THE UNIVERSITY OF ZAMBIA
SCHOOL OF ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING

Strength of Materials II – MEC 3352

PRINCIPAL STRAINS IN 3 - DIMENSIONS

PRINCIPAL PLANES

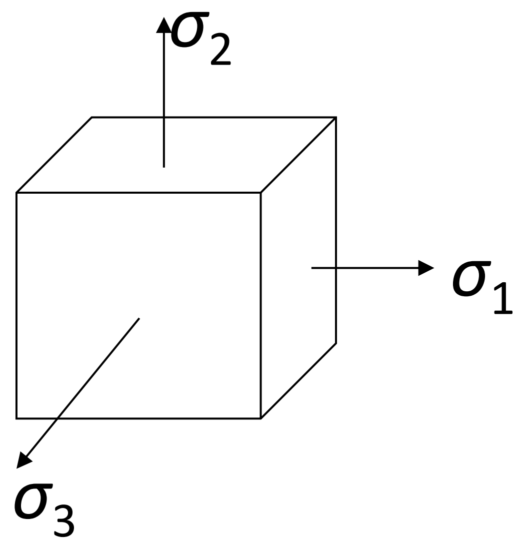
The principal planes can be described in the following manner:

- A plane is said to be a principal plane, when the shear stress acting on that plane is zero.
- The converse of the above statement is also true:
- If the shear stress on a given plane is zero, then that plane must be a principal plane.

- A point subjected to plane stress has three principal stresses: the two in-plane principal stresses σ_1 and σ_2, and a third principal stress σ_3, which acts in the out-of-plane direction.

3D PRINCIPAL STRAINS

From the 2D case, it can be shown that the principal strains in 3D σ_1, σ_2 and σ_3 are:



$$\varepsilon_1 = \sigma_1/E - \nu\sigma_2/E - \nu\sigma_3/E \qquad (1)$$

$$\varepsilon_2 = \sigma_2/E - \nu\sigma_3/E - \nu\sigma_1/E \qquad (2)$$

$$\varepsilon_3 = \sigma_3/E - \nu\sigma_1/E - \nu\sigma_2/E \qquad (3)$$

PRINCIPAL STRESSES DETERMINED FROM PRINCIPAL STRAINS

3D Stress System:

We re-writing equations (1), (2) and (3) as follows:

$$E\varepsilon_1 = \sigma_1 - \nu\sigma_2 - \nu\sigma_3 \qquad (4)$$

$$E\varepsilon_2 = \sigma_2 - \nu\sigma_3 - \nu\sigma_1 \qquad (5)$$

$$E\varepsilon_3 = \sigma_3 - \nu\sigma_1 - \nu\sigma_2 \qquad (6)$$

- Subtracting (5) from (4):

$$E(\varepsilon_1 - \varepsilon_2) = (\sigma_1 - \sigma_2)(1 + \nu) \qquad (7)$$

PRINCIPAL STRESSES DETERMINED FROM PRINCIPAL STRAINS

From (1) and (3), we eliminate σ_3:

$$E(\varepsilon_1 + \nu\varepsilon_3) = \sigma_1(1 - \nu^2) - \sigma_2(1 + \nu)\,\nu \qquad (8)$$

Multiplying (7) by ν and subtracting from (8):

$$\begin{aligned} E[(1- \nu)\varepsilon_1 + \nu(\varepsilon_2 + \varepsilon_3)] &= \sigma_1(1 - \nu^2) - \sigma_2(1 + \nu)\,\nu - (\sigma_1 - \sigma_2)(1 + \nu)\nu \\ &= \sigma_1(1 - \nu - 2\nu^2) \\ &= \sigma_1(1 + \nu)(1 - 2\nu) \end{aligned}$$

PRINCIPAL STRESSES DETERMINED FROM PRINCIPAL STRAINS

Re-arranging:

$$\sigma_1 = \frac{E[(1-\upsilon)\varepsilon_1 + \upsilon(\varepsilon_2 + \varepsilon_3)]}{(1+\upsilon)(1-2\upsilon)} \tag{9}$$

