THE UNIVERSITY OF ZAMBIA
SCHOOL OF ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING

MEC 3352 – STRENGTH OF MATERIALS II

THEORIES OF FAILURE

THEORIES OF FAILURE

Introduction

- The strength of every material may be expressed in terms of the value of stress or strain at which the material is capable of withstanding a certain amount of loading without failing.
- Beyond this value of stress or strain the materials will fracture or fail to carry the load.

Failure Criterion

- This is the criterion used to explain the theory behind the failure.

Importance of Theories of Failure

- Theories of failure are useful in the following ways:
  1) Used in the design of structural components and the calculation of safety factor.
  2) They are a guide to materials development.
  3) Used to determine weak and strong directions in the material.

THEORIES OF FAILURE

Some Failure Modes

Yielding

- A process of global permanent plastic deformation.
- It results in the change of the geometry of the material.

Fracture:

- A process in which cracks propagates through the material to the extent that it breaks apart.

Buckling

- The loss of equilibrium in a structural member.
- Compressive loading may lead to buckling in columns.

Creep

- Failure occurring due to elevated temperatures.
- Material’s load carrying capacity decreased.

THEORIES OF FAILURE

- When some external load is applied on a material, stresses and strains are produced in the body.
- Within the elastic limit, the stresses are directly proportional to the strains.
- Within the elastic region, the material returns to original shape when the load is removed.
- When the material is loaded beyond the elastic limit, permanent deformation occurs
- Whenever permanent deformation occurs, the body is said to have failed.
- Remember that failure does not mean rapture of the material!
- The Question is: What is the failure criterion?
- The actual value of that particular factor which corresponds to the onset of failure; usually taken to be the value it reaches in simple tension.

THEORIES OF FAILURE

- Five principal theories of failure can be identified.
- In these failure theories, it is assumed that σ is the tensile stress at the elastic limit in simple tension, and σ_1, σ_2 and σ_3 are the principal stresses in any complex stress system.
- According to the theories of failure, failure takes place when a certain limiting value is reached.
- The Five principal theories of failure are:
    1) Maximum Principal Stress
    2) Maximum Shear Stress
    3) Maximum Strain Energy
    4) Maximum Shear Strain Energy
    5) Maximum Principal Strain

THEORIES OF FAILURE

(1) Maximum Principal Stress Theory (Rankine's Theory)

Failure occurs when the maximum principal stress in the complex system reaches the value of the maximum stress at the elastic limit in simple tension, i.e.

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{\left[\left(\sigma_x - \sigma_y\right)^2 + 4\tau^2\right]} = \sigma \qquad 2.1$$

Where, σ is the stress in simple tension

where σ_x, σ_y and τ are the stresses on given planes in the complex stress system.

Graphical Representation: Maximum Principal Stress Theory

- We consider a 2D case in which σ_1, σ_2 exist and $\sigma_3 = 0$.
- In this theory of failure, we assume the elastic limit is the same in both tension and compression.
- We, therefore, have: $\sigma_t = \sigma_c = \sigma$ where σ is the stress at the elastic limit in simple tension or compression

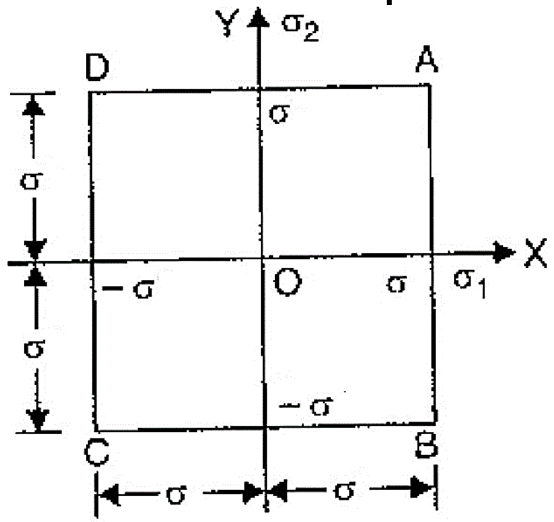

Figure 2.1: Graphical representation of Maximum Principal Stress theory

- According to this theory, failure occurs if a point (σ_1, σ_2) lies outside the square ABCD

THEORIES OF FAILURE

(2) Maximum Shear Stress or Stress Difference Theory (Guest and Tresca's Theory)

Failure occurs when the maximum shear stress in the complex system reaches the value of the maximum shear stress in simple tension at the elastic limit, i.e.

$$\frac{\sigma_1 - \sigma_2}{2} = \tau_{\max} = \frac{1}{2}\sqrt{\left[\left(\sigma_x - \sigma_y\right)^2 + 4\tau^2\right]} = \frac{\sigma}{2} \qquad 2.2a$$

