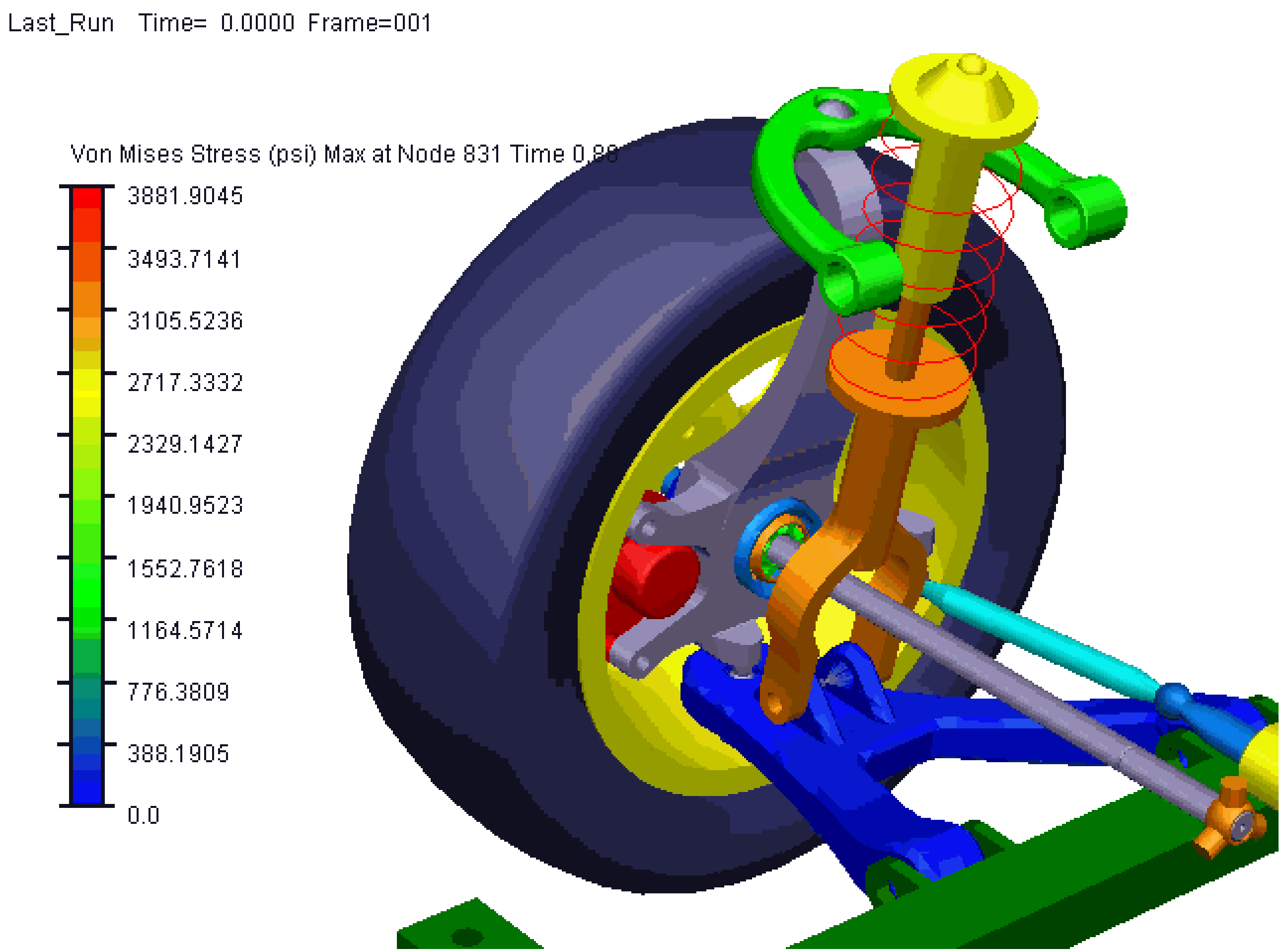