Similarly

$$\sigma_2 = \frac{E[(1-\upsilon)\varepsilon_2 + \upsilon(\varepsilon_3 + \varepsilon_1)]}{(1+\upsilon)(1-2\upsilon)} \tag{10}$$

and

$$\sigma_3 = \frac{E[(1-\upsilon)\varepsilon_3 + \upsilon(\varepsilon_1 + \varepsilon_2)]}{(1+\upsilon)(1-2\upsilon)} \tag{11}$$

2D STRESS SYSTEM:

For a 2D system, σ_3 = 0, and equations (4), (5) and (6) reduce to:

$$E\varepsilon_1 = \sigma_1 - \nu\sigma_2 \qquad (12)$$

$$E\varepsilon_2 = \sigma_2 - \nu\sigma_1 \qquad (13)$$

Solving (12) and (13) for σ_1 and σ_2 gives

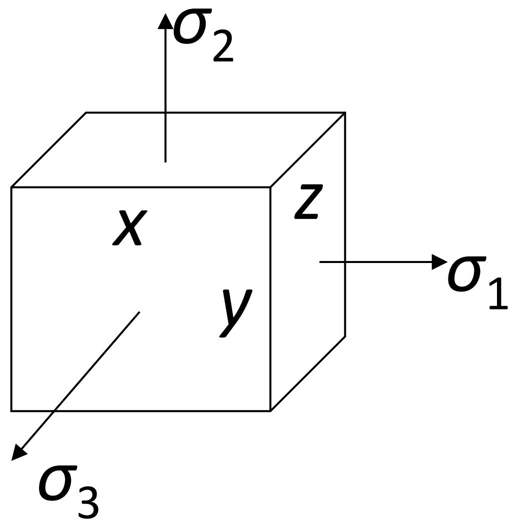
$$\sigma_1 = \frac{E(\varepsilon_1 + \upsilon\varepsilon_2)}{(1-\upsilon^2)} \qquad (14)$$

and

$$\sigma_2 = \frac{E(\varepsilon_2 + \upsilon\varepsilon_1)}{(1-\upsilon^2)} \qquad (15)$$

VOLUMETRIC STRAIN

- Consider a block with sides x, y and z acted upon by principal stresses σ_1, σ_2 and σ_3.
- Let corresponding linear strains be ε_1, ε_2 and ε_3.



- Corresponding resultant dimensions are:

$$(x + \varepsilon_1 x), (y + \varepsilon_2 y) \text{ and } (z + \varepsilon_3 z)$$

or $$x(1 + \varepsilon_1), y(1 + \varepsilon_2) \text{ and } z(1 + \varepsilon_3)$$

Volumetric strain, is given as:

$$\varepsilon_v = \frac{Change\ in\ volume}{Original\ volume}$$

VOLUMETRIC STRAIN

$$\varepsilon_v = \frac{x(1+\varepsilon_1).y(1+\varepsilon_2).z(1+\varepsilon_3) - xyz}{xyz}$$

$$= (1+\varepsilon_1).(1+\varepsilon_2).(1+\varepsilon_3) - 1$$

- Expanding and neglecting second order terms – since strains are small:

$$\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \qquad (16)$$

- Expressing the strains in terms of principal stresses (using Equations (1), (2) and (3)):

$$Volumetric\,strain, \varepsilon_v = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1-2\upsilon)}{E} \qquad (17)$$

Examples

Example 001

An aluminium alloy block is subjected to a uniform pressure of p = 35 MPa as shown in Figure Q1. Taking E = 73 GPa; ν = 0.33, determine the

i) change in lengths of sides *AB*, *BC*, and *BD*. [6 marks]

ii) change in volume of the block. [2 marks]

