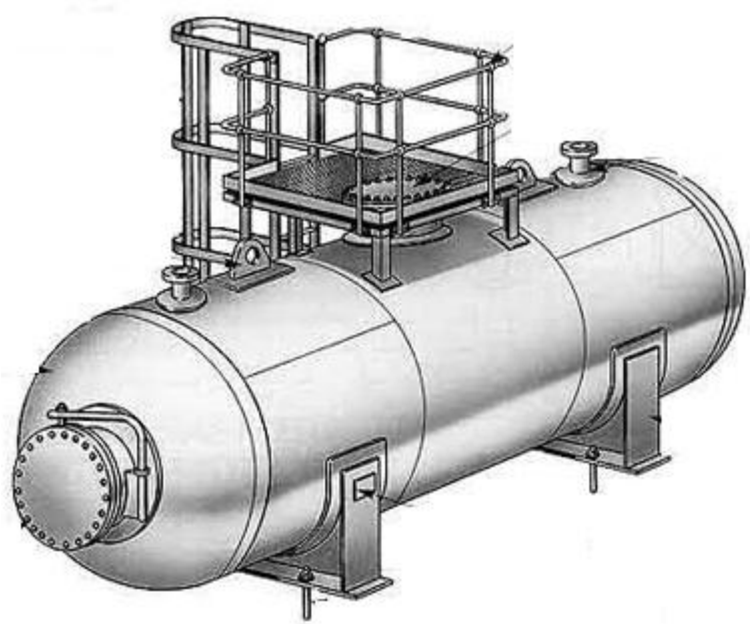
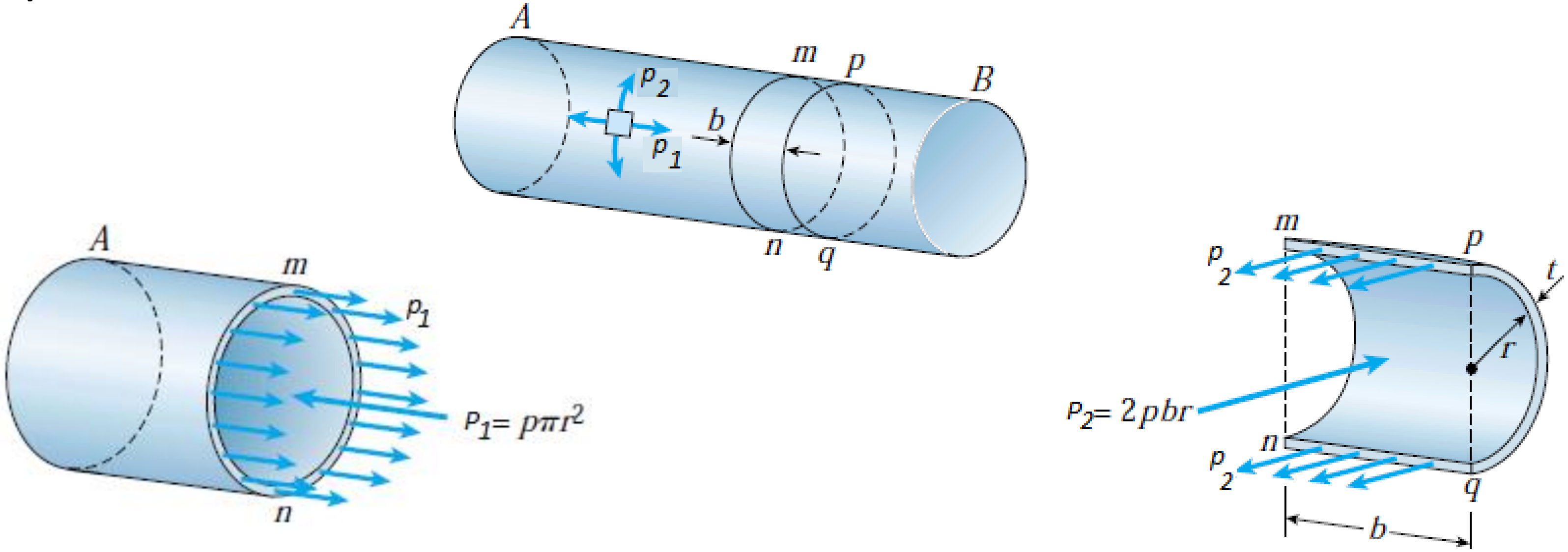
Thin Cylindrical Shells and Spheres



