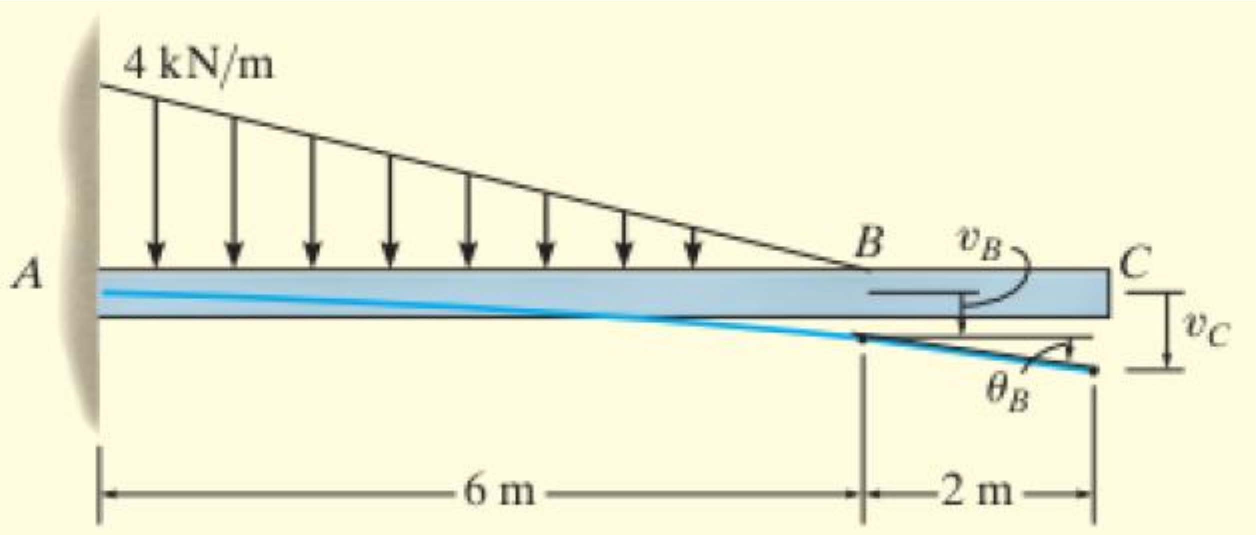
Deflection of Straight Beams



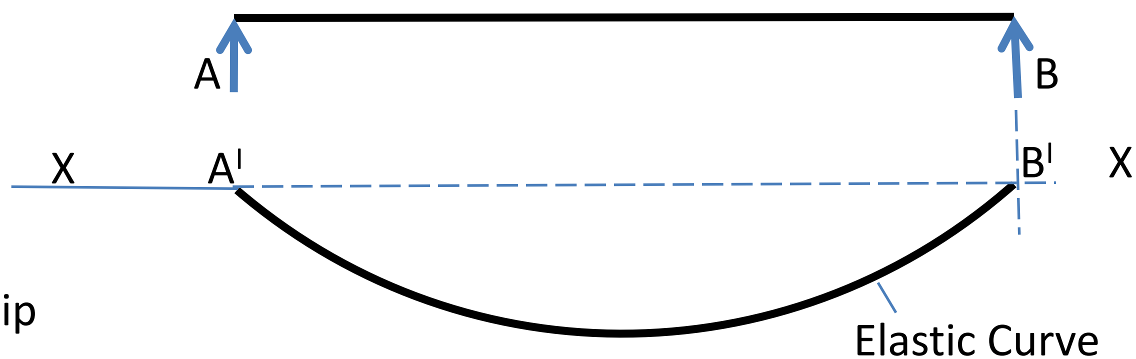
Deflection of a point – distance between its position before and after loading.

Slope at a section in deflected beam – The angle, in radians, which the tangent at the section makes with the original axis of the beam.

Stiffness of a beam – Ratio of max deflection of beam to its span.

