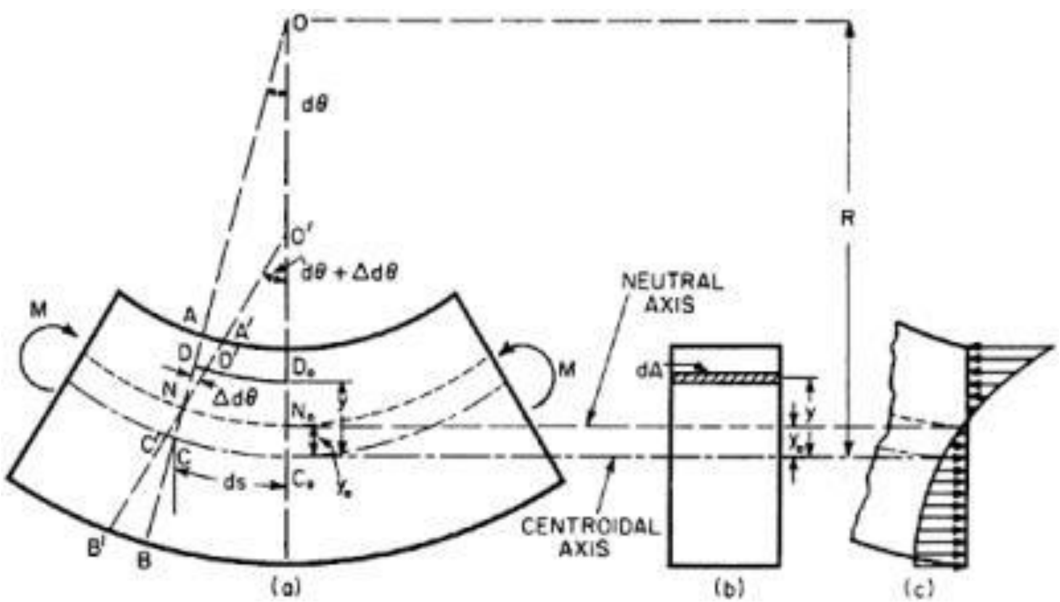
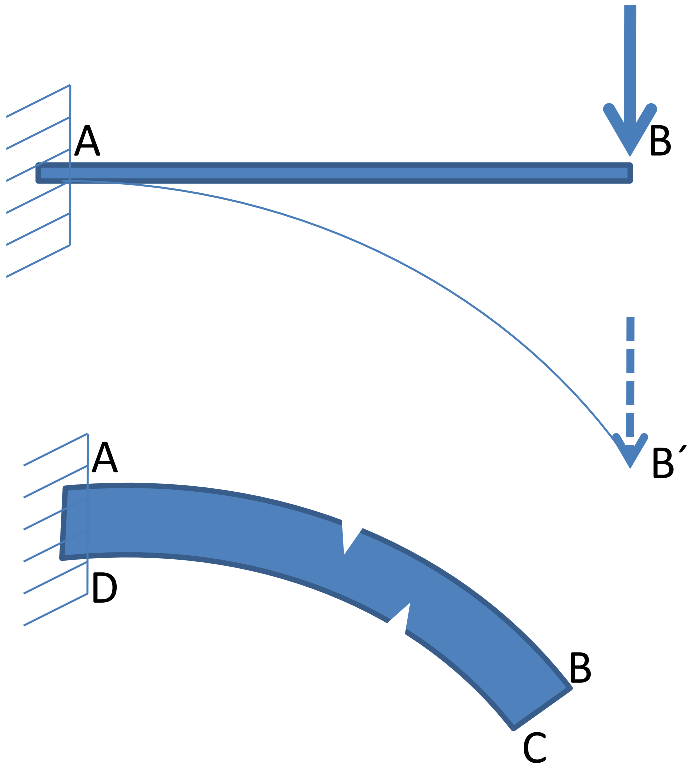
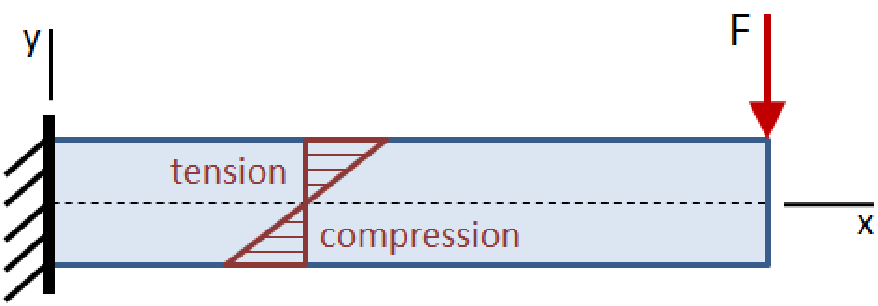
Beam Stresses



Consider a cantilever carrying load at free end.