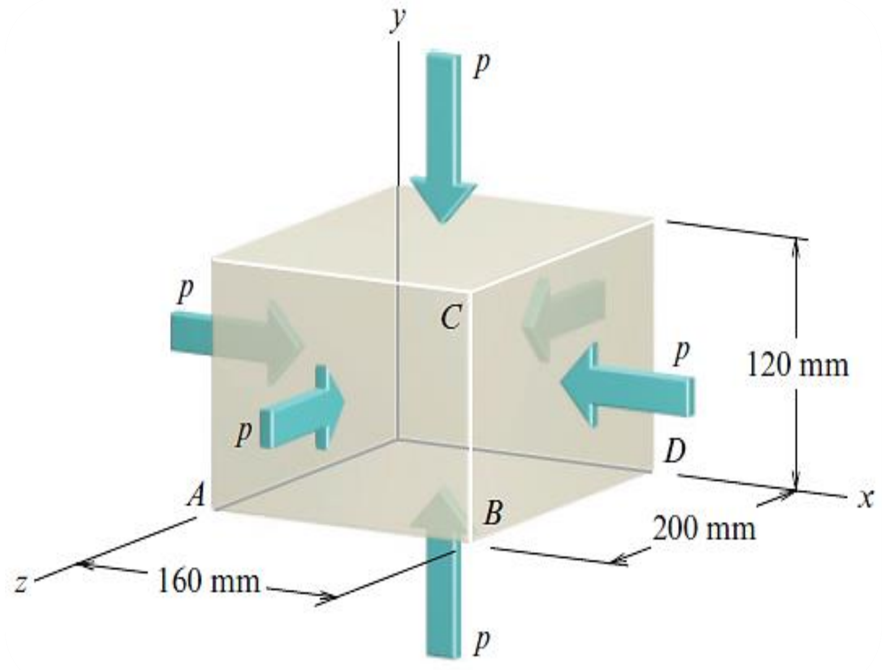


Figure Q1: Block under hydrostatic pressure

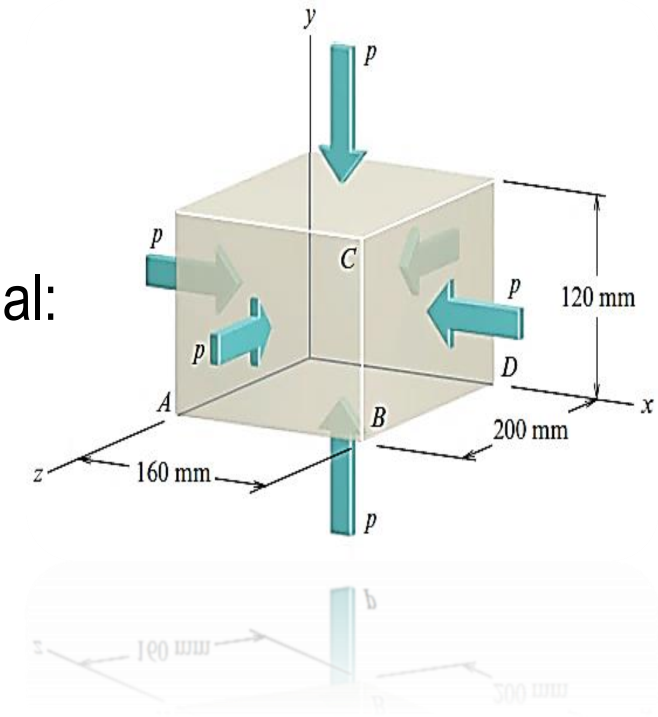
Example 001

Change in length of sides *AB*, *BC*, and *BD*

Normal stresses:

The normal stresses (principal stresses) are equal:

$$\sigma_x = \sigma_y = \sigma_z = -p = -35 \text{ MPa}$$



Normal strains:

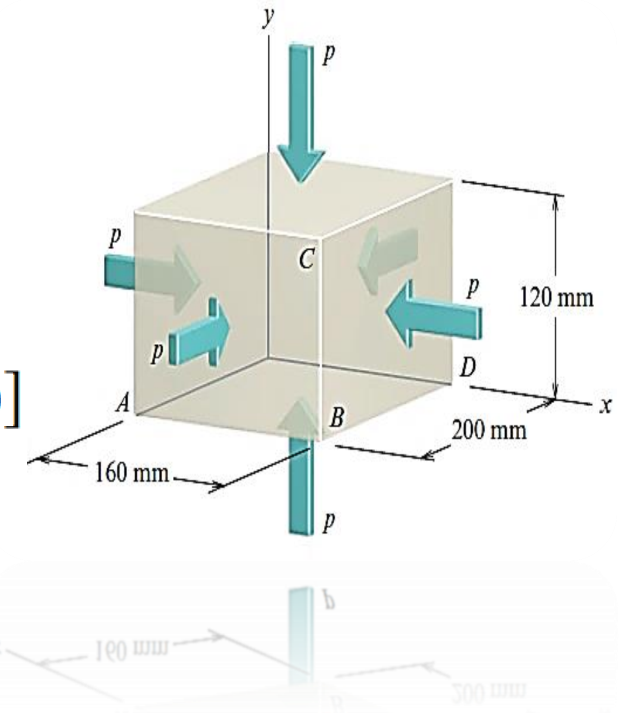
We have ε_x for a hydrostatic stress state expressed as:

$$\begin{aligned}\varepsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ &= \frac{1}{E}[-p - \nu(-p - p)] \\ &= -\frac{p}{E}(1 - 2\nu)\end{aligned}$$

We get the same expression for ε_y and ε_z

Example 001

For a pressure p = 35 MPa, the strains in the aluminium alloy block are:



$$\varepsilon_x = \varepsilon_y = \varepsilon_z = -\frac{35\ \text{MPa}}{73{,}000\ \text{MPa}}[1 - 2(0.33)]$$

$$= -163.0 \times 10^{-6}\ \text{mm/mm}$$

Deformations:

$$\delta_{AB} = (160\ \text{mm})(-163.0 \times 10^{-6}\ \text{mm/mm}) = -0.0261\ \text{mm}$$

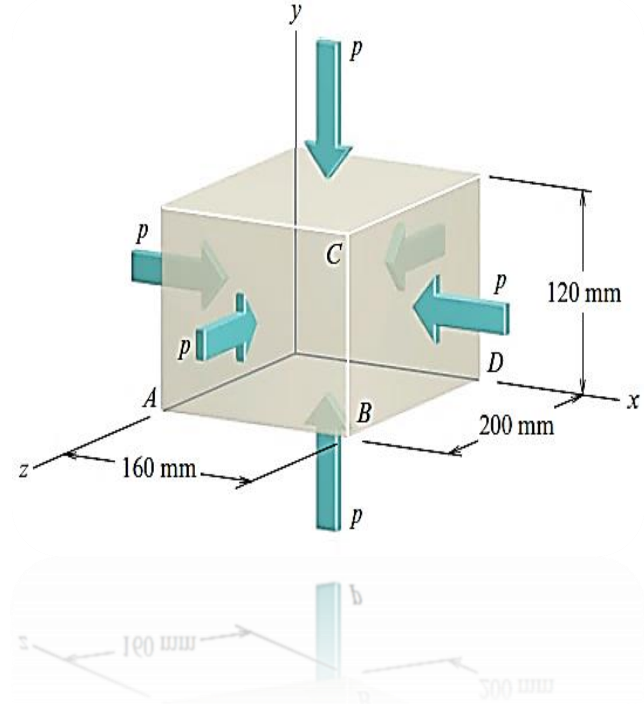
$$\delta_{BC} = (120\ \text{mm})(-163.0 \times 10^{-6}\ \text{mm/mm}) = -0.01956\ \text{mm}$$

$$\delta_{BD} = (200\ \text{mm})(-163.0 \times 10^{-6}\ \text{mm/mm}) = -0.0326\ \text{mm}$$

Example 001

Volumetric Strain:

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$$
$$= 3(-163.0 \times 10^{-6})$$
$$= -489.0 \times 10^{-6}$$



Initial volume of the block:

$$V = (160\ \text{mm})(120\ \text{mm})(200\ \text{mm})$$
$$= 3.84 \times 10^6\ \text{mm}^3$$

Change in Volume of the Block

$$\Delta V = \varepsilon_v . V$$
$$= (-489.0 \times 10^{-6})(3.84 \times 10^6\ \text{mm}^3)$$
$$= -1{,}878\ \text{mm}^3$$

Note that volume of the block has decreased under hydrostatic pressure.

Examples

Example 002

A 75 mm marble cube shown in Figure Q2 has the measured compressive strains of $\varepsilon_x = -650 \times 10^{-6}$ and $\varepsilon_y = \varepsilon_z = -370 \times 10^{-6}$. Taking E = 55 GPa; $\nu = 0.22$, determine the following:

i) normal stresses σ_x, σ_y, and σ_z acting on the x, y, and z faces of the cube. [9 marks]

ii) maximum shear stress in the material. [3 marks]

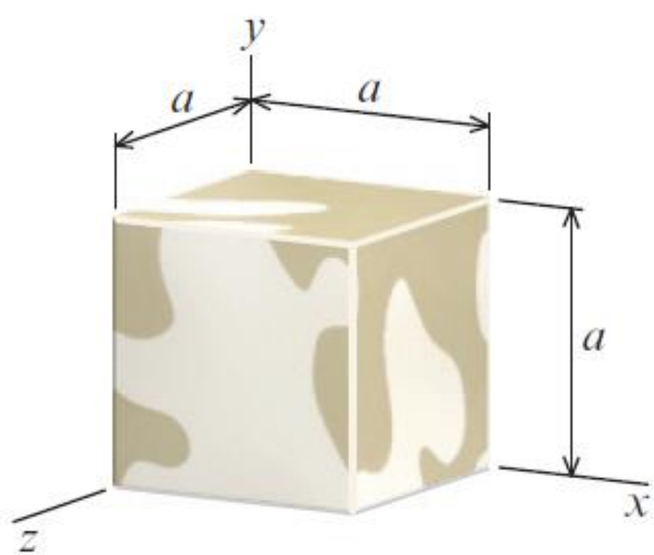


Figure Q2: Marble cube

Examples 002

Normal Stresses

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)]$$

$$= \frac{55{,}000 \text{ MPa}}{(1+0.22)[1-2(0.22)]}[(1-0.22)(-650) + (0.22)(-370-370)](10^{-6})$$

$$= -53.9 \text{ MPa}$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z)]$$

$$= \frac{55{,}000 \text{ MPa}}{(1+0.22)[1-2(0.22)]}[(1-0.22)(-370) + (0.22)(-650-370)](10^{-6})$$

$$= -41.3 \text{ MPa}$$

Examples 002

Normal Stresses

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)]$$

$$= \frac{55{,}000 \text{ MPa}}{(1+0.22)[1-2(0.22)]}[(1-0.22)(-370) + (0.22)(-650-370)](10^{-6})$$

$$= -41.3 \text{ MPa}$$

Maximum shear stress

- There are no shear stresses acting on the *x*, *y*, or *z* faces of the marble cube
- Consequently, σ_x, σ_y, and σ_z must be principal stresses:

$$\sigma_x = -53.9 \text{ MPa}$$

$$\sigma_y = -41.3 \text{ MPa}$$

$$\sigma_z = -41.3 \text{ MPa}$$

STRAIN ENERGY

- The three (3) principal stresses ($\sigma_1, \sigma_2, \sigma_3$) applied on the principal planes result in (3) three principal strains ($\varepsilon_1, \varepsilon_2, \varepsilon_3$).
- Work is done on the material during the deformation process as the stresses are gradually applied.
- During the deformation process the material's energy levels changes.
- The change is due to the energy absorbed by the material during the loading process.
- This change in the material's internal energy level leads us to the concept of strain energy.