Relationship between Curvature, Deflection and Slope.

Due to imposed load, let the beam AB bend to the curve $A^I B^I$.



From the relationship

$$\frac{\sigma}{r} = \frac{E}{R} = \frac{M}{I}$$

We have

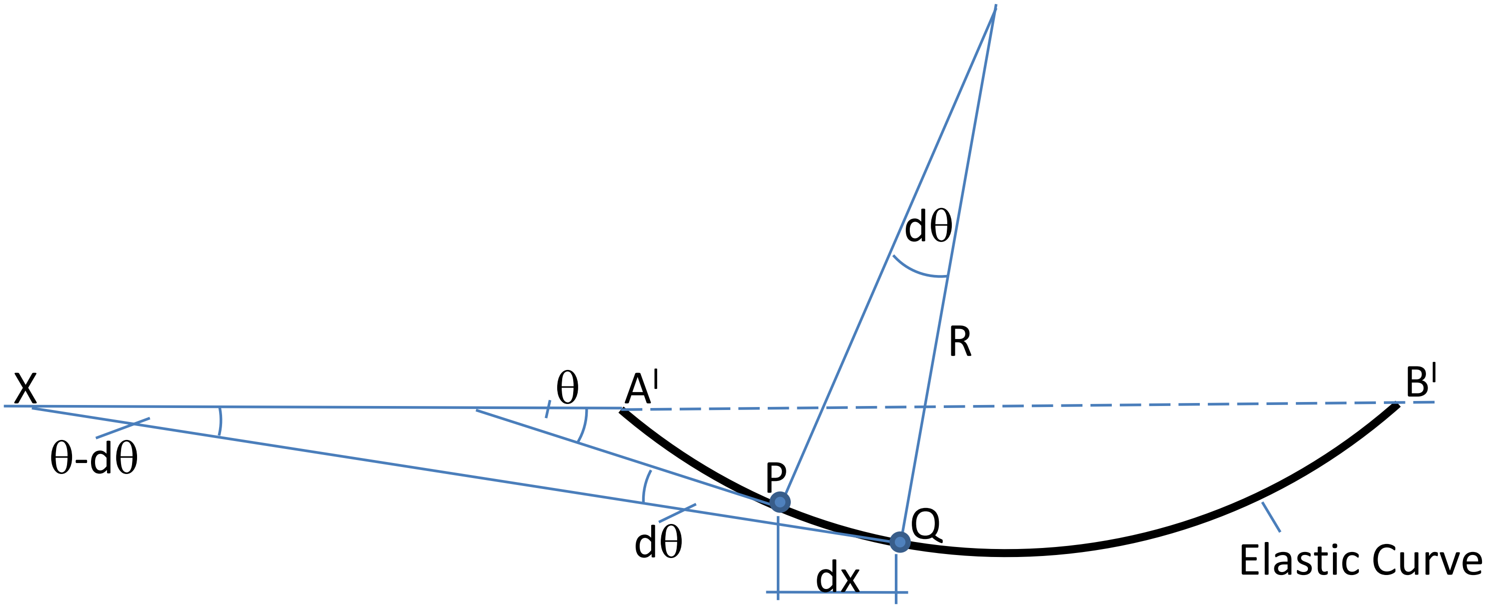
$$\frac{1}{R} = \frac{M}{EI} \text{..............(i)}$$

Of which

EI – Flexural rigidity

R - Radius of curvature

M – Bending moment causing deflection of the beam



Consider two points P and Q on the elastic curve.

Let θ - Angle made by tangent at P with X-axis .

$d\,\theta$ - Angle between normals to the curve at P and Q.

