THE UNIVERSITY OF ZAMBIA
SCHOOL OF ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING

MEC 3352 – STRENGTH OF MATERIALS II

Principal Strains in Three Dimensions

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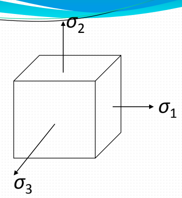
The 3D Principal Strains

From the 2D case, it can be shown that the principal strains in 3D σ_1, σ_2 and σ_3 are:



- $\varepsilon_1 = \sigma_1/E - \nu\sigma_2/E - \nu\sigma_3/E$ (1)
- $\varepsilon_2 = \sigma_2/E - \nu\sigma_3/E - \nu\sigma_1/E$ (2)
- $\varepsilon_3 = \sigma_3/E - \nu\sigma_1/E - \nu\sigma_2/E$ (3)

Principal Stresses Determined from Principal Strains

(a) 3D Stress System:

We re-writing eqs. (1), (2) and (3):

- $E\varepsilon_1 = \sigma_1 - \nu\sigma_2 - \nu\sigma_3$ (4)
- $E\varepsilon_2 = \sigma_2 - \nu\sigma_3 - \nu\sigma_1$ (5)
- $E\varepsilon_3 = \sigma_3 - \nu\sigma_1 - \nu\sigma_2$ (6)
- Subtracting (5) from (4):
  - $E(\varepsilon_1 - \varepsilon_2) = (\sigma_1 - \sigma_2)(1 + \nu)$ (7)

From (1) and (3), we eliminate σ_3:

$$E(\varepsilon_1 + v\varepsilon_3) = \sigma_1(1 - v^2) - \sigma_2(1 + v)\, v \qquad (8)$$

Multiplying (7) by v and subtracting from (8):

$$\begin{aligned} E[(1- v)\varepsilon_1 + v(\varepsilon_2 + \varepsilon_3)] &= \sigma_1(1 - v^2) - \sigma_2(1 + v)\, v - (\sigma_1 - \sigma_2)(1 + v)v \\ &= \sigma_1(1 - v - 2v^2) \\ &= \sigma_1(1 + v)(1 - 2v) \end{aligned}$$

Rearranging:

$$\sigma_1 = \frac{E[(1-\upsilon)\varepsilon_1 + \upsilon(\varepsilon_2 + \varepsilon_3)]}{(1+\upsilon)(1-2\upsilon)} \qquad (9)$$

Similarly

$$\sigma_2 = \frac{E[(1-\upsilon)\varepsilon_2 + \upsilon(\varepsilon_3 + \varepsilon_1)]}{(1+\upsilon)(1-2\upsilon)} \qquad (10)$$

and

$$\sigma_3 = \frac{E[(1-\upsilon)\varepsilon_3 + \upsilon(\varepsilon_1 + \varepsilon_2)]}{(1+\upsilon)(1-2\upsilon)} \qquad (11)$$

(b) *2D stress system.*

If $\sigma_3 = 0$, (4), (5) and (6) reduce to:

$$E\varepsilon_1 = \sigma_1 - v\sigma_2 \qquad (12)$$

$$E\varepsilon_2 = \sigma_2 - v\sigma_1 \qquad (13)$$

Solving (12) and (13) for σ_1 and σ_2 gives

$$\sigma_1 = \frac{E(\varepsilon_1 + \upsilon\varepsilon_2)}{(1-\upsilon^2)} \qquad (14)$$

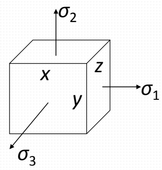
and

$$\sigma_2 = \frac{E(\varepsilon_2 + \upsilon\varepsilon_1)}{(1-\upsilon^2)} \qquad (15)$$

Volumetric Strain

Consider a block with sides *x*, *y* and *z* acted upon by principal stresses σ_1, σ_2 and σ_3.

Let corresponding linear strains be ε_1, ε_2 and ε_3.



Corresponding resultant dimensions are:

$x + \varepsilon_1 x,\ y + \varepsilon_2 y$ and $z + \varepsilon_3 z$

or $x(1 + \varepsilon_1),\ y(1 + \varepsilon_2)$ and $z(1 + \varepsilon_3)$

Volumetric strain, is given as: $\varepsilon_v = \dfrac{Change\,in\ volume}{Original\,volume}$

$$\varepsilon_v = \frac{x(1+\varepsilon_1).y(1+\varepsilon_2).z(1+\varepsilon_3) - xyz}{xyz}$$

$$= (1+\varepsilon_1).(1+\varepsilon_2).(1+\varepsilon_3) - 1$$

Expanding and neglecting second order terms – since strains are small:

$$\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \qquad (16)$$

Expressing the strains in terms of principal stresses (using Eqs. (1), (2) and (3)):

$$Volumetric\,strain,\ \varepsilon_v = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\upsilon)}{E} \qquad (17)$$

Strain Energy

Strain energy, *U* = work done by a system of stresses in straining a material.