STRAIN ENERGY

- We define strain energy as the energy absorbed by the material during the loading process.
- This strain energy is also defined as the work done by the load; provided no energy is added to or subtracted from the material in the form of heat.
- Strain energy is also known as internal work to distinguish it from external work.
- We now consider a simple bar subjected to a tensile force F.
- We then consider a small element on the bar of dimensions dx, dy and dz.

STRAIN ENERGY

- By definition, strain energy (U), is the work done by a system of stresses in straining a material.
- Consider a cube (or element) under a system of principal stresses σ_1, σ_2 and σ_3.
- For the corresponding strains of ε_1, ε_2 and ε_3, the work done is:

$$U = \sum \frac{1}{2} \sigma \varepsilon$$

- The work done (if the stresses are gradually applied).

$$U = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$$

- Using Equations (1), (2) and (3):

$$\varepsilon_1 = \sigma_1/E - \nu\sigma_2/E - \nu\sigma_3/E \qquad (1)$$

$$\varepsilon_2 = \sigma_2/E - \nu\sigma_3/E - \nu\sigma_1/E \qquad (2)$$

$$\varepsilon_3 = \sigma_3/E - \nu\sigma_1/E - \nu\sigma_2/E \qquad (3)$$

- We therefore, get the following expression for the work done:

$$U = \frac{1}{2E}[\sigma_1(\sigma_1 - \upsilon\sigma_2 - \upsilon\sigma_3) + \sigma_2(\sigma_2 - \upsilon\sigma_3 - \upsilon\sigma_1) + \sigma_3(\sigma_3 - \upsilon\sigma_1 - \upsilon\sigma_2)]$$

STRAIN ENERGY

$$U = \left(\frac{1}{2E}\right)\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \qquad (18)$$

per unit volume

For a 2D stress system, $\sigma_3 = 0$

$$U = \left(\frac{1}{2E}\right)\left[\sigma_1^2 + \sigma_2^2 - 2v(\sigma_1\sigma_2)\right] \qquad (19)$$

per unit volume.

We can express the principal stresses as follows:

$$\sigma_1 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{3}(\sigma_1 - \sigma_2) + \frac{1}{3}(\sigma_1 - \sigma_3) \qquad (20)$$

$$\sigma_2 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{3}(\sigma_2 - \sigma_3) + \frac{1}{3}(\sigma_2 - \sigma_1) \qquad (21)$$

$$\sigma_3 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{3}(\sigma_3 - \sigma_1) + \frac{1}{3}(\sigma_3 - \sigma_2) \qquad (22)$$

SHEAR STRAIN ENERGY

- Remember that shear stresses on principal planes are ZERO.
- Thus, under the action of the mean ($\bar{\sigma}$) stress, there WILL BE volumetric strain but with NO distortion of shape (i.e. no shear stress anywhere).

$$\bar{\sigma} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

- The strain energy under this mean stress acting in each direction can be obtained from the general formula, Equation (18):

$$U = \left(\frac{1}{2E}\right)\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\upsilon(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \qquad (18)$$

- Using the mean stress, equation (18) and equations (20) to (22) we express the volumetric strain energy as follows:

$$U = \left(\frac{3}{2E}\right)\left[\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right]^2 \cdot (1 - 2\upsilon) \qquad (23)$$

- The other terms in the re-arrangement of σ_1, σ_2 and σ_3 are proportional to the **maximum shear stress values** in the three planes, and will cause a distortion of the shape.