Every section of cantilever subjected to SF and BM caused my applied load.

•Cut on upper part sure to widen due to tensile force.

•AB becomes longer.

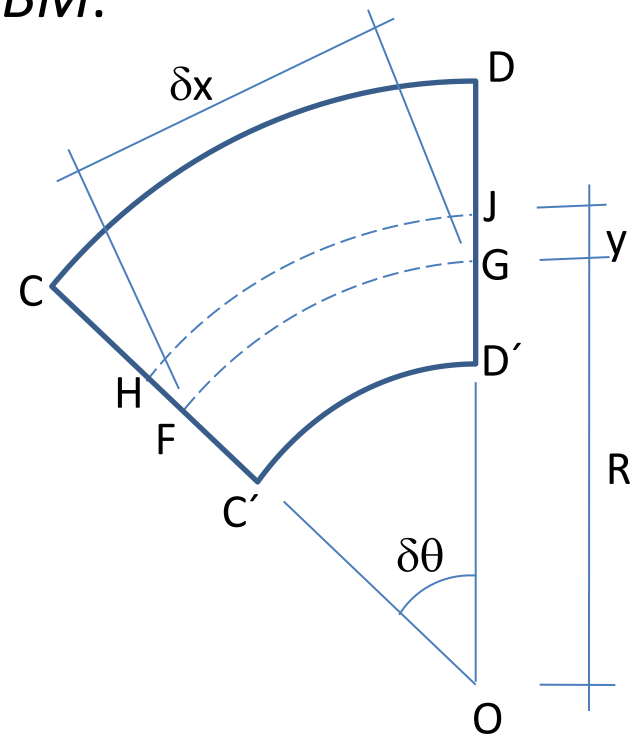
•while lower cut closes up due to compressive force.

•CD become shorter.

•Neutral plane – Plane where exists neither compression nor tension

(a) Bending Equation

Consider a length *CD* = *x* of the cantilever bent to circular arc due to *BM*.



Let FG be neutral plane

Consider a fibre HJ at distance y from N.A.

Since CC’ and DD’ shall remain plane after bending. Then CC’ and DD’ shall meet at O - Centre of Curvature. OG – Radius of Curvature.

Now $FG = R\delta\theta$

and $HJ = (R + y)\delta\theta$

$$\therefore \frac{HJ - FG}{FG} = Strain = \frac{y}{R}$$

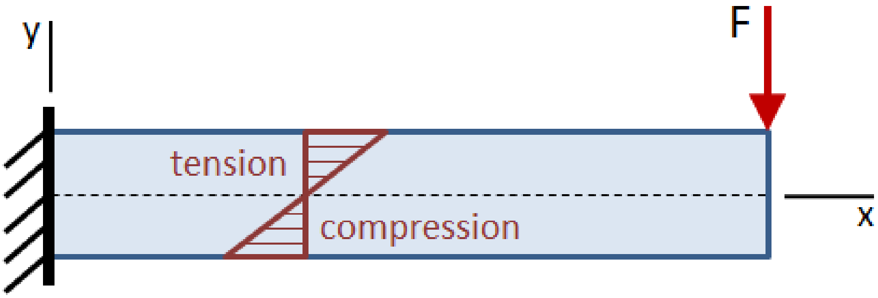
Assuming material follows Hook's law and *E* is the same both in compression and in tension.

$$\therefore \qquad \frac{Stress}{E} = Strain \text{ or } \frac{\sigma}{E} = \frac{y}{R} \qquad \text{or} \qquad \frac{\sigma}{y} = \frac{E}{R} \qquad .....(1)$$