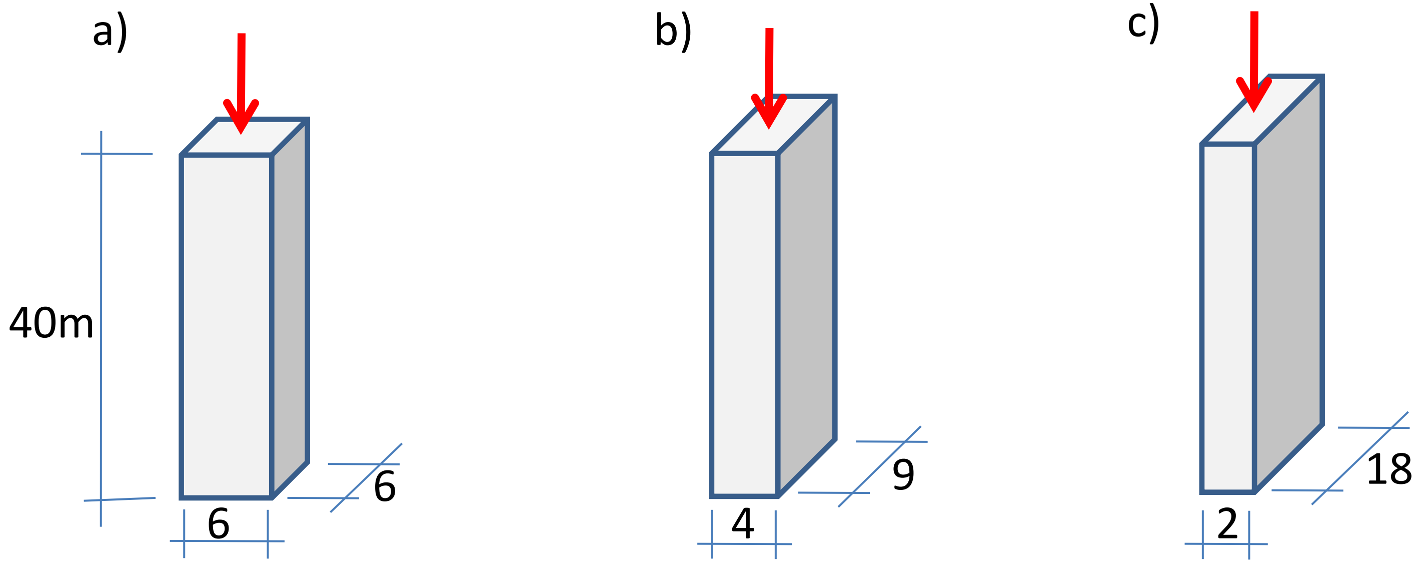
Columns & Struts