- Water supply main pipes, steam boilers, gas pipes and air vessels etc. are subjected to internal fluid pressures.
- Pressures are taken to be uniformly distributed over internal surfaces.
- **Wall thickness** of pressure vessels are **so small** compared to the large cross-sectional internal diameter.
- If wall thickness is equal to or **less** than **1/20** of the internal diameter the vessel $\Rightarrow$ **Thin walled**.
- Otherwise $\Rightarrow$ **Thick walled**.
- Lecture is concerned with **stresses arising** from uniform internal pressure acting on a cylinder or a sphere or cylinder with spherical ends
- The following is assumed:
    1) **Radial stresses** in cylinder walls are **neglected**
    2) There are no longitudinal stays in the cylinder
    3) Stresses are **uniformly distributed** through the wall.

Thin Cylindrical Circular Shell

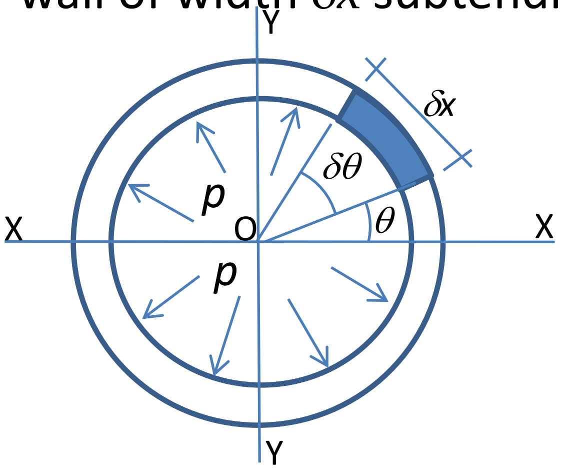
Consider the effect of an internal pressure *p* due to a fluid enclosed in along cylinder closed at both ends.



- Pressure acting at the ends is transmitted to the walls producing **longitudinal tensile stress** p_1 in the walls.
- Internal pressure acting on the long sided of the cylinder give rise to ***circumferential stress*** p_2 called **Hoop stresses**.

a) Circumferential Stress or Hoop Stress

Consider an elementary portion of the cylinder wall of width δx subtending an angle $\delta\theta$ at the centre and θ with X – X.



Let

r - Internal radius

l - overall length of cylinder.

P'- Total internal pressure.

$$P^{/} = p * \text{area of elementary portion}$$

$$= p * \delta x * l$$

$$= p * r\delta\theta * l \qquad \left(\because \quad \frac{\delta x}{r} = \delta\theta \right)$$

$\therefore$ Vertical component of this force $= pr * \delta\theta * l \sin\theta$

$\therefore$ Total vertical upward force on the semicircular cylindrical shell above X-X

$$= \int_0^{\pi} prl \sin\theta \, \delta\theta$$

$$= prl[-\cos\theta]_0^{\pi} = p * 2r * l$$

$$= p * \text{horizontal projected area of the shell}$$

- Similarly, the vertical downward force on the semicircular portion of the cylindrical below X-X is also ***p x 2r x l*** .
- These two equal and opposite forces tend to burst the shell longitudinally at the plane X-X.
- This bursting force is balanced by resisting **Hoop stress** created.

$$\therefore \qquad p2rl = 2 p_h t l$$

$$or \qquad p_h = \frac{pr}{t} \qquad \ldots(1)$$

b) Longitudinal Stress

Consider a shell of wall thickness *t* and radius *r* subjected to internal pressure *p.*