Under a given condition of loading for a given beam

$\frac{E}{R}$ is constant

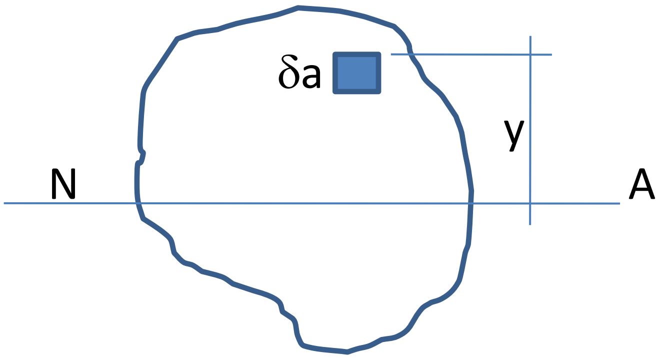
and as such bending stresses $\sigma \propto y$.



(b) Position of Neutral Axis

Consider an elementary area δa at distance y from N.A. and experiencing bending stress σ.

If δP – force on elementary area



$$\delta P = \sigma \delta a$$

$\therefore$ Force on entire cross-section

$$P = \sum \sigma \delta a$$

From equilibrium considerations, there is no force on the section i.e. Total sum of compressive and tensile on section = 0

$$P = \sum \sigma \delta a = 0$$

But from the relation

$$\frac{\sigma}{y} = \frac{E}{R} \qquad ....(1)$$

We have

$$\sigma = \frac{Ey}{R}$$

$\therefore$

$$\frac{Ey}{R}\sum \delta a = 0 \qquad \text{or} \qquad \frac{E}{R}\sum y\delta a = 0$$

Since $\frac{E}{R}$ is a constant for a particular section, we have

$$\sum y\delta a = 0$$

It means that the moment of the whole section area about the N.A. is zero. But this is the condition for N.A. to be the centroidal axis.

In conclusion

$\therefore$ The N.A. always passes through the centroid of the section of the beam.

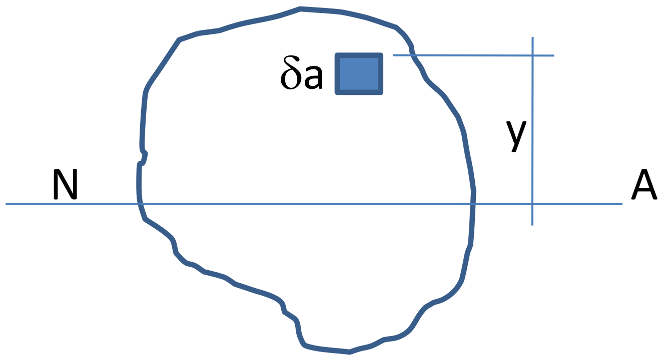
(c) **Moment of Resistance**

- From equilibrium considerations, the compressive and tensile forces about N.A. must be equal in magnitude.
- Have moments about N.A.
- The algebraic sum of these moment must be equal and opposite to the B.M. at the section.

Moment of resistance – the algebraic sum of moments about N.A. of the internal forces developed in a beam due to bending.

Calculated basing on **max permissible bending stress**.

Force acting on elementary area δa is



$$= \sigma \delta a$$

∴ Its moment about N.A.

$$= (\sigma \delta a \cdot y)$$

∴ Moment of resistance (M.R.)

$$= \int \sigma y \delta a$$

But $$\sigma = \frac{Ey}{R}$$

∴ $$M.R. = \frac{E}{R}\int y^2 \delta a = \frac{EI}{R}$$

But M.R. is the same as B.M.