Now $$PQ = Rd\theta$$

Or $$\frac{1}{R} = \frac{d\theta}{PQ} = \frac{d\theta}{dx}$$

(For very small deflections, PQ = *dx*)

Slope at P = θ; at Q = $(\theta + d\theta)$. Slope decreases with increase in dx

$$\therefore \quad \frac{d\theta}{dx} \quad \text{is -ve}$$

Hence $$\frac{1}{R} = -\frac{d\theta}{dx}$$

But $$\frac{dy}{dx} = \tan\theta = \theta \qquad \text{(small angle } \theta\text{)}$$

Therefore $$\frac{d^2y}{dx^2} = \frac{d\theta}{dx} = -\frac{1}{R} = -\frac{M}{EI}$$

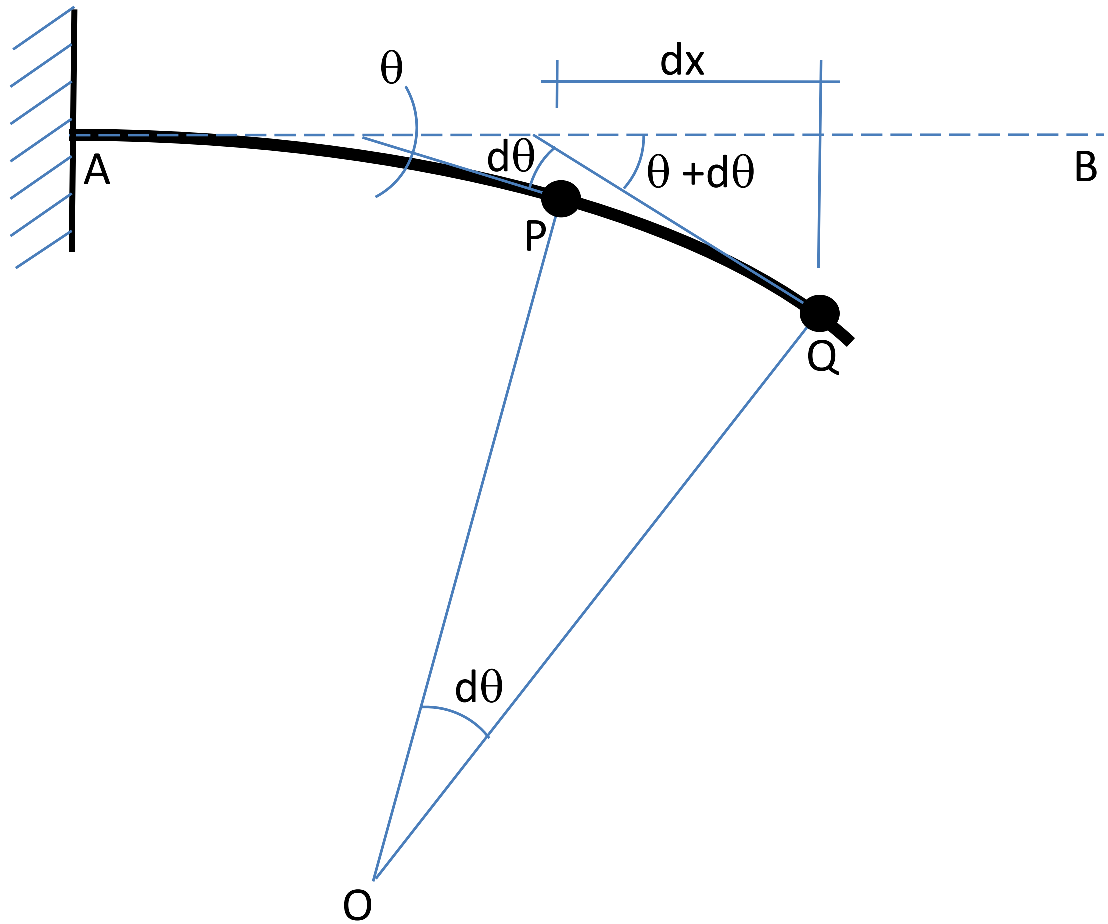
Therefore $$EI\,\frac{d^2y}{dx^2} = -M$$

Since

$\frac{d\theta}{dx}$ is -ve, so $\frac{d^2y}{dx^2}$ is -ve.

