MEC 3352 – STRENGTH OF MATERIALS II

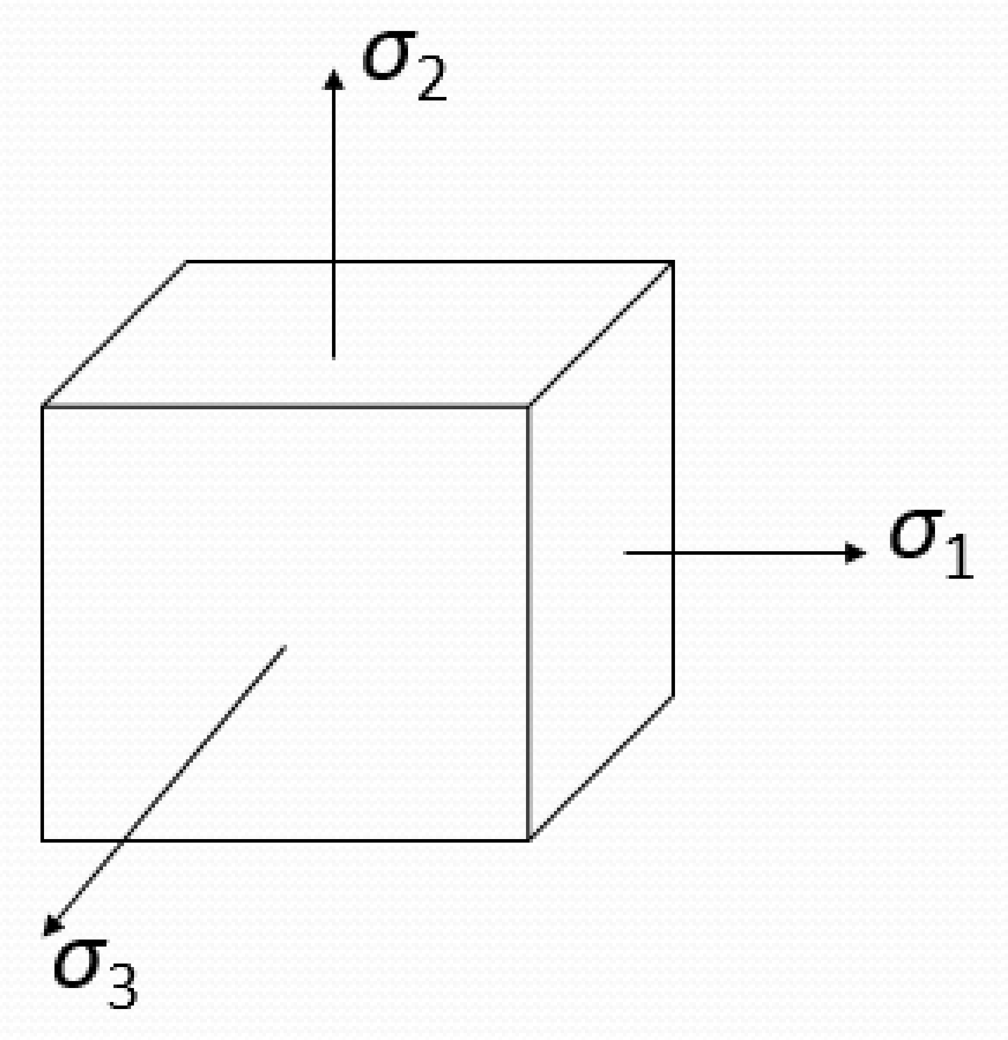
Principal Strains in Three Dimensions

G M Munakaampe

PRINCIPAL STRAINS IN THREE DIMENSIONS

From the 2D case, it can be shown that the principal strains in the directions σ_1, σ_2 and σ_3 are:

- $\varepsilon_1 = \sigma_1/E - v\sigma_2/E - v\sigma_3/E$ (1)
- $\varepsilon_2 = \sigma_2/E - v\sigma_3/E - v\sigma_1/E$ (2)
- $\varepsilon_3 = \sigma_3/E - v\sigma_1/E - v\sigma_2/E$ (3)



Principal Stresses Determined from Principal Strains

(a) 3D stress system. Re-writing Eqs. (1), (2) and (3):

- $E\varepsilon_1 = \sigma_1 - v\sigma_2 - v\sigma_3$ (4)
- $E\varepsilon_2 = \sigma_2 - v\sigma_3 - v\sigma_1$ (5)
- $E\varepsilon_3 = \sigma_3 - v\sigma_1 - v\sigma_2$ (6)

- Subtracting (5) from (4):
  - $E(\varepsilon_1 - \varepsilon_2) = (\sigma_1 - \sigma_2)(1 + v)$ (7)

From (1) and (3), eliminating σ_3:

$$E(\varepsilon_1 + v\varepsilon_3) = \sigma_1(1 - v^2) - \sigma_2(1 + v)v \qquad (8)$$