$$\therefore \qquad M = \frac{EI}{R}$$

$$\text{Or} \qquad \frac{M}{I} = \frac{E}{R}$$

From (1), we have

$$\frac{E}{R} = \frac{\sigma}{y}$$

$$\therefore \qquad \boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

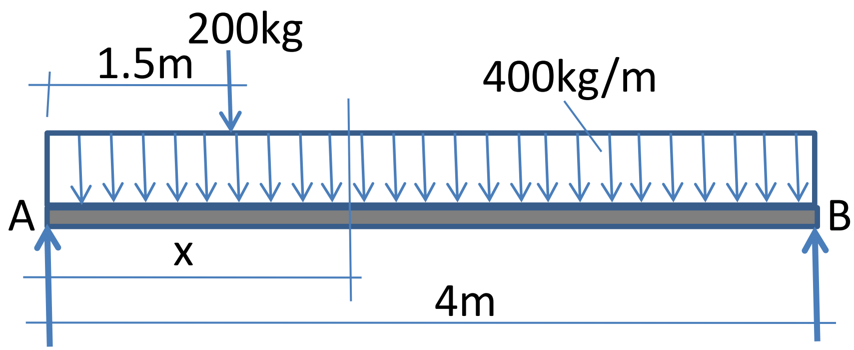
The above relation is known as the **Bending equation**.

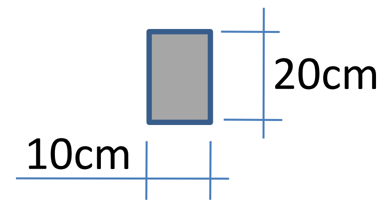
Assumptions in Theory of Bending

1. Transverse sections of the beam that were **plane** before bending **remain so** even after bending.
2. The material of beam is **homogeneous** and follow Hook's law.
3. The beam is **initially straight** and of constant cross-section.
4. The **moduli** of elasticity for tension and for compression are the **same**.
5. The stresses in beam do **not exceed elastic limit**.
6. The beam is subjected to **pure bending** (there being only bending moment and no shearing force) and therefore **bending in an arc** of circle.
7. **Radius** of curvature is **large** compared to dimensions of the section.
8. The plane of the loading contains a principal axis of the beam cross-section and the load act perpendicular to beam axis.

Example

A 4 metre long beam with rectangular section of 10cm width and 20cm depth is simply supported at the ends. If it is loaded with a uniformly distributed load of 400kg/m throughout the span and a concentrated load P = 200kg placed at a distance of 1.5m from one end. Determine the maximum bending stress in the beam.





Support reactions

$$R_B \cdot 4 = 200 \cdot 1.5 + 400 \cdot 4 \cdot \frac{4}{2}$$

$$\therefore R_B = \underline{875}kg \quad \text{and} \quad R_A = 200 + 400 \cdot 4 - 875 = \underline{925}kg$$

B.M. at a distance x from A is

$$M_x = +925x - 200(x-1.5) - 400\frac{x^2}{2}$$

For it to be maximum we have

$$\frac{dM_x}{dx} = 0$$

(Since the rate of change of B.M at any section represent the S.F at that section)

$$\therefore \quad +925 - 200 - 200 \cdot 2x = 0$$

$$\therefore \quad x = 1.8125\,m \quad \text{from} \quad \text{A}$$

$$\therefore \quad M_{\max} = 925 \cdot 1.8125 - 200(1.8125 - 1.5) - 400 \cdot \frac{1.8125^2}{2}$$

$$= +957.03\,kg.m = \underline{95703\,kg.cm}$$

Moment of inertia for the given section is

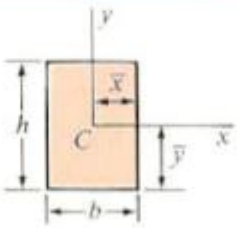
$$I = \frac{bd^3}{12} = \frac{10 \cdot 20^3}{12} = 6666.667\,cm^4$$

$$y = \frac{d}{2} = \frac{20}{2} = 10\,cm$$

The maximum bending stress σ is given by the relation

$$\sigma = \frac{My}{I} = \frac{95703 \cdot 10}{6666.667} = 143.554\,\mathrm{kg/\ cm^2}$$

1



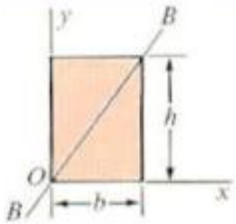
Rectangle (Origin of axes at centroid.)

$$A = bh \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{2}$$

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12}$$

$$I_{xy} = 0 \qquad I_p = \frac{bh}{12}(h^2 + b^2)$$

2

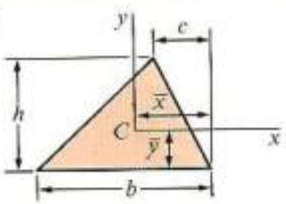


Rectangle (Origin of axes at corner.)