The B.M causing deflection is +ve.

But in the case below B.M is –ve and the slope at Q is more than at P.



At P the slope is θ whereas at Q is (θ+dθ)

$\therefore$ Here $\quad \dfrac{d\theta}{dx}$ is + ve and so is $\dfrac{d^2y}{dx^2}$

In both cases arc *PQ = R*$d\theta$

$\cong$ *PQ = dx* (since the arc is very small)

Therefore $\quad Rd\theta = dx$

$$\text{or } \frac{d\theta}{dx} = \frac{1}{R} \quad \text{or} \quad \frac{1}{R} = \frac{d^2y}{dx^2} = -\frac{M}{EI}$$

(Compare with eqn (i), M is –ve for cantilever)

Therefore $\quad EI\dfrac{d^2y}{dx^2} = -M$

When M is +ve (case of beams) $\frac{d^2y}{dx^2}$ is –ve

M is –ve (case of cantilever) $\frac{d^2y}{dx^2}$ is +ve.

General equation of deflection

$$EI\frac{d^2y}{dx^2} = -M$$

By integrating it once we get $\frac{dy}{dx}$ - the slope equation.

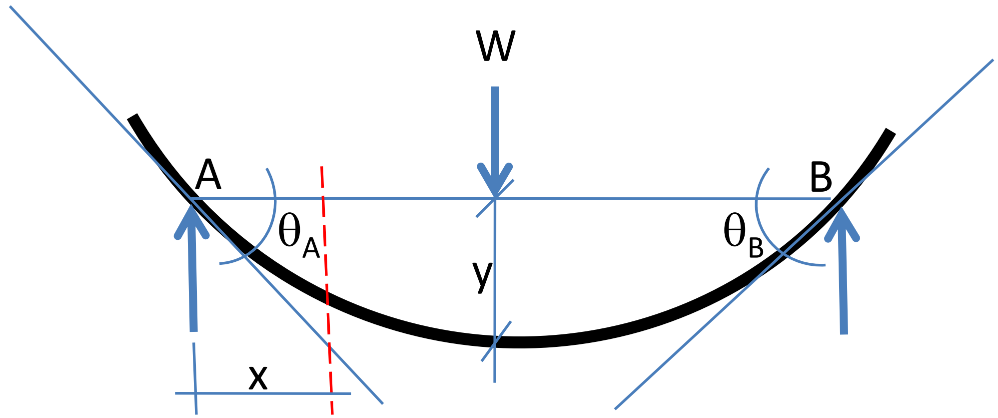
By integrating it twice we get **y** - the deflection.

The above equation is known as **Differential Equation of Flexure**.

Sign Convention

1. When measuring along the beam from left to right, x is taken as +ve.
2. Deflection y is +ve downwards.
3. Bending moment M is positive (+ve) when sagging.
4. Slope θ is +ve if while going from left to right along the beam tangent to the elastic curve while inclined downwards.

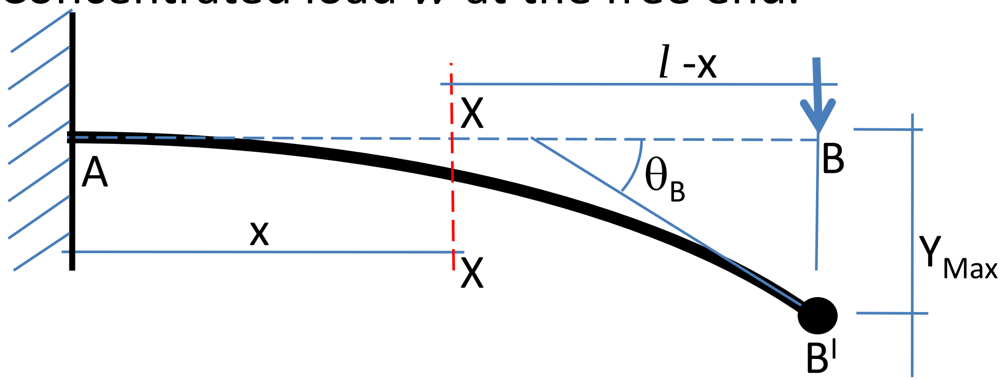
Below: Distance x; deflection y; bending moment M and slope θ_A are positive. Whereas θ_B is –ve.