Resultant force of the internal pressure at the end is $P = p * \pi r^2$

If p_l - Longitudinal resisting stress developed in the shell, then

$$P = p_l * 2\pi r * t$$

$$or \quad p * \pi r^2 = p_l * 2\pi r * t$$

$$\therefore \qquad p_l = \frac{p * \pi r^2}{2\pi r * t} = \frac{pr}{2t} \qquad \ldots(2)$$

Hoop stress p_h is thus twice the longitudinal stress. As such, of the two stress Hoop stress p_h is the maximum.

c) Maximum Shear Stress

At any point on the circumference of the cylindrical shell there is set of two mutually perpendicular stresses p_h and p_l which are principal stresses and as such the planes in which these act are the principal planes.

∴ The maximum shear stress is

$$q_{\max} = \frac{p_h - p_l}{2}$$

$$= \frac{\frac{pr}{t} - \frac{pr}{2t}}{2} = \frac{pr}{4t} \qquad \ldots(3)$$

Design of Thin Cylindrical Shells

If wall thickness of a thin cylindrical shell is desired to be determined so that it can withstand a given internal pressure *p* then we have to ensure that the maximum stress developed in the shell does not exceed the allowable tensile stress $[p_t]$ for the material of the shell.

Since the hoop stress is bigger, so we design the shell based on it.

But

$$p_h = \frac{pr}{t}, \qquad \text{where t - required shell thickness.}$$

But p_h is not to exceed p_t

$$\therefore \qquad \frac{pr}{t} \le p_t$$

$$\text{or} \qquad t \ge \frac{pr}{[p_t]}$$

Strains in Thin Cylindrical Shell due to Internal Pressure

If

t – thin cylinder wall thickness

r – cylinder radius

p – internal fluid pressure

Hoop stress and longitudinal stress:

$$p_h = \frac{pr}{t} \qquad \text{and} \qquad p_l = \frac{pr}{2t}$$

If for the material that the shell is made of

$$\text{Poisson's ratio} = \frac{1}{m}$$

$$\text{and} \quad \text{Modulus of elasticity} = E$$

then circumferential or Hoop strain is :

$$e_h = \frac{1}{E}\left(p_h - \frac{1}{m} p_l \right)$$

Substituting values for p_h *and* p_l we have:

$$e_h = \frac{1}{E}\left(\frac{pr}{t} - \frac{pr}{2tm}\right)$$

$$= \frac{pr}{tE}\left(1 - \frac{1}{2m}\right) \qquad \ldots(5)$$

Longitudinal strain

$$e_l = \frac{1}{E}\left(p_l - \frac{1}{m}p_h\right)$$

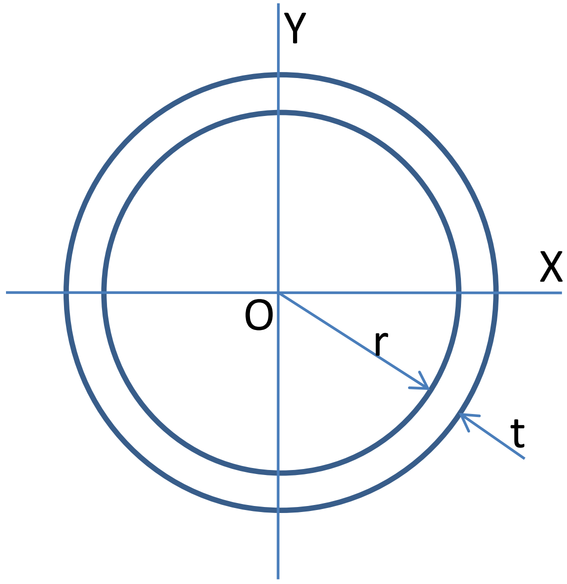
Substituting values for p_h *and* p_l we have:

$$e_l = \frac{1}{E}\left(\frac{pr}{2t} - \frac{pr}{tm}\right)$$

$$= \frac{pr}{tE}\left(\frac{1}{2} - \frac{1}{m}\right) \qquad \ldots(6)$$

Strain along OX and OY is equal and is e_h whereas along axis of the cylinder it is e_l .