Multiplying (7) by v and subtracting from (8):

$$E[(1- v)\varepsilon_1 + v(\varepsilon_2 + \varepsilon_3)] = \sigma_1(1 - v^2) - \sigma_2(1 + v)v - (\sigma_1 - \sigma_2)(1 + v)v$$

$$= \sigma_1(1 - v - 2v^2)$$

$$\boldsymbol{= \sigma_1(1 + v)(1 - 2v)}$$

Rearranging:

$$\sigma_1 = \frac{E[(1-\nu)\varepsilon_1 + \nu(\varepsilon_2 + \varepsilon_3)]}{(1+\nu)(1-2\nu)} \tag{9}$$

Similarly

$$\sigma_2 = \frac{E[(1-\nu)\varepsilon_2 + \nu(\varepsilon_3 + \varepsilon_1)]}{(1+\nu)(1-2\nu)} \tag{10}$$

and

$$\sigma_3 = \frac{E[(1-\nu)\varepsilon_3 + \nu(\varepsilon_1 + \varepsilon_2)]}{(1+\nu)(1-2\nu)} \tag{11}$$

(b) *2D stress system.*

If $\sigma_3 = 0$, (4), (5) and (6) reduce to:

$$E\varepsilon_1 = \sigma_1 - \nu\sigma_2 \quad (12)$$

$$E\varepsilon_2 = \boldsymbol{\sigma_2} - \nu\sigma_1 \quad (13)$$

Solving (12) and (13) for σ_1 and σ_2 gives

$$\sigma_1 = \frac{E(\varepsilon_1 + \nu\varepsilon_2)}{(1-\nu^2)} \quad (14)$$

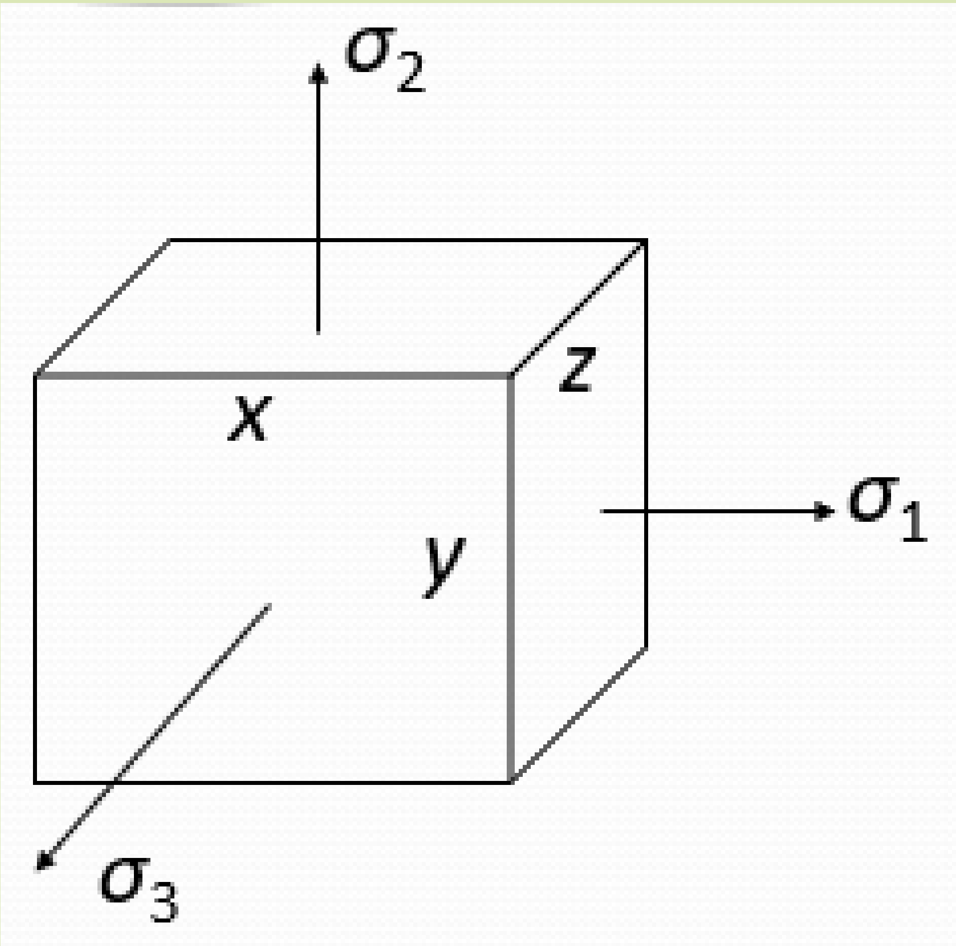
and

$$\sigma_2 = \frac{E(\varepsilon_2 + \nu\varepsilon_1)}{(1-\nu^2)} \quad (15)$$

Volumetric Strain

Consider a block with sides x, y and z acted upon by principal stresses σ_1, σ_2 and σ_3.