Standard Cases using the deflection equation of flexure.

1. **Cantilever**

a) Concentrated load w at the free end.



AB = l

Consider a section X-X at a distance x from the fix end A.

$$M_x = -w(l-x)$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = -M = w(l-x) = wl - wx$$

Integrating we have

$$EI\frac{dy}{dx} = wlx - \frac{wx^2}{2} + c_1$$

where $\mathbf{C_1}$ – constant of integration

At **A** where **x=0** the slope $\dfrac{dy}{dx}$ is zero, therefore $\mathbf{C_1 = 0}$

Hence
$$EI\,\frac{dy}{dx} = wlx - \frac{wx^2}{2} \quad \ldots \text{(i)}$$

For slope at **B** where **x =** ***l***

$$\theta_B = \frac{dy}{dx} = \frac{1}{EI}\left(wl * l - \frac{wl^2}{2} \right) = \frac{wl^2}{2EI} \quad \ldots \text{(a)}$$

For deflection, integrate equation (i)

$$EIy = \frac{wx^2 l}{2} - \frac{wx^3}{6} + c_2 \qquad \text{where } c_2 \text{ - const of integration}$$

Deflection at **A** is **zero**, thus **y=0** when **x=0** therefore **C_2 = 0**

Hence
$$EIy = \frac{wlx^2}{2} - \frac{wx^3}{6} \quad \ldots \text{(ii)}$$

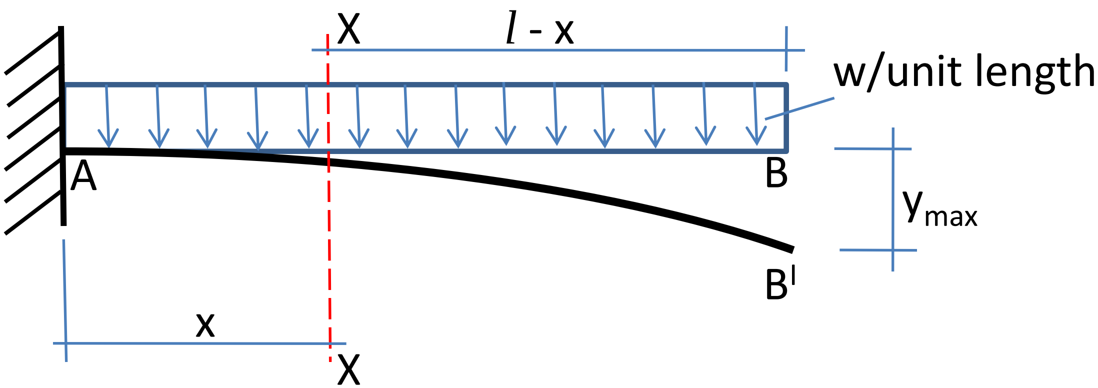
For deflection at B where x = l

$$y_B = \frac{1}{EI}\left(\frac{wl * l^2}{2} - \frac{wl^3}{6}\right) = \frac{wl^3}{3EI} \qquad \ldots(b)$$

Eqns (i) and (ii) $\Rightarrow$ Slope and deflection resp.

Eqns (a) and (b) $\Rightarrow$ Max values of slope and deflection at the free end resp.

2) Carrying u.d.l at the of w/unit length over entire span.