Load carried by a short column

$$P = \sigma A \qquad \text{where } \sigma \text{ - Stress intensity}$$
$$\text{A - Cross - sectional Area}$$

- If σ is the ultimate crush stress for the column, then P shall be the crushing load for the column.
- Columns do not generally fail only due to crushing, they buckle too.



- Keeping cross-section area the same but varying the side.
- Column c) likely to fail by buckling.

- Also the load carrying capacity reduces with increase in height.
- $\therefore$ shall depend on slenderness of the column.

$$\text{Slenderness Ratio} = \frac{\text{Height of Column}}{\text{Least Side of Column}}$$

Also

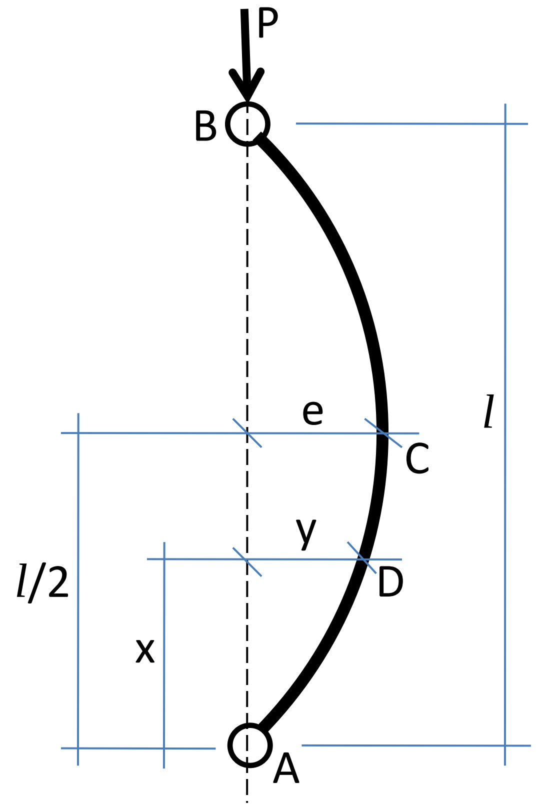
$$= \frac{\text{Height of Column}}{\text{Least radius of Gyration (r)}} = \frac{l}{r_{x,y}}$$

$$\text{Where } r_{x,y} = \sqrt{\frac{I_{x,y}}{A}}, \; \mathrm{I_{x,y}} \text{ - Corresponding Moment of Inertia}$$

$$\text{A - Cross-sectional Area}$$

- Apart from the slenderness, the load carrying capacity depends upon its end condition too to fix the column.
- At a certain critical load a column shall start bending after which failure of column is certain.

Consider a straight column AB of length l that buckles under load P such that the mid-point C has a max eccentricity e.



Both ends hinged

Section C is subjected to max moment

$$M = P \text{ x } e$$

The max and min stress intensifies due to the combination effect of direct & bending stresses σ_d and σ_b are

$$\sigma_{\max} = \sigma_d + \sigma_b = \frac{P}{A} + \frac{Pe}{Z}$$

$$\sigma_{\min} = \sigma_d - \sigma_b = \frac{P}{A} - \frac{Pe}{Z}$$

Z – section modulus

Column shall fail when either σ_{max} reaches the ultimate crush stress OR when σ_{min} reaches the ultimate tensile strength of the column material.

- Column that fails primarily due to direct stress is called a short column.
- Column that fails primarily due to bending stress is called a long column.
- Short columns are those whose length is less than 8 times the diameter (or the smallest side).

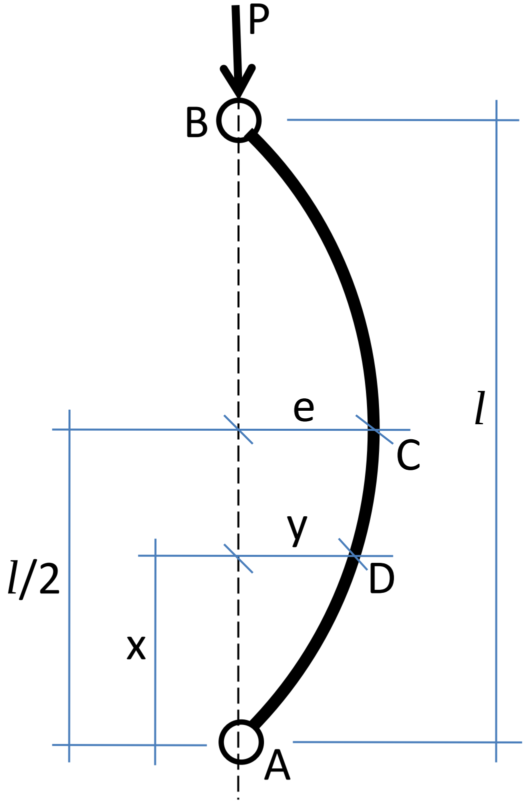
Euler's formulation

His analysis is based on following assumptions:

a) The columns are initially straight.
b) The columns are made of homogeneous material.
c) The columns carry perfectly axial loads.
d) The columns have uniform cross-sectional areas throughout.
e) The columns are long compared to lateral dimensions.
f) Self weight of the columns is neglected.
g) The columns fail only due to buckling.
h) Shortening of columns due to direct loading is neglected.
i) Stresses do not exceed the limit stress of the material.

