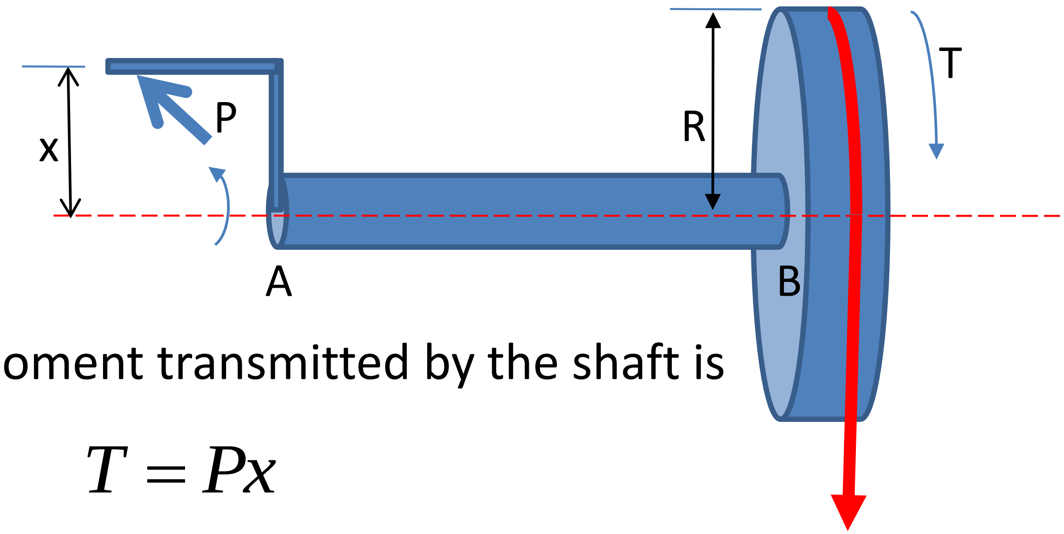
Torsion



Turning moment transmitted by the shaft is

$$T = Px$$

- The turning moment T – also known as ***torque*** or ***twist moment****.*
- Under torsion, every part of shaft is subjected to shear stress.

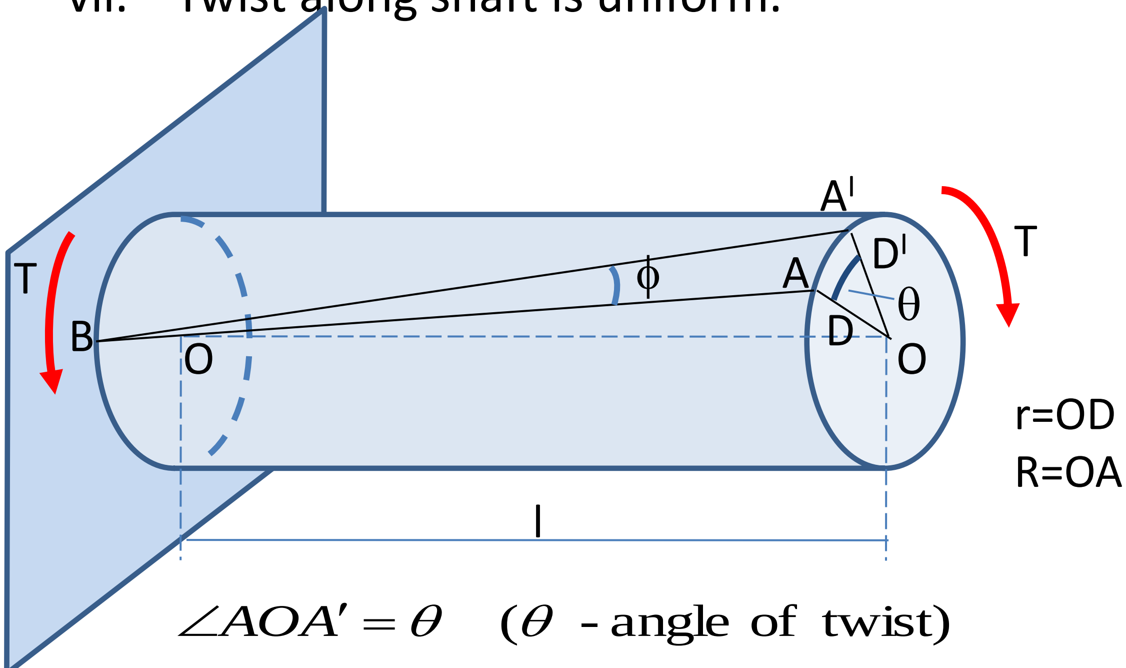
To derive torsion formulae, following is assumed:

i. Circular sections remain circular after twisting.
ii. Plane sections remain plane and do not twist or warp.
iii. All diameters of normal cross-section remain straight after twisting.
iv. Stresses do not exceed proportional limit.

v. Shaft is loaded by twist couples in planes that are perpendicular to the axis of the shaft.

vi. Material of the shaft is uniform throughout .

vii. Twist along shaft is uniform.



Consider shaft of length *l* and radius *R* subjected to torque *T*. Let AB be parallel to shaft axis, takes new position A'B.

$$\angle AOA' = \theta \quad (\theta \text{ - angle of twist})$$

$$\angle ABA' = \phi = \frac{AA'}{AB} \quad (\phi \text{ - shear strain on the surface of the shaft})$$

$$AA' = R\theta = l\phi \quad \left(\text{From the relationship} \quad \theta = \frac{l}{r}\right)$$

$$\therefore \phi = \frac{R\theta}{l} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If q_s – shear stress on outer surface of shaft

C - Modulus of rigidity of shaft material.

Then $$C = \frac{q_s}{\phi} \qquad \ldots\text{(i)}$$

$$\therefore \quad \phi = \frac{q_s}{C}$$

But from equation (1) above: $$\phi = \frac{R\theta}{l}$$

$$\therefore \quad \frac{R\theta}{l} = \frac{q_s}{C} \quad \text{Or} \quad \frac{C\theta}{l} = \frac{q_s}{R} \qquad \ldots\text{(ii)}$$

The point D at the radius r from centre takes the position D^{l} after twisting.

Strain will be as in (i):

$$DD' = r\theta = l\phi$$

$$\therefore \quad \phi = \frac{r\theta}{l}$$

If q be the stress at DD^{l} then as above

$$\frac{q}{C} = \frac{r\theta}{l}$$

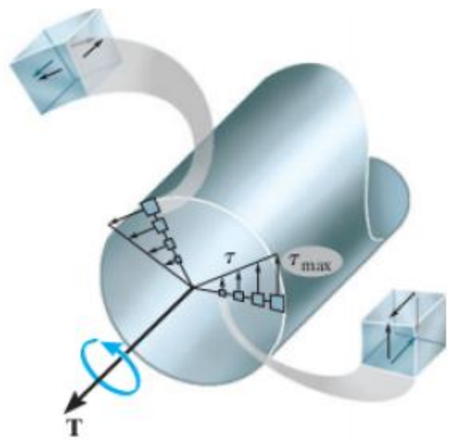
$$\therefore \quad \frac{q}{r} = \frac{C\theta}{l} \qquad \ldots\text{(iii)}$$

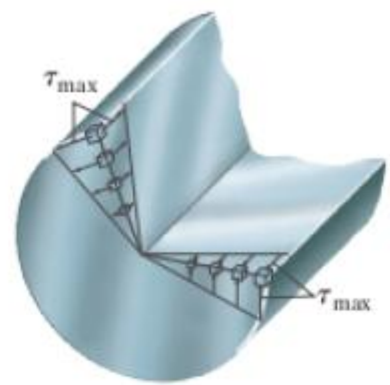
From equations (ii) and (iii) above

$$\frac{q}{r} = \frac{C\theta}{l} = \frac{q_s}{R} \qquad \ldots(2) \qquad \left(or \quad \frac{q}{r} = \frac{G\theta}{l} = \frac{\tau}{R} \right)$$