Consider a section X – X at distance x from fixed end A.

$$M_x = \frac{-w(l-x)^2}{2}$$

$$\therefore \qquad EI\frac{d^2y}{dx^2} = -M = \frac{w(l-x)^2}{2} = \frac{w}{2}\left(l^2 - 2lx + x^2\right)$$

Integrating both sides we get:

$$EI\frac{dy}{dx} = \frac{w}{2}\left[l^2x - \frac{2lx^2}{2} + \frac{x^3}{3}\right] + c_1$$

Now at point A the slope is zero, $\therefore$ putting $\dfrac{dy}{dx} = 0$

when **x = 0** $\Rightarrow$ **C_1 = 0**

$$EI\frac{dy}{dx} = \frac{w}{2}\left[l^2x - lx^2 + \frac{x^3}{3}\right] \qquad \ldots\text{(i)}$$

For slope at B, put **x = *l***

$$\theta_B = \frac{dy}{dx} = \frac{w}{2EI}\left(l^2 * l - l * l^2 + \frac{l^3}{3}\right) = \frac{wl^3}{6EI} = \frac{Wl^2}{6EI}$$

where

$$W = wl \qquad \ldots\text{(a)}$$

Integrating eqn (i) for deflection, we have:

$$EIy = \frac{w}{2}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right) + c_2$$

Deflection y at **A** is **zero**. Thus **y = 0** when **x = 0**, $\therefore$ **C_2 = 0**

Hence

$$EIy = \frac{w}{2}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right) \quad \ldots\text{(ii)}$$

For deflection at **B**, put **x = *l***

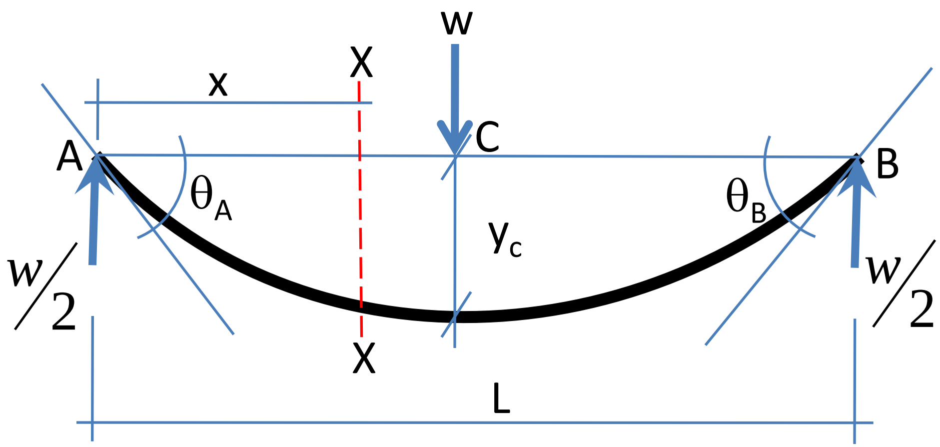
$$y_B = \frac{w}{2EI}\left(\frac{l^2 * l^2}{2} - \frac{l * l^3}{12}\right) = \frac{wl^4}{8EI} = \frac{Wl^3}{8EI} \quad \ldots\text{(b)}$$

where

$$W = wl$$

2) Simply Supported Beam

a) Point load at mid-span



Consider a section X –X at a distance x from the support A but within the portion AC. By symmetry the support reactions at A and B are equal to

$$R_A = R_B = \frac{w}{2} \qquad \therefore \quad M_x = \frac{w}{2}x$$

But

$$EI\frac{d^2y}{dx^2} = -M \quad \Rightarrow \quad -\frac{w}{2}x = EI\frac{d^2y}{dx^2}$$

Integrating above expression for the slope, we have:

$$EI\,\frac{dy}{dx} = -\frac{wx^2}{4} + c_1 \qquad \text{where } c_1 \text{ - integration const.}$$