1) Both Ends Hinged

Consider a long column of effective length l , hinged at both ends crippling under an axial load, called Euler's Crippling load P_E . Assume a small internal force buckles the column. Let the eccentricity at any D at a height x from A be y. The moment at the point is



Both ends hinged

$$M = +Py$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = -Py \quad \text{OR} \quad EI\frac{d^2y}{dx^2} + Py = 0 \quad \ldots(1)$$

If we set

$$\frac{P}{EI} = k^2 \quad \ldots(2)$$

then the differential equation of deflection becomes

$$\frac{d^2y}{dx^2} + k^2 y = 0 \quad \ldots(3)$$

- When solving the above equation, we need merely find a function which when differentiated twice and added to itself (times a constant) is equal to zero.
- Evidently either ***sin kx*** or ***cos kx*** possesses this property. In fact, a combination of these terms in the form,

$$y(x) = C\sin kx + D\cos kx \quad \ldots \ (4)$$

- May be taken as solution of eqn (3). This may be readily checked by substituting y(x) into eqn (3).

To determine C and D: at **A**, **y = 0** when **x = 0**. substituting these values in eqn (4), we obtain,

$$0 = 0 + D \qquad \text{or} \qquad D = 0$$

At **B**, **y = 0** when **x = *l*** . Substituting these values in eqn (4) with **D = 0**, we obtain,

$$0 = C \sin kl$$

Evidently either **C =0** or **sin *kl* = 0**. But if **C = 0** then **y** is everywhere **zero** and we have only the trivial case of a straight bar which is configured prior to the occurrence of buckling. Therefore we take

$$\sin kl = 0 \quad \ldots \quad (5)$$

For this to be true, we have

$$kl = n\pi \quad (n = 1,2,3\ldots) \quad \ldots(6)$$

Substituting $k^2 = \frac{P}{EI}$ in eqn (6), we find

$$\sqrt{\frac{P}{EI}}\, l = n\pi \qquad \text{or} \qquad \boxed{\mathrm{P} = \frac{\mathrm{n}^2\pi^2 EI}{l^2}} \qquad \ldots(7)$$

The smallest value of this load evidently occurs when **n=1** . Then we have the so-called first mode of buckling where the **critical load** is given by

$$\boxed{P_{cr} = \frac{\pi^2 EI}{l^2}} \quad \ldots\ (8)$$

This is ***Euler's buckling load for pin-ended column***. The deflection shape corresponding to this load is