Consider a cube acted upon by a system of principal stresses σ_1, σ_2 and σ_3. For the corresponding strains of ε_1, ε_2 and ε_3,

$$U = \sum \frac{1}{2}\sigma\varepsilon$$

The work done (if the stresses are gradually applied).

$$U = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$$

$$= [\sigma_1(\sigma_1 - \upsilon\sigma_2 - \upsilon\sigma_3) + \sigma_2(\sigma_2 - \upsilon\sigma_3 - \upsilon\sigma_1) + \sigma_3(\sigma_3 - \upsilon\sigma_1 - \upsilon\sigma_2)] \quad \text{from (1), (2) and (3).}$$

$$U = \left(\frac{1}{2E}\right)\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\upsilon\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right] \quad (18)$$

per unit volume.

For a 2D stress system, $\sigma_3 = 0$

$$U = \left(\frac{1}{2E}\right)\left[\sigma_1^2 + \sigma_2^2 - 2\upsilon\left(\sigma_1\sigma_2\right)\right] \quad (19)$$

per unit volume.

Shear Strain Energy

Writing

$$\sigma_1 = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) + \frac{1}{3}\left(\sigma_1 - \sigma_2\right) + \frac{1}{3}\left(\sigma_1 - \sigma_3\right) \quad (20)$$

$$\sigma_2 = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) + \frac{1}{3}\left(\sigma_2 - \sigma_3\right) + \frac{1}{3}\left(\sigma_2 - \sigma_1\right) \quad (21)$$

$$\sigma_3 = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) + \frac{1}{3}\left(\sigma_3 - \sigma_1\right) + \frac{1}{3}\left(\sigma_3 - \sigma_2\right) \quad (22)$$

Then, under the action of the mean stress there, WILL BE volumetric strain with NO distortion of shape (i.e. no shear stress anywhere).

$\therefore$ The strain energy under this mean stress acting in each direction is obtained from the general formula, Eq. (18), and may be called *the volumetric strain energy,*

$$U = \left(\frac{3}{2E}\right)\left[\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right]^2 \cdot \left(1 - 2\upsilon\right)$$

Giving

$$U = \left(\frac{1}{6E}\right)\left[\sigma_1 + \sigma_2 + \sigma_3\right]^2 \cdot \left(1 - 2\upsilon\right) \quad (23)$$

The other terms in the rearrangement of σ_1, σ_2 and σ_3 are proportional to the maximum shear stress values in the three planes, and will cause a distortion of the shape.

Let U_s = Shear strain energy = Total strain energy – Volumetric strain energy

$$U_s = \left(\frac{1}{2E}\right)\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\upsilon\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right] - \left(\frac{1}{6E}\right)\left[\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2 \cdot (1 - 2\upsilon)\right]$$

$$= \left(\frac{1}{6E}\right)\left[\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)(3 - 1 + 2\upsilon) - \left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)(6\upsilon + 2 - 4\upsilon)\right]$$

$$= \left(\frac{1+\upsilon}{6E}\right)\left[2\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - 2\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]$$

$$U_s = \left(\frac{1+\upsilon}{12G}\right)\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right] \quad (24)$$

since $E = 2G(1 + \upsilon)$.

The quantities in brackets are each twice the maximum shear stress in their respective planes .

In pure shear system (stress τ), the principal stresses are $\pm\tau$, 0 (review maximum shear stresses).

$$\therefore \quad U_s = \left(\frac{1+\upsilon}{12G}\right)\left[(2\tau)^2 + (-\tau)^2 + (-\tau)^2\right]$$

(Since $\sigma_1 = \tau$, $\sigma_2 = -\tau$ when $\sigma_3 = 0$)

$$U_s = \frac{\tau^2}{2G} \quad (25)$$

(Compare with strain energy in direct shear stress)

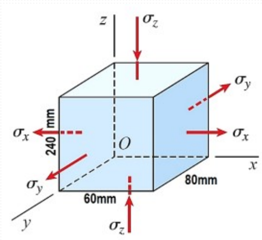
Examples

Example 001

A block of steel measuring 240 mm x 16 mm x 25 mm, is subjected to a tensile force of 40kN in the direction of its length. Given that the modulus of elasticity (*E*) and the Poisson's ratio (ν) are 200kN/mm^2 and 0.33 respectively. Compute the change in the volume of the block.

Examples

Example 002



A rectangular block measuring 240 mm x 80mm x 60mm is loaded as shown in the diagram, and F_x = 80kN, F_y= 100kN, F_z = 40kN. Compute the change in volume, the bulk modulus and modulus of rigidity. Take E = 205kN/mm^2 and ν = 0.36.

ME 3352: Strength of Materials II

Enjoy MEC 3352.

Grazie