From above, shear stress varies linearly along each radial line from zero to max

$$q_{s\,Max} = \frac{C\theta R}{l}$$

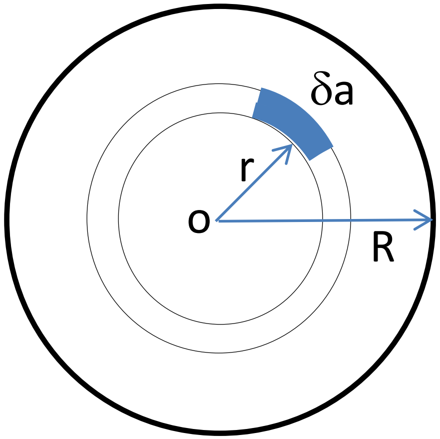




Moment of Resistance

Circular section of shaft of radius *R* and length *l* subjected to torque *T* .

Consider an element area δa at distance r from the axis of shaft.



Shear Stress at element δa is:

$$q = \frac{rq_s}{R} \qquad \ldots\text{(from eqn 2)}$$

∴ Total shear resistance offered by the area δa:

$$= q * \delta a$$

$$= \frac{rq_s}{R}\delta a$$

Moment of resistance offered by δa: $= Force * r = \frac{rq_s \delta a}{R} * r = \frac{r^2 q_s}{R}\delta a$

Total resistance offered by the whole section is:

$$T = \frac{q_s}{R} \sum r^2 * \delta a$$

$\sum r^2 * \delta a$ - Polar moment of inertia of the section and its denoted by “J”.

$$\therefore \qquad T = \frac{q_s}{R} * J$$

$$or \qquad \frac{T}{J} = \frac{q_s}{R}$$

But from equation 2

$$\frac{q_s}{R} = \frac{q}{r} = \frac{C\theta}{l}$$

Therefore $$\frac{q}{r} = \frac{q_s}{R} = \frac{C\theta}{l} = \frac{\mathrm{T}}{\mathrm{J}} \quad \ldots\ldots(3)$$

For Solid circular shafts:

$$J = \frac{\pi D^4}{32}$$

For Hollow circular shafts:

$$J = \frac{\pi\left(D_o^{\ 4} - D_i^{\ 4}\right)}{32}$$

Horse Power Transmitted by a Shaft

Let a shaft subjected to an average torque T kg.m rotate at N r.pm and transmit P horse power.

Angle turned by shaft per rev. = 2π radians.

Angle turned by shaft per minute in N rev. = 2π N radians

Work done by shaft per minute = Torque x angle turned in one minute

= T x 2π N kg.m

Power transmitted by shaft:

$$P = \frac{\text{Work done per minute in kg.m}}{4500}$$

$$P = \frac{2\pi NT}{4500} \quad \text{H.P.} \quad ....(4)$$

Since 1kW = 102 kg m/sec

= 102 x 60 kg m/minute.

$\therefore$ Power transmitted by shaft

$$= \frac{2\pi NT}{102 * 60} kW$$

$$= \frac{\pi NT}{3060} kW \quad ....(5)$$

Comparison Between Solid and Hollow Shafts

(a) By Weight for Equal Strength.