$$y(x) = C \sin\left(\sqrt{\frac{P}{EI}}\, x\right) \quad \ldots(9)$$

Substituting eqn (8) into (9), we obtain

$$y(x) = C \sin \frac{\pi x}{l} \quad \ldots(10)$$

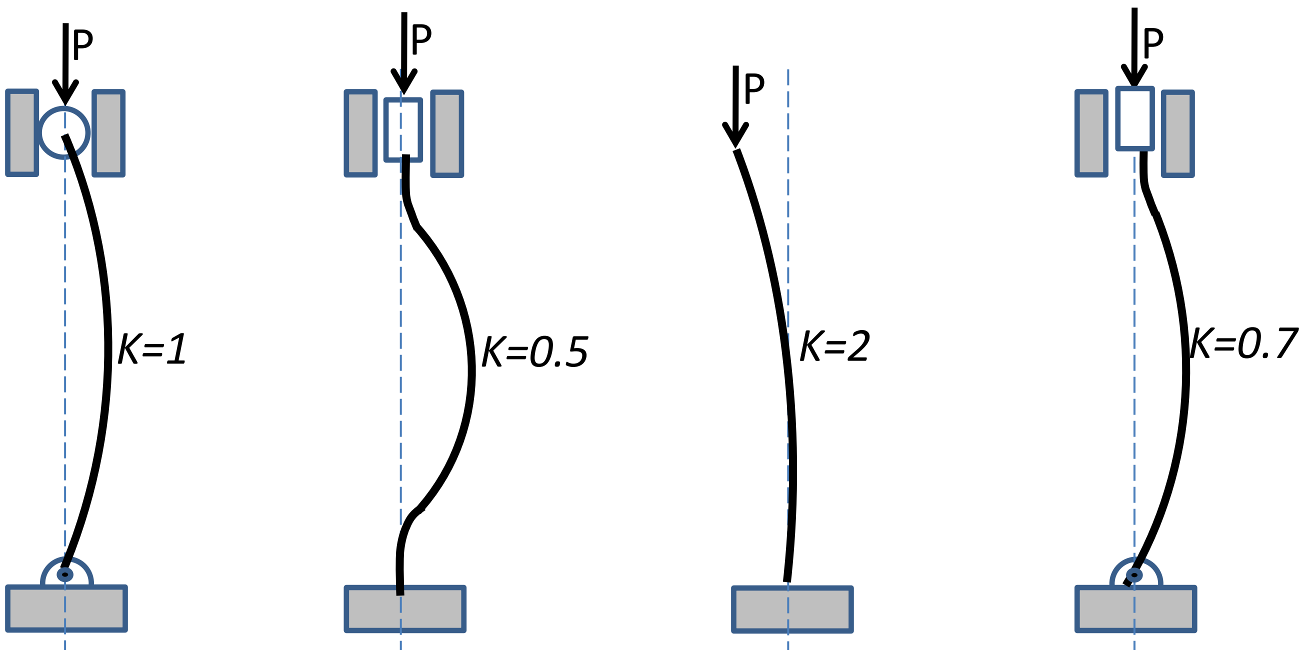
Note that the deflected shape is a Sine function

Eqn (8) may be modified to the form

$$P_{cr} = \frac{\pi^2 EI}{(Kl)^2} \quad \ldots(11)$$

Where Kl – an **effective length** of the column, defined to be a portion of the deflected bar between points corresponding to zero curvature.

As such,



(a) Pinned ends

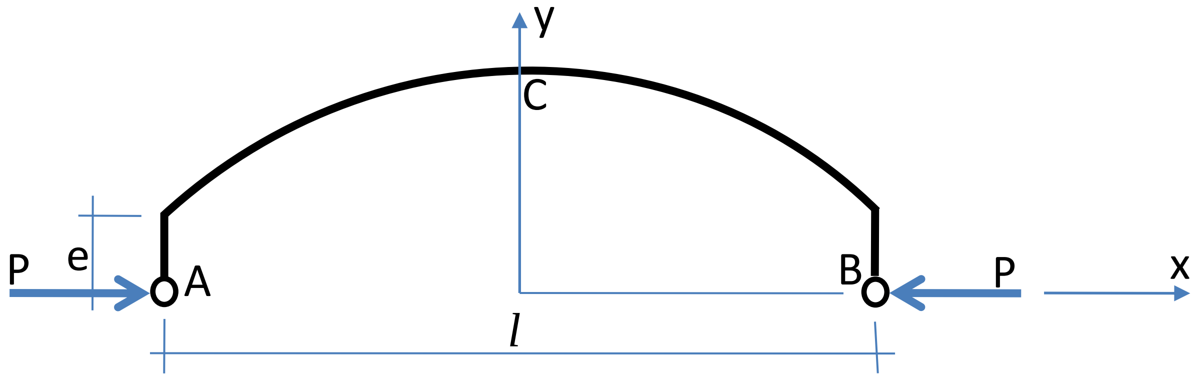
(b) Clamped ends

(c) One end free

(d) Clamped end, pinned end

Eccentrically loaded Columns

Assume an initially straight, pin-ended column subjected to an axial compressive force applied with know eccentricity e.



