THE UNIVERSITY OF ZAMBIA
SCHOOL OF ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING

MEC 3352 – STRENGTH OF MATERIALS II

Theories of Failure

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THEORIES OF FAILURE

- A material ‘fails’ when permanent deformation occurs, when the elastic limit is exceeded (not necessarily at rapture).

- In simple tension, it is assumed that the elastic limit is associated with a definite value of tensile stress.

- But in reality, other quantities such as shear stress, strain energy assume values that could be responsible for failure.

- In a complex stress system these quantities can be calculated from the known stresses and material constants.

Theories of Failure

- The major problem is to decide which quantity is the failure criterion.
- ***The actual value of that particular factor which corresponds to the onset of failure is usually taken to be the value it reaches in simple tension.***
- Five principal theories of failure are outlined below.
- In these theories, it is assumed that σ is the tensile stress at the elastic limit in simple tension, and σ_1, σ_2 and σ_3 are the principal stresses in any complex system.

(1) Maximum Principal Stress Theory (Rankine's Theory)

Failure occurs when the maximum principal stress in the complex system reaches the value of the maximum stress at the elastic limit in simple tension, i.e.

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{\left[\left(\sigma_x - \sigma_y\right)^2 + 4\tau^2\right]} = \sigma \quad \text{in simple tension.} \quad (26)$$

where σ_x, σ_y and τ are the stresses on given planes in the complex system.

(2) Maximum Shear Stress or Stress Difference Theory (Guest and Tresca's Theory)

Failure occurs when the maximum shear stress in the complex system reaches the value of the maximum shear stress in simple tension at the elastic limit, i.e.

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}\sqrt{\left[\left(\sigma_x - \sigma_y\right)^2 + 4\tau^2\right]} = \frac{\sigma}{2} \quad \text{in simple tension.} \tag{27a}$$

Or

$$\sigma_1 - \sigma_2 = \sigma \tag{27b}$$

(3) Strain Energy Theory (Haigh's Theory).

Based on the argument that as strains are reversible up to the elastic limit, the energy absorbed by the material should be a single-value function at failure, independent of the stress system causing it, i.e. strain energy per unit volume causing failure is equal to the strain energy at the elastic limit in simple tension.

$$\left(\frac{1}{2E}\right)[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] = \left(\frac{\sigma^2}{2E}\right) \qquad (28a)$$

or

$$\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma^2 \qquad (28b)$$

(4) *Shear Strain Energy Theory* (von Mises and Hencky's Theory)

At failure, the shear strain energy in the complex system are equal to the shear strain energy in simple tension, i.e.

$$\left(\frac{1}{12G}\right)[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]=\left(\frac{\sigma^2}{6G}\right) \qquad (29a)$$

Or

$$[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]=2\sigma^2 \qquad (29b)$$

The value in the simple tension case is found by putting the principal stresses equal to σ, 0, 0.

(5) *Maximum Principal Strain Theory* (St. Venant's Theory)

If ε_1 is the maximum strain in the complex system, then according to this theory, it is equal to the strain in simple tension at the elastic limit, i.e.:

$$\varepsilon_1 = \left(\frac{1}{E}\right)(\sigma_1 - \nu\sigma_2 - \nu\sigma_3) = \left(\frac{\sigma}{E}\right) \qquad (30a)$$