$$I_x = \frac{bh^3}{3} \qquad I_y = \frac{hb^3}{3}$$

$$I_{xy} = \frac{b^2h^2}{4} \qquad I_p = \frac{bh}{3}(h^2 + b^2) \qquad I_{BB} = \frac{b^3h^3}{6(b^2 + h^2)}$$

3



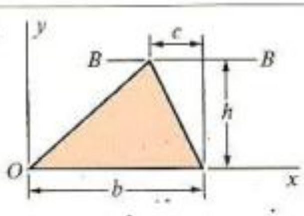
Triangle (Origin of axes at centroid.)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b + c}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{bh}{36}(b^2 - bc + c^2)$$

$$I_{xy} = \frac{bh^2}{72}(b - 2c) \qquad I_p = \frac{bh}{36}(h^2 + b^2 - bc + c^2)$$

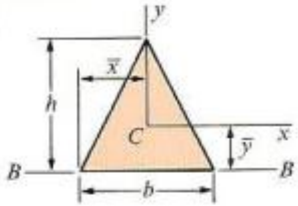
4



Triangle (Origin of axes at vertex.)

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{bh}{12}(3b^2 - 3bc + c^2)$$

$$I_{xy} = \frac{bh^2}{24}(3b - 2c) \qquad I_{BB} = \frac{bh^3}{4}$$

5 

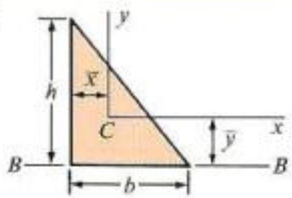
Isosceles triangle (Origin of axes at centroid.)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{48} \qquad I_{xy} = 0$$

$$I_P = \frac{bh}{144}(4h^2 + 3b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

(Note: For an equilateral triangle, $h = \sqrt{3}b/2$.)

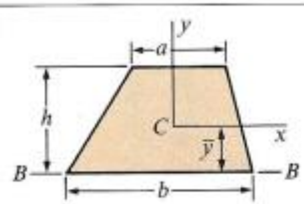
6 

Right triangle (Origin of axes at centroid.)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{36} \qquad I_{xy} = -\frac{b^2h^2}{72}$$

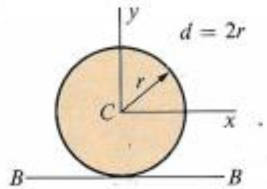
$$I_P = \frac{bh}{36}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

8 

Trapezoid (Origin of axes at centroid.)

$$A = \frac{h(a + b)}{2} \qquad \bar{y} = \frac{h(2a + b)}{3(a + b)}$$

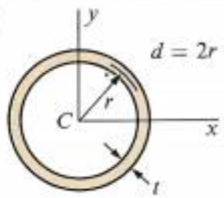
$$I_x = \frac{h^3(a^2 + 4ab + b^2)}{36(a + b)} \qquad I_{BB} = \frac{h^3(3a + b)}{12}$$

9


Circle (Origin of axes at center.)

$A = \pi r^2 = \frac{\pi d^2}{4} \qquad I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$

$I_{xy} = 0 \qquad I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad I_{BB} = \frac{5\pi r^4}{4} = \frac{5\pi d^4}{64}$

10


Circular ring (Origin of axes at center.)
Approximate formulas for case when t is small.

$A = 2\pi r t = \pi d t \qquad I_x = I_y = \pi r^3 t = \frac{\pi d^3 t}{8}$

$I_{xy} = 0 \qquad I_p = 2\pi r^3 t = \frac{\pi d^3 t}{4}$

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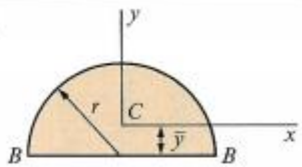
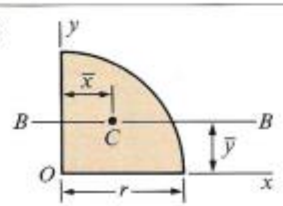


Semicircle (Origin of axes at centroid.)

$A = \frac{\pi r^2}{2} \qquad \bar{y} = \frac{4r}{3\pi}$

$I_x = \frac{(9\pi^2 - 64)r^4}{72\pi} \approx 0.1098 r^4 \qquad I_y = \frac{\pi r^4}{8}$

$I_{xy} = 0 \qquad I_{BB} = \frac{\pi r^4}{8}$

12


Quarter circle (Origin of axes at center of circle.)