The differential equation of a bar in its deflected configuration if

$$EI\frac{d^2y}{dx^2} = -Pe \quad \dots (11)$$

Which has a standard solution

$$y = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right) \quad \dots (12)$$

Since y = e at each ends, x=-l/2 and x=l/2 , the value of the two constants of integration are readily found to be

$$C_1 = 0 \qquad C_2 = \frac{e}{\cos\left(\sqrt{\frac{P}{EI}}\frac{l}{2}\right)} \qquad \dots (13)$$

Therefore , the deflection curve of the bent bar is

$$y = \frac{e}{\cos\left(\sqrt{\frac{P}{EI}}\frac{l}{2}\right)}\cos\left(\sqrt{\frac{P}{EI}}x\right) \qquad \dots (14)$$

The max value of deflection occurs at x = 0

$$y_{\max} = e\sec\left(\sqrt{\frac{P}{EI}}\frac{l}{2}\right) \qquad \dots (15)$$

Introducing the value of the critical load P_{cr} as given by eqn (8)

$$y_{\max} = e\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) \qquad \ldots (16)$$

The max compressive stress occurs on the concave side of the bar at C and its given by

$$\sigma_{\max} = \boxed{\frac{P}{A}} + \boxed{\frac{M_{\max} y_c}{I}} = \frac{P}{A} + \frac{Pe\,y_c}{I}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) \qquad \ldots (17)$$

y_c – distance from neutral axis to outer fibre of bar.

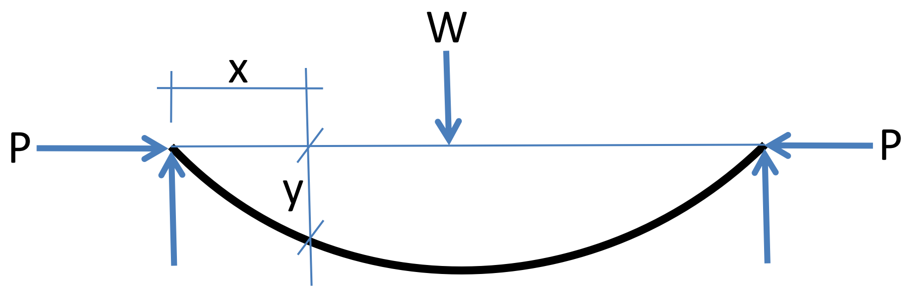
If we introduce the radius of gyration *r* of the cross-section, this becomes:

$$\sigma_{\max} = \frac{P}{A} + \frac{M_{\max} y_c}{I} = \boxed{\frac{P}{A}\left[1 + \frac{e\,y_c}{r^2}\sec\left(\frac{l}{2r}\sqrt{\frac{P}{AE}}\right)\right]} \qquad \ldots (18)$$

Struts with transverse loading also

a) Point load at mid span

Let the strut AB of length l, hinged at ends, carry transverse load W at mid span and a thrust at ends. Let this strut so deflect that at x from support A the deflection is y.



B.M at distance x from A is $+\left(Py + \frac{W}{2}x\right)$

$$\therefore \qquad EI\frac{d^2y}{dx^2} = -\left(Py + \frac{W}{2}x\right)$$

$$or \qquad \frac{d^2y}{dx^2} + \frac{P}{EI}y + \frac{W}{2EI}x = 0$$

Solution to above differential equation is:

$$y = C_1 \cos\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}}x\right) - \frac{Wx}{2P}$$

To determine constants C_1 and C_2 apply end condition at A where

y = 0 when x = 0, therefore $C_1 = 0$

Thus $$y = C_2 \sin\left(\sqrt{\frac{P}{EI}}x\right) - \frac{Wx}{2P}$$

$$\therefore \qquad \frac{dy}{dx} = C_2 \sqrt{\frac{P}{EI}} \cos\left(\sqrt{\frac{P}{EI}}x\right) - \frac{W}{2P}$$