Or

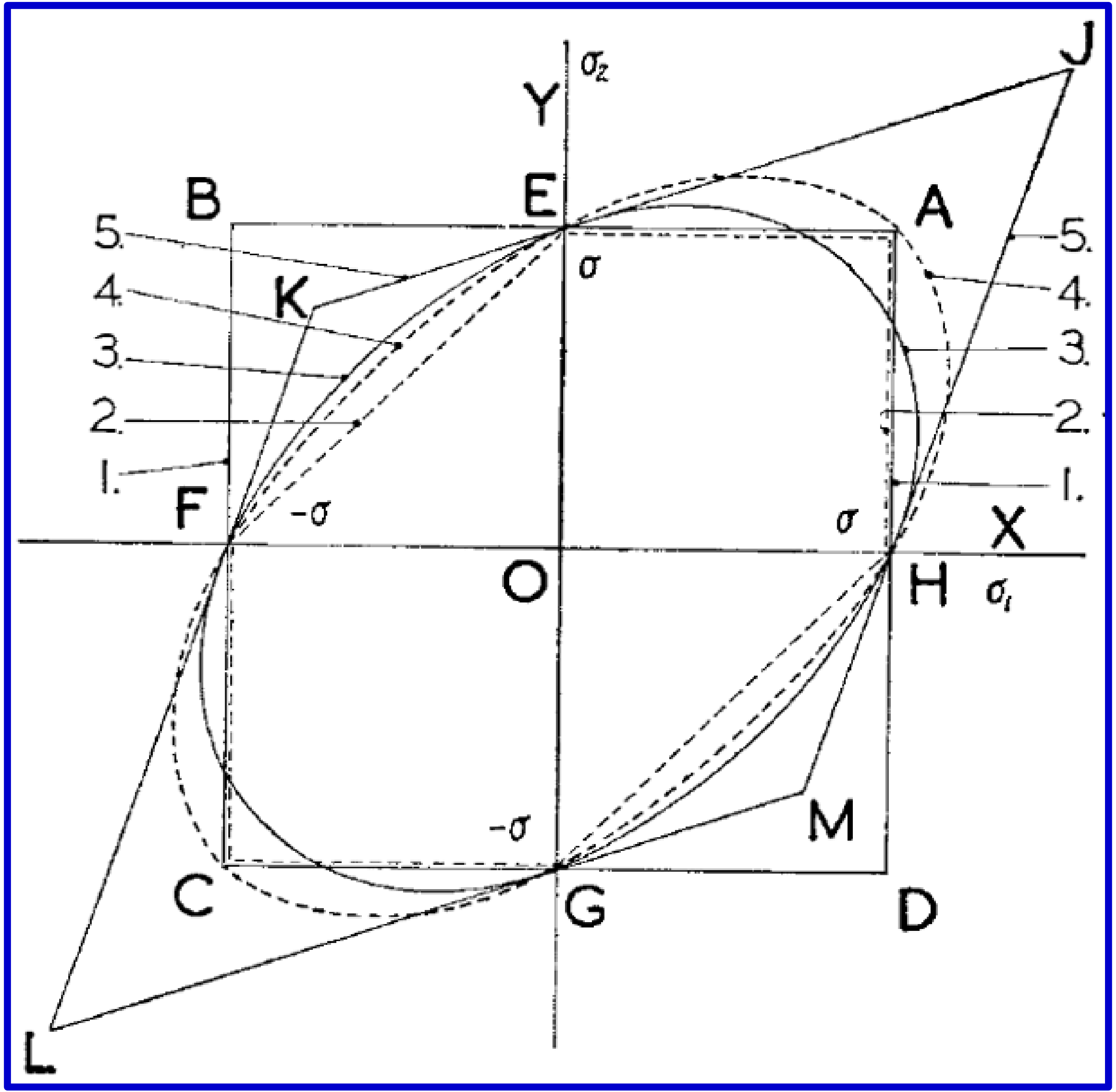
$$\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \sigma \qquad (30b)$$

Graphical Representation

- Considering a two-dimensional stress system, the limits of principal stresses can be shown graphically according to the different theories.

  - σ_1 is on axis OX, and is +ve to the right.
  - σ_2 is on axis OY, and is +ve upwards.
  - $\sigma_3 = 0$.

- It can be assumed that the elastic limit σ is the same in tension and in compression.



1. Maximum Principal Stress (*Rankine's*) Theory.
2. Maximum Shear Stress or Stress Difference (*Guest and Tresca's*) Theory
3. Strain Energy (*Haigh's*) Theory
4. Shear Strain Energy (*von Mises and Hencky's*) Theory
5. Maximum Principal Strain (*St. Venant's*) Theory

Graphical Representation of the five theories of Failure.