$\therefore$ Volumetric strain



$$e_v = e_x + e_y + e_z$$

$$But \quad e_x = e_y = e_h$$

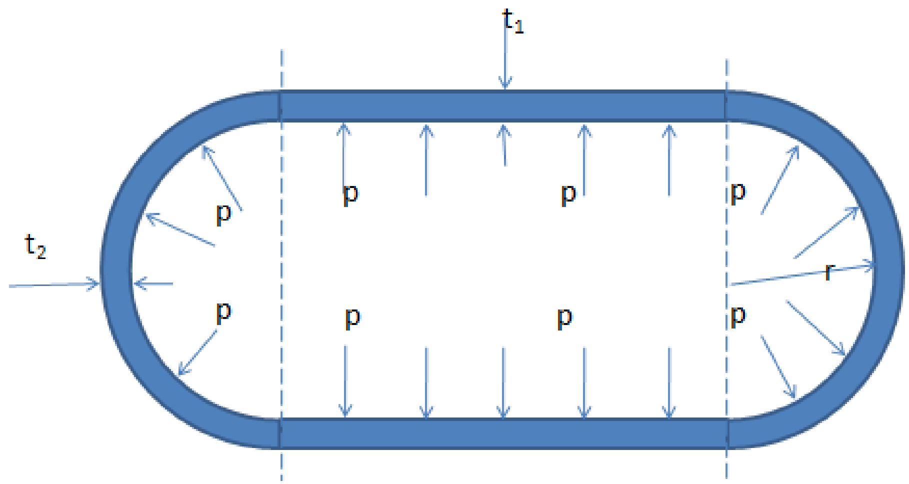
$$\therefore \quad e_v = 2e_h + e_l$$

$$= \frac{2pr}{tE}\left(1 - \frac{1}{2m}\right) + \frac{pr}{tE}\left(\frac{1}{2} - \frac{1}{m}\right)$$

$$= \frac{pr}{tE}\left(\frac{5}{2} - \frac{2}{m}\right) \qquad \text{.......(7)}$$

Thin Cylindrical Shells with Hemispherical Ends

Let ***r*** be the radius of the shell and respective wall thickness of the cylindrical and spherical ends be t_1 and t_2 with internal pressure ***p*** .



If ***p*** be internal pressure shell is subjected to, then the hoop strain:

For cylindrical portion is:

$$e_h = \frac{pr}{t_1 E}\left(1 - \frac{1}{2m}\right) \qquad \ldots\text{(i)} \qquad (\because (5))$$

For hemispherical ends is:

$$e = \frac{pr}{2t_2 E}\left(1 - \frac{1}{m}\right) \qquad \ldots\text{(ii)}$$

So that there is no distortion at the junction of cylindrical and hemispherical portions, the hoop strains in the two have to be equal,