At mid span where

$$x = \frac{l}{2}; \qquad \frac{dy}{dx} \text{ is zero}$$

$$\therefore \qquad 0 = C_2\sqrt{\frac{P}{EI}}\cos\frac{l}{2}\sqrt{\frac{P}{EI}} - \frac{W}{2P}$$

$$\therefore \qquad C_2 = \frac{W}{2P}\sqrt{\frac{EI}{P}}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}$$

Thus $$y = \frac{W}{2P}\sqrt{\frac{EI}{P}}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}\sin x\sqrt{\frac{P}{EI}} - \frac{Wx}{2P} \quad \ldots(a)$$

B.M at distance x from A is

$$M_x = Py + \frac{Wx}{2}$$

$$= \frac{W}{2}\sqrt{\frac{EI}{P}}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}\sin x\sqrt{\frac{P}{EI}} - \frac{Wx}{2P} + \frac{Wx}{2}$$

$$= \frac{W}{2}\sqrt{\frac{EI}{P}}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}\sin x\sqrt{\frac{P}{EI}} \quad \ldots(b)$$

The **B.M.** shall be **max** at mid span where **x = *l*/2**

$$M_x = \frac{W}{2}\sqrt{\frac{EI}{P}}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}\sin\frac{l}{2}\sqrt{\frac{P}{EI}}$$

$$= \boxed{\frac{W}{2}\sqrt{\frac{EI}{P}}\tan\frac{l}{2}\sqrt{\frac{P}{EI}}} \qquad \ldots(c)$$

The direct stress is

$$\boxed{\sigma_d = \frac{P}{A}}$$

The max bending stress is

$$\sigma_{b_{\max}} = \frac{M_{\max} y_c}{I}$$

y_c – Dist from N.A to extreme end of column

$$= \frac{W}{2I} y_c \sqrt{\frac{EI}{P}}\tan\frac{l}{2}\sqrt{\frac{EI}{P}} \qquad \ldots\ (d)$$

Therefore, max bending stress is:

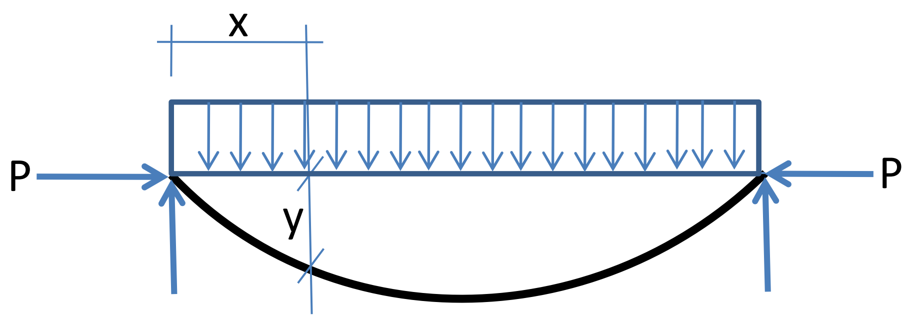
$$\boxed{\sigma_{\max} = \sigma_d + \sigma_{b_{\max}}}$$

$$= \frac{P}{A} + \frac{W}{2I} y_c \sqrt{\frac{EI}{P}} \tan \frac{l}{2}\sqrt{\frac{P}{EI}}$$

$$= \frac{1}{A}\left(P + \frac{W y_c}{2r^2}\sqrt{\frac{EI}{P}} \tan \frac{l}{2}\sqrt{\frac{P}{EI}} \right) \qquad \ldots \text{(d)}$$

$$(\because \quad I = Ar^2)$$

$$(r - \text{Radius of Gyration})$$



Consider a strut AB of length l hinged at both ends carrying a u.d.l of w/unit length. Let the deflection at a distance x from A be y.