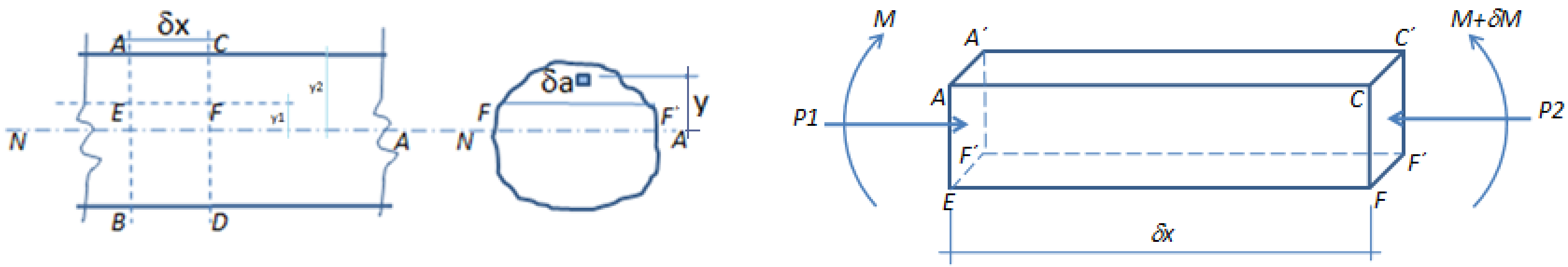
$A = \frac{\pi r^2}{4} \qquad \bar{x} = \bar{y} = \frac{4r}{3\pi}$

$I_x = I_y = \frac{\pi r^4}{16} \qquad I_{xy} = \frac{r^4}{8}$

$I_{BB} = \frac{(9\pi^2 - 64)r^4}{144\pi} \approx 0.05488 r^4$

Shear Stress Distribution

Consider two cross-sections *AB* and *CD* of a loaded beam such that the distance between the sections is δx. Let *M* and *M*+*δM* be the bending moments at sections *AB* and *CD* respectively.



Consider an elementary area δa at a height *y* from the N.A. If σ1 is the bending stress on the elementary area on face AA´EE´ and σ2 is the bending stress on the elementary area on face CC´FF´ . Then

$$\sigma_1 = \frac{My}{I} \qquad \text{and} \qquad \sigma_2 = \frac{(M + \delta M)y}{I}$$

The thrust on the elementary area δa on face AA'EE' of section AB is

$$\sigma_1 \delta a = \frac{My}{I} \delta a$$

And the thrust on the elementary area δa on face CC'FF' of section CD is

$$\sigma_2 \delta a = \frac{(M + \delta M)y}{I} \delta a$$

The thrust on AA'EE' is

$$P_1 = \sum_{y_1}^{y_2} \frac{My}{I}$$

And the thrust on CC'FF' is

$$P_2 = \sum_{y_1}^{y_2} \frac{(M + \delta M)y}{I}$$

The resultant thrust experienced by the portion of beam between the two sections AB and CD and planes AC and EF is $(P_2 - P_1)$ from the right to the left. This resultant thrust causes shearing of the portion of beam at the plane EF. This shearing force is resisted by the shear stress τ generated on surface EE'F'F

$$\therefore \quad \tau b\,\delta x = \sum\left[\frac{(M+\delta M)}{I} - \frac{M}{I}\right] y\,\delta a$$

Where *b* is the width of FF'

$$= \frac{\delta M}{I}\sum_{y_1}^{y_2} y\,\delta a$$

$$\therefore \quad \tau = \frac{\delta M}{I b\,\delta x}\sum_{y_1}^{y_2} y\,\delta a$$

Since the rate of change of B.M at any section represent the S.F at that section $\frac{dM}{dx} = F$

and $\sum_{y_1}^{y_2} y\,\delta a$ - moment of area of cross-section of beam above plane FF' about N.A, therefore

$$\tau = \frac{F}{Ib} A\bar{y} \quad \text{.... (3)}$$

A - area of section

$\bar{y}$ - distance of centroid of element from N.A.