Let:

Outer diam. of hollows shaft = D_o

Inside diam. of hollow shaft = D_i

Diam. of solid shaft = D_o

Length of both shafts = l

Density of shafts materials = δ

Weight of hollow shaft = W_H

Weight of solid shaft = W_S

Now
$$\frac{W_H}{W_S} = \frac{\pi\left(D_o{}^2 - D_i{}^2\right)*l*\delta l\ 4}{4*\pi D^2*l*\delta} = \frac{\left(D_o{}^2 - D_i{}^2\right)}{D^2} = \frac{D_o{}^2\left(1-n^2\right)}{D^2}$$

$$\ldots\ldots\ldots(i)$$

Since torque transmitted by both shafts is equal,

$$\therefore \qquad \frac{q_s}{\frac{D_o}{2}} * \frac{\left(D_o{}^2 - D_i{}^2\right)}{32} = \frac{q_s}{\frac{D_o}{2}} * \frac{\pi D^4}{32} \qquad \left(\because T = \frac{q}{R} J\right)$$

Or $$\frac{D_o{}^4 - D_i{}^4}{D_o} = \frac{D^4}{D} = D$$

But $$D_i = nD_o$$

$\therefore$ $$D_o{}^3\left(1 - n^4\right) = D^3$$

Or $$\frac{D_o}{D} = \frac{1}{\left(1 - n^4\right)^{1/3}}$$

$\therefore$ $$\frac{D_o{}^2}{D^2} = \frac{1}{\left(1 - n^4\right)^{2/3}}$$

Substituting in eqn (i) above $$\frac{W_H}{W_S} = \frac{\left(1 - n^2\right)}{\left(1 - n^4\right)^{2/3}}$$ -By weight for equal strength comparison)

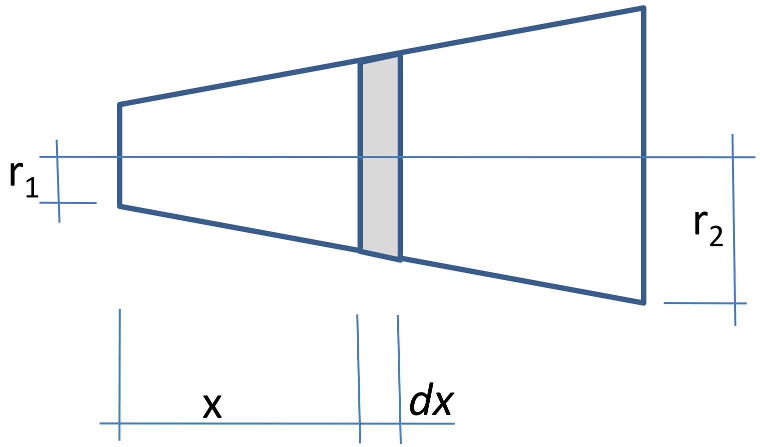
(b) Strengths of two given Rods.

Note:

- The one capable of transmitting more torque is stronger.
- If lengths, weights, materials of two shafts are the same and the torques transmitted by both are equal, then the one in which maximum shear stress developed on the surface is weaker of the two shafts.

Shafts of Tapering Section

Tapered circular shaft of length *l* subjected to torque *T*.



Consider a section of length *dx* at distance x from the smaller end.

From the relationship $$\frac{C\theta}{l} = \frac{T}{J}$$

We have $$\theta = \frac{Tl}{CJ}$$

If $d\theta$ be the angle of twist and r radius section, Then

$$d\theta = \frac{T * dx}{CJ} = \frac{Tdx}{C\,\dfrac{\pi d^4}{32}} = \frac{Tdx}{\pi C\,\dfrac{(2r)^4}{32}} = \frac{2Tdx}{\pi C r^4}$$

But $$r = r_1 + x\,\frac{r_2 - r_1}{l}$$

$$= (r_1 + kx) \qquad \text{where } k = \frac{r_2 - r_1}{l} \qquad \text{(a constant for a given shaft)}$$

$$\therefore \qquad d\theta = \frac{2Tdx}{\pi C (r_1 + kx)^4}$$

For the whole shaft

$$\theta = \int_0^l \frac{2Tdx}{\pi C (r_1 + kx)^4} = \ldots\ldots = \frac{2Tl}{3\pi C}\left[\frac{\left(r_1^{\,2} + r_1 r_2 + r_2^{\,2}\right)}{r_1^{\,3} r_2^{\,3}}\right] \quad \ldots(1)$$