B.M. at the section is

$$M_x = Py + \frac{wl}{2}x - \frac{wx^2}{2}$$

Therefore

$$EI\frac{d^2y}{dx^2} = -Py - \frac{wlx}{2} + \frac{wx^2}{2}$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = -\frac{wlx}{2EI} + \frac{wx^2}{2EI}$$

Solution to the differential equation is:

$$y = C_1 \cos\sqrt{\frac{P}{EI}}x + C_2 \sin\sqrt{\frac{P}{EI}}x - \frac{wlx}{2P} - \frac{wEI}{P^2} + \frac{wx^2}{2P}$$

To determine C_1 apply end condition at A where y = 0 when x = 0.

Therefore

$$0 = C_1 - \frac{wEI}{P^2} \qquad \text{or} \qquad C_1 = \frac{wEI}{P^2}$$

Thus

$$y = \frac{wEI}{P^2}\cos\sqrt{\frac{P}{EI}}x + C_2 \sin\sqrt{\frac{P}{EI}}x - \frac{wlx}{2P} - \frac{wEI}{P^2} + \frac{wx^2}{2P}$$

Therefore

$$\frac{dy}{dx} = -\frac{wEI}{P^2}\sqrt{\frac{P}{EI}}\sin\sqrt{\frac{P}{EI}}x + C_2\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}x - \frac{wl}{2P} + \frac{wx}{P}$$

At mid span where x = $l/2$; dy/dx = 0

$$0 = -\frac{wEI}{P^2}\sqrt{\frac{P}{EI}}\sin\frac{l}{2}\sqrt{\frac{P}{EI}} + C_2\sqrt{\frac{P}{EI}}\cos\frac{l}{2}\sqrt{\frac{P}{EI}} - \frac{wl}{2P} + \frac{wl}{2P}$$

Therefore,

$$C_2 = \frac{wEI}{P^2}\tan\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)$$

Thus

$$y = \frac{wEI}{P^2}\cos\sqrt{\frac{P}{EI}}x + \frac{wEI}{P^2}\tan\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)\sin\sqrt{\frac{P}{EI}}x - \frac{wlx}{2P} - \frac{wEI}{P^2} + \frac{wx^2}{2P} \quad \ldots(\text{e})$$

Deflection is max at mid span when x = *l*/2

Therefore,

$$y_{\max} = \frac{wEI}{P^2}\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)\cos\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) + \frac{wEI}{P^2}\tan\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)\sin\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)$$

$$-\frac{wl^2}{P^2} - \frac{wEI}{P^2} + \frac{wl^2}{8P}$$

$$= \frac{wEI}{P^2}\cos\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) + \frac{wEI}{P^2}\frac{\sin^2\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)}{\cos\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)} - \frac{wl^2}{8P} - \frac{wEI}{P^2}$$

$$y_{max} = \frac{wEI}{P^2}\sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) - \frac{wl^2}{8P} - \frac{wEI}{P^2}$$

$$= \frac{wEI}{P^2}\left[\sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) - 1\right] - \frac{wl^2}{8P} \qquad \dots \text{(f)}$$

Max B.M is at mid span at x = *l*/2

Therefore,

$$M_{max} = Py_{max} + \frac{wl}{2}\left(\frac{l}{2}\right) - \frac{w}{2}\left(\frac{l}{2}\right)^2 = Py_{max} + \frac{wl^2}{8}$$

$$= \frac{wEI}{P}\left[\sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) - 1\right] - \frac{wl^2}{8} + \frac{wl^2}{8}$$

$$= \frac{wEI}{P}\left[\sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) - 1\right] \qquad \dots \text{(g)}$$

Direct stress is

$$\sigma_d = \frac{P}{A}$$

and max bending stress is

$$\sigma_{b_{\max}} = \frac{M_{\max} y_c}{I} = \frac{wE}{P}\left[\sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}} - 1\right)\right] y_c \qquad \ldots(h)$$

Max compressive stress is

$$\sigma_{\max} = \sigma_d + \sigma_{b_{\max}}$$

$$= \frac{P}{A} + \frac{wE}{P} y_c \left[\sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}} - 1\right)\right] \qquad \ldots(i)$$