SHEAR STRAIN ENERGY

$$U = \left(\frac{1}{6E}\right)[\sigma_1 + \sigma_2 + \sigma_3]^2 \cdot (1 - 2\upsilon)$$

- Let U_s = Shear strain energy

  U_s = [Total strain energy] – [Volumetric strain energy]

$$U_s = \left(\frac{1}{2E}\right)[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\upsilon(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

$$-\left(\frac{1}{6E}\right)[(\sigma_1 + \sigma_2 + \sigma_3)^2 \cdot (1 - 2\upsilon)]$$

$$= \left(\frac{1}{6E}\right)\begin{bmatrix} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)(3 - 1 + 2\upsilon) \\ -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)(6\upsilon + 2 - 4\upsilon) \end{bmatrix}$$

$$= \left(\frac{1 + \upsilon}{6E}\right)[2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad (24)$$

SHEAR STRAIN ENERGY

- Remember that:

$$\mathrm{E} = 2\mathrm{G}(1 + \nu).$$

- Thus:

$$U_s = \left(\frac{1+\upsilon}{12G}\right)[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

SHEAR STRAIN ENERGY

The quantities in brackets are each twice the maximum shear stress in their respective planes .

In pure shear system (stress τ), the principal stresses are $\pm\tau$, 0 (**review maximum shear stresses**).

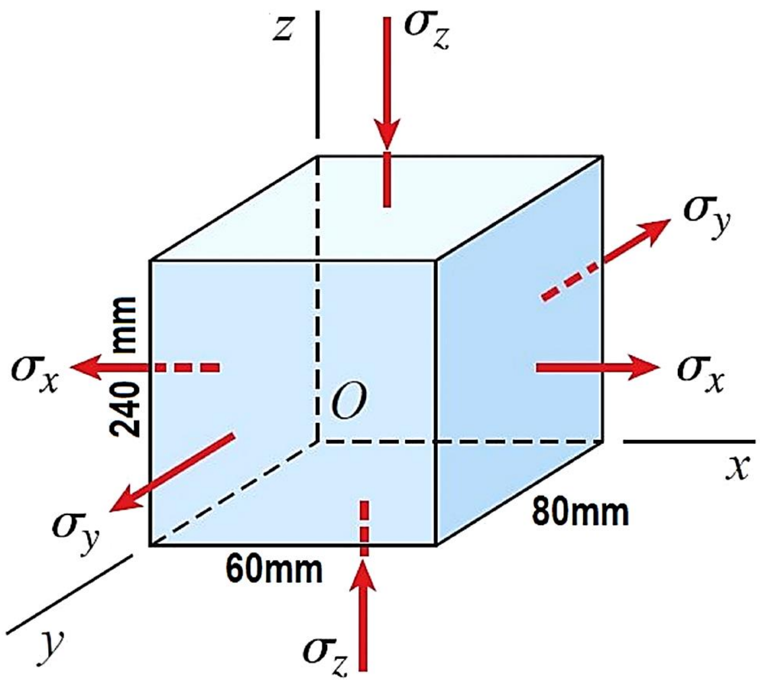
$$\therefore \qquad U_s = \left(\frac{1+\upsilon}{12G}\right)\left[(2\tau)^2 + (-\tau)^2 + (-\tau)^2\right]$$

(Since $\sigma_1 = \tau$, $\sigma_2 = -\tau$ when $\sigma_3 = 0$)

$$U_s = \frac{\tau^2}{2G} \qquad (25)$$

(Compare with strain energy in direct shear stress)

Examples



Example 003

A rectangular block measuring 240 mm x 80mm x 60mm is loaded as shown in the diagram, and F_x = 80kN, F_y= 100kN, F_z = 40kN. Compute the change in volume, the bulk modulus and modulus of rigidity. Take E = 200kN/mm^2 and ν = 0.3

Examples

Example 004

A block of steel measuring 240 mm x 16 mm x 25 mm, is subjected to a tensile force of 40kN in the direction of its length. Given that the modulus of elasticity (*E*) and the Poisson's ratio (ν) are 200kN/mm^2 and 0.33 respectively. Compute the change in the volume of the block.

Examples

Example 005

a) Derive an expression for strain energy induced in a material per unit volume, under gradual tensile loading.

b) Calculate the strain energy stored in a bar 3 m long and 40mm in diameter when subjected to a tensile load of 80kN. Take E = 210kN/mm^2.

Grazie