At mid span C, the slope is zero

$$ie. \quad \frac{dy}{dx} = 0 \quad \text{where} \quad x = \frac{l}{2}$$

$$\therefore \quad EI\,\frac{dy}{dx} = \frac{-w\left(\frac{l}{2}\right)^2}{4} + c_1 = 0 \quad \Rightarrow \quad c_1 = \frac{wl^2}{16}$$

$$\therefore \quad EI\,\frac{dy}{dx} = -\frac{wx^2}{4} + \frac{wl^2}{16}$$

For slope at **A** where **x = 0**

$$\theta_A = \frac{dy}{dx} = \frac{wl^2}{16EI} \qquad \text{By symmetry}\ \theta_B = -\theta_A = -\frac{wl^2}{16EI}$$

For deflection, further integrate expression (i) above:

$$EIy = -\frac{wx^3}{12} + \frac{wl^2 x}{16} + c_2 \qquad \text{where } c_2 - \text{Integration const.}$$

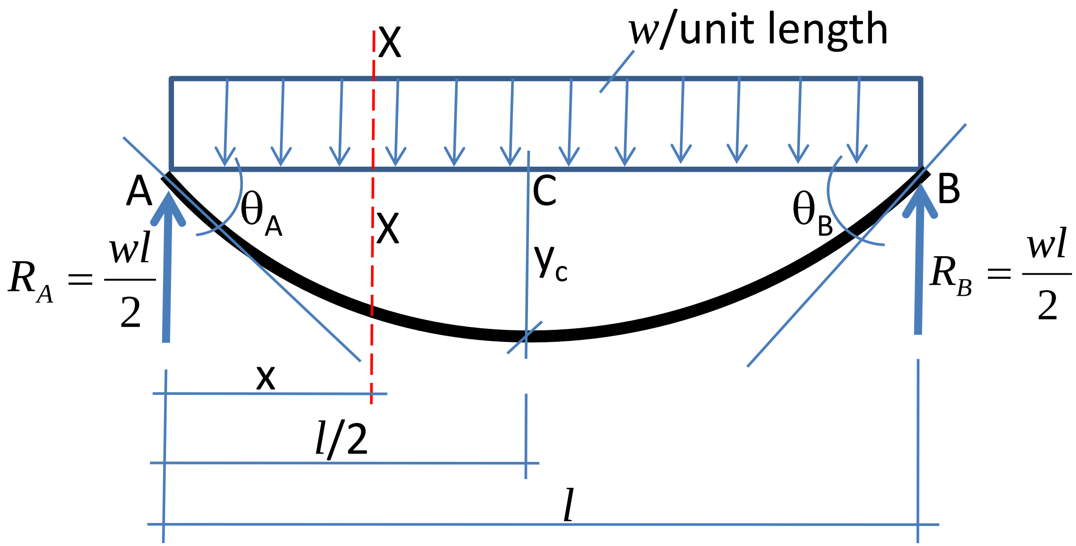
At **A** deflection is **zero**, ie. **Y = 0** when **x = 0**. $\therefore$ **C_2 = 0**

$$\therefore \quad EIy = -\frac{wx^3}{12} + \frac{wl^2 x}{16} \quad \ldots\text{(ii)}$$

For deflection at C , put $x = \dfrac{l}{2}$

$$y_c = \frac{1}{EI}\left[\frac{-w\left(\frac{l}{2}\right)^3}{12} + \frac{wl^2\left(\frac{l}{2}\right)}{16}\right] = \frac{wl^3}{48EI}$$

b) U.D.L of w/unit length over the while span



Consider a section at distance x from the support A and within portion AC. By symmetry, support reactions at A and B are each equal to $\frac{wl}{2} = R_A = R_B$

$$\therefore \qquad M_x = \frac{wlx}{2} - \frac{wx^2}{2}$$

But $$EI\frac{d^2y}{dx^2} = -M = -\frac{wlx}{2} + \frac{wx^2}{2}$$

Integrating the above for slope, we have:

$$EI\,\frac{dy}{dx} = -\frac{wlx^2}{4} + \frac{wx^3}{6} + c_1$$

At mid-point, slope is zero ie.