(Figure 3.33, Ryder, p. 58)

(1) ***Maximum principal stress*** equal numerically to the elastic limit. This produces square *ABCD* whose sides are defined by

$$\sigma_1 = \sigma; \qquad \sigma_2 = \sigma; \qquad \sigma_1 = -\sigma; \qquad \text{and} \quad \sigma_2 = -\sigma \tag{31}$$

(2) ***Maximum shear stress*** equals numerically to the value in simple tension (½σ). Where the principal stresses are alike, the greatest maximum shear stress is ½σ_1 (or ½σ_2), obtained by taking half the difference between the principal stresses σ_1 and 0, or σ_2 and 0. This produces lines

$$\frac{\sigma_1}{2} = \frac{\sigma}{2}; \quad \frac{\sigma_2}{2} = \frac{\sigma}{2}; \quad \frac{\sigma_1}{2} = -\frac{\sigma}{2}; \quad \text{and} \quad \frac{\sigma_2}{2} = -\frac{\sigma}{2} \tag{32}$$

in the first and third quadrants (*HA*, *AE*, *FC*, *CG*).

When the principal stresses are of opposite type, maximum shear stress is

$$\frac{\sigma_1 - \sigma_2}{2} = \pm \frac{\sigma}{2} \tag{33}$$

completing the figure in the second and fourth quadrants with the lines *EF* and *GH*. The boundary is then *AEFCGHA*.

(3) In the two-dimensional case, ***the strain energy theory*** is defined by an ellipse with axes at 45° to *OX* and *OY*; the equation is

$$\sigma_1^2 + \sigma_2^2 - 2\nu\, \sigma_1\sigma_2 = \sigma^2. \tag{34}$$

It passes through the points *E*, *F*, *G* and *H*.

(4) ***The shear strain energy theory*** results in an ellipse similar to (3), defined by

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma^2. \tag{35}$$

(5) ***The principal strains*** are

$$\left(\frac{1}{E}\right)(\sigma_1 - \nu\sigma_2) \quad \text{and} \quad \left(\frac{1}{E}\right)(\sigma_2 - \nu\sigma_1) \tag{36}$$

and failure is assumed to occur when either of these values reaches $\pm\sigma/E$. For like principal stresses the lines *HJ*, *JE*, *FL* and *LG* are produced by the equation

$$\sigma_1 - \nu\sigma_2 = \sigma; \quad \sigma_2 - \nu\sigma_1 = \sigma; \quad \sigma_1 - \nu\sigma_2 = -\sigma; \quad \text{and} \quad \sigma_2 - \nu\sigma_1 = -\sigma \tag{37}$$

respectively. For unlike stresses *EK*, *KF*, *GM* and *MH* complete the figure.

NOTES:

1. From experiments done on various stress systems such as tubes under internal pressure, end loads and torsion; also on different materials, NO conclusive evidence has been produced in favour of any one theory.

2. Cause of failure depends not only on the properties of the material but also on the stress system to which it is subjected, and it may not be possible to embody the results for all cases in one comprehensive formula.

3. Generally, however, it has been observed that:

- For brittle materials (cast iron) – the maximum principal stress theory should be used.

- For ductile materials – the maximum shear stress or strain energy theories give a good approximation, though the shear strain energy theory is to be preferred, particularly when the mean principal stress is compressive.

- The maximum strain energy theory should not be used in general, as it only gives results in particular cases.

- Since the shear stress or strain energy theories depend only on stress differences, they are independent of the value of the mean stress and imply that a material will not fail under a “hydrostatic” stress system (i.e. $\sigma_1 = \sigma_2 = \sigma_3$).

- In practice the effect of such a stress system, if tensile, is to produce a brittle type of fracture in a normally ductile material.

- Conversely, a tri-axial compressive system will produce a ductile type fracture in a normally brittle material.

- In general the tendency to ductility is increased as the ratio of maximum shear stress to maximum tensile stress under load in increased.

The End

ME 3352: Strength of Materials II

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