Let corresponding linear strains be ε_1, ε_2 and ε_3.



Corresponding resultant dimensions are:

$$x + \varepsilon_1 x, \quad y + \varepsilon_2 y \quad \text{and} \quad z + \varepsilon_3 z$$

or $$x(1 + \varepsilon_1), \quad y(1 + \varepsilon_2) \quad \text{and} \quad z(1 + \varepsilon_3)$$

Let ***volumetric strain***, be $$\varepsilon_v = \frac{\text{Change in volume}}{\text{Original volume}}$$

$$\varepsilon_V = \frac{x(1+\varepsilon_1) + y(1+\varepsilon_2) + z(1+\varepsilon_3)}{xyz}$$

$$= (1+\varepsilon_1) + (1+\varepsilon_2) + (1+\varepsilon_3) - 1$$

Expanding and neglecting second order terms – since strains are small:

$$\varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \qquad (16)$$

Expressing the strains in terms of principal stresses (using Eqs. (1), (2) and (3)):

$$\text{Volumetric Strain}, \varepsilon_V = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1-2\nu)}{E} \qquad (17)$$

Strain Energy

Strain energy, U = work done by a system of stresses in straining a material. Consider a cube acted upon by a system of principal stresses σ_1, σ_2 and σ_3. For the corresponding strains of ε_1, ε_2 and ε_3,

The work done $U = \sum \frac{1}{2}\sigma\varepsilon$ (if the stresses are gradually applied).

$$U = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$$

$$= [\sigma_1(\sigma_1 - v\sigma_2 - v\sigma_3) + \sigma_2(\sigma_2 - v\sigma_3 - v\sigma_1) + \sigma_3(\sigma_3 - v\sigma_1 - v\sigma_2)]$$

from (1), (2) and (3).

$$U = \left(\frac{1}{2E}\right)[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad \text{per unit volume.} \tag{18}$$

For a 2D stress system, $\sigma_3 = 0$

$$\boldsymbol{U = \left(\frac{1}{2E}\right)\left[\sigma_1^2 + \sigma_2^2 - 2\nu(\sigma_1\sigma_2)\right]} \quad \text{per unit volume.} \tag{19}$$

Shear Strain Energy

Writing

$$\sigma_1 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{3}(\sigma_1 - \sigma_2) + \frac{1}{3}(\sigma_1 - \sigma_3) \qquad (20)$$

$$\sigma_2 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{3}(\sigma_2 - \sigma_3) + \frac{1}{3}(\sigma_2 - \sigma_1) \qquad (21)$$

$$\sigma_3 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{3}(\sigma_3 - \sigma_1) + \frac{1}{3}(\sigma_3 - \sigma_2) \qquad (22)$$

Then under the action of the mean stress there WILL BE volumetric strain with NO distortion of shape (i.e. no shear stress anywhere).

∴ The strain energy under this mean stress acting in each direction is obtained from the general formula, Eq. (18), and may be called *the volumetric strain energy*,

$$U=\left(\frac{3}{2E}\right)\left[\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right]^2\cdot(1-2\nu)$$

Giving

$$\boldsymbol{U=\left(\frac{1}{6E}\right)[\sigma_1+\sigma_2+\sigma_3]^2\cdot(1-2\nu)} \qquad (23)$$

The other terms in the rearrangement of σ_1, σ_2 and σ_3 are proportional to the maximum shear stress values in the three planes, and will cause a distortion of the shape.

Let U_s = Shear strain energy = Total strain energy – Volumetric strain energy

$$U_s = \left(\frac{1}{2E}\right)[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\upsilon(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

$$- \left(\frac{1}{6E}\right)[(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \cdot (1 - 2\nu)]$$

$$= \left(\frac{1}{6E}\right)[(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)(3 - 1 + 2\nu) - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)(6\nu + 2 - 4\nu)]$$

$$= \left(\frac{1 + \nu}{6E}\right)[2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

$$\boldsymbol{U_s = \left(\frac{1}{12G}\right)\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \qquad (24)$$

since $E = 2G(1 + \upsilon)$.

The quantities in brackets are each twice the maximum shear stress in their respective planes .

In pure shear system (stress τ), the principal stresses are $\pm\tau$, 0 (review maximum shear stresses).

$$\therefore \qquad U_s = \left(\frac{1}{12G}\right)\left[(2\tau)^2 + (-\tau)^2 + (-\tau)^2\right]$$

(Since $\sigma_1 = \tau$, $\sigma_2 = -\tau$ when $\sigma_3 = 0$)

$$\boldsymbol{U_s = \frac{\tau^2}{2G}} \qquad (25)$$

(Compare with strain energy in direct shear stress)

The End

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For it is, this far, the best there can be among courses!!!