$$e_h = e \qquad \text{.... eqns (i) and (ii)}$$

Assuming that both the portions are made of the same material, then

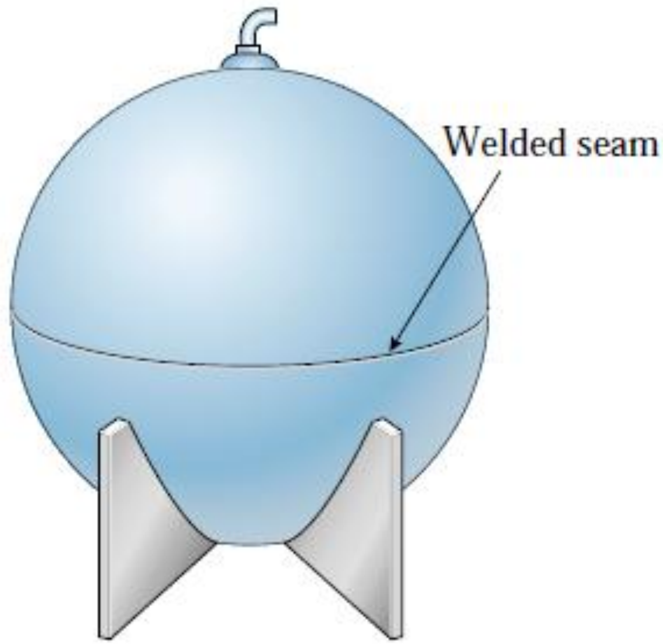
$$\frac{pr}{t_1 E}\left(1-\frac{1}{2m}\right) = \frac{pr}{2t_2 E}\left(1-\frac{1}{m}\right)$$

$$\therefore \qquad \frac{t_2}{t_1} = \frac{\left(1-\frac{1}{m}\right)}{2\left(1-\frac{1}{2m}\right)} = \frac{(m-1)}{(2m-1)} \qquad \text{...(8)}$$

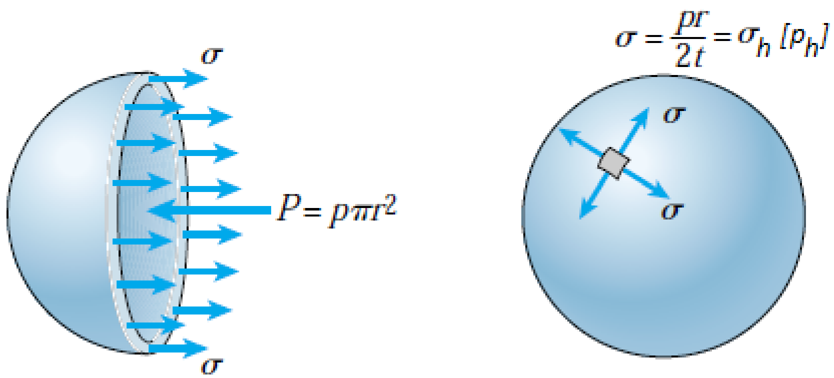
$$or \qquad t_2 : t_1 \;::\; (m-1):(2m-1)$$

t_2 shall always be less that ***t_1*** regardless the value of ***m,*** the hemispherical end is always thinner than cylindrical portion.

Thin Spherical Shell



Consider a diametral plane through the shell, internal radius ***r*** and wall thickness ***t*** subjected to internal pressure ***p***.



Resultant force of internal pressure normal to the diametral plane is

$$P = p * \pi r^2$$

For equilibrium, we equally have resisting force

$$P = p_h * 2\pi r * t$$

Therefore

$$p * \pi r^2 = p_h * 2\pi r * t$$

or $$p_h = \frac{p * \pi r^2}{2\pi r * t} = \frac{pr}{2t} \qquad \ldots(9)$$

Since stress p_h is the same in all the direction X,Y and Z-axis, therefore strain too in all the three planes are same.

The diameter of the shell in all diametral plane is strained by an amount:

$$e = \frac{1}{E}\left(p_h - \frac{1}{m}p_h\right) = \frac{p_h}{E}\left(1 - \frac{1}{m}\right)$$

$$= \frac{pr}{2tE}\left(1 - \frac{1}{m}\right) \qquad \dots(10) \qquad \left(\because p_h = \frac{pr}{2t}\right)$$

Since stress p_h is the same in all the direction X,Y and Z-axis, therefore strain too in all the three planes are same.

Therefore, volumetric strain

$$e_v = e_x + e_y + e_z = e + e + e = 3e$$

$$\frac{3p_h}{E}\left(1 - \frac{1}{m}\right) = \frac{3pr}{2tE}\left(1 - \frac{1}{m}\right) \qquad \dots(11)$$

End of Lecture