$$\frac{dy}{dx} = 0 \qquad \text{where} \quad x = \frac{l}{2}$$

$$\therefore \quad -\frac{wl}{4}\left(\frac{l}{2}\right)^2 + \frac{w}{6}\left(\frac{l}{2}\right)^3 + c_1 \qquad \Rightarrow \quad c_1 = +\frac{wl^3}{24}$$

$$\therefore \quad EI\,\frac{dy}{dx} = -\frac{wlx^2}{4} + \frac{wx^3}{6} + \frac{wl^3}{24} \qquad \text{... (i)}$$

For slope at A, put x = 0

$$\theta_A = \frac{wl^3}{24EI} \qquad \text{and by symmetry}$$

$$\theta_A = -\theta_B$$

$$\theta_A = \frac{wl^3}{24EI} = \frac{Wl^2}{24EI} \quad \ldots \text{ (ii) where } W = wl$$

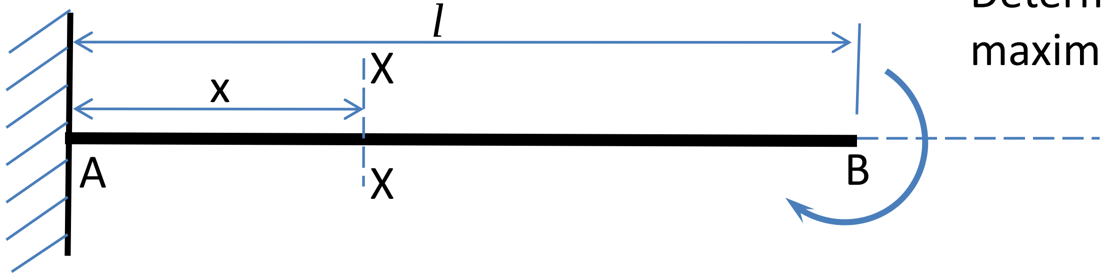
For deflection, further integrate expression (i) above:

$$EIy = -\frac{wlx^3}{12} + \frac{wx^4}{24} + \frac{wl^3x}{24} + c_2 \qquad \begin{bmatrix} y_A = 0 \\ at\ x = 0 \\ c_2 = 0 \end{bmatrix}$$

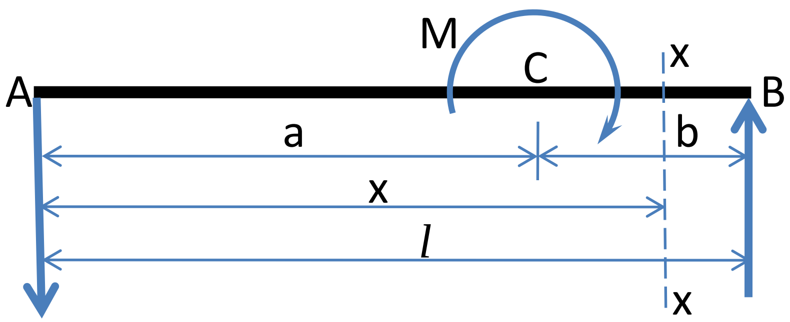
For deflection at C put $x = l/2$

$$y_c = \frac{1}{EI}\left[-\frac{wl}{12}\left(\frac{l}{2}\right)^3 + \frac{w}{24}\left(\frac{l}{2}\right)^4 + \frac{wl^3}{24}\left(\frac{l}{2}\right)\right] \quad \Rightarrow \quad \frac{5wl^4}{384EI} = \frac{5Wl^3}{384EI}$$

Moments applied on a beam



Determine the slope and maximum deflection at B



Determine the slope and maximum deflection at C