Where σ is the stress in simple tension

$$\sigma_1 - \sigma_2 = \sigma \qquad 2.2b$$

The **maximum-shear-stress theory** or **Tresca yield criterion** may be expressed as follows:

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_Y}{2} \qquad 2.2c$$

Where σ_Y is the yield stress in simple tension

THEORIES OF FAILURE

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_Y}{2} \qquad 2.2c$$

The maximum-shear-stress theory for *plane stress* can be expressed for any two in-plane principal stresses σ_1 and σ_2 by the following criteria:

$$\left.\begin{aligned} |\sigma_1| &= \sigma_Y \\ |\sigma_2| &= \sigma_Y \end{aligned}\right\} \quad \sigma_1, \sigma_2 \text{ have same signs}$$

$$|\sigma_1 - \sigma_2| = \sigma_Y\} \quad \sigma_1, \sigma_2 \text{ have opposite signs}$$

Remember that failure occurs when the material is loaded beyond the yield stress σ_Y

Graphical Representation: Maximum Shear Stress Theory

- For a 2D case failure criterion for this theory is $\frac{\sigma_1}{2} = \frac{\sigma}{2}$ and $\frac{\sigma_2}{2} = \frac{\sigma}{2}$
- The values are obtained by taking half the difference between σ_1 and 0, as well $\sigma_2$and 0.

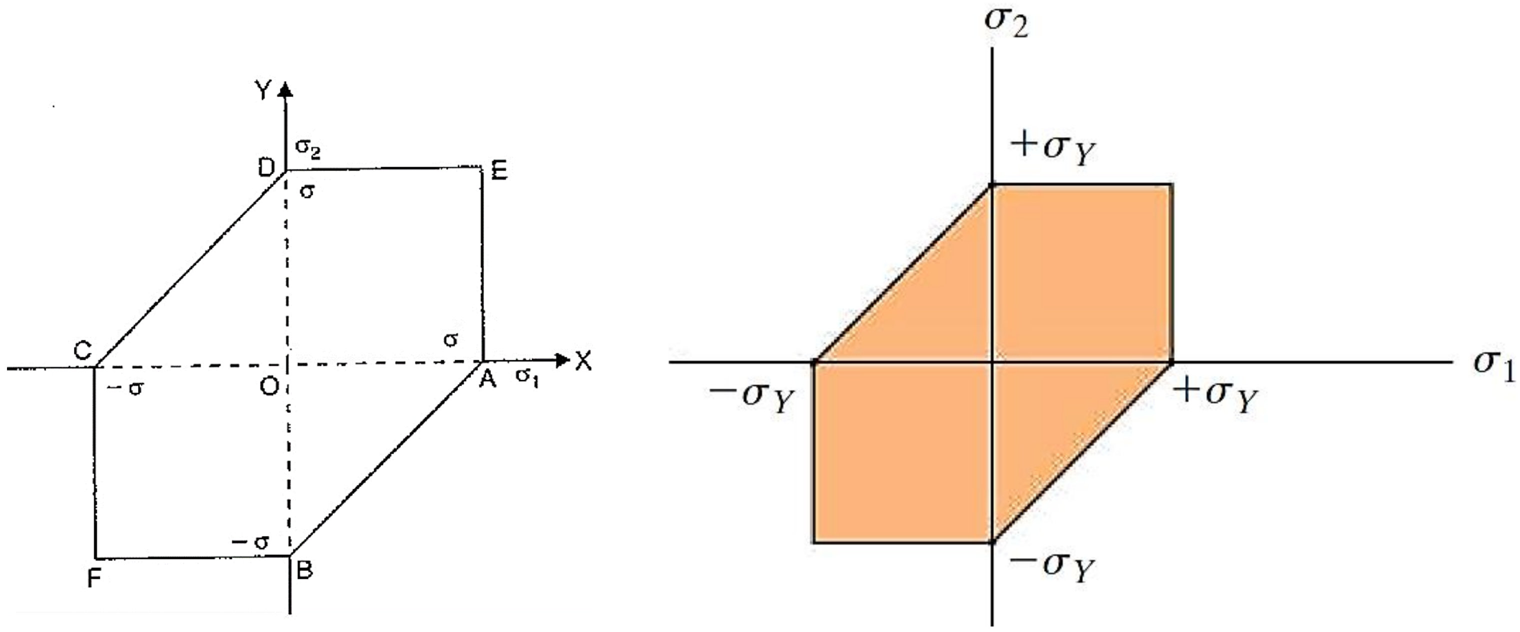

Figure 2.2: Graphical representation of Maximum Shear Stress theory

- We use the failure criterion to plot ABFCDE.
- Failure occurs if a point (σ_1, σ_2) lies outside ABFCDE.

THEORIES OF FAILURE

Example 2001

The in plane principal stresses acting on an element of material on the surface of the shaft in torsion are 9.56 MPa and -28.66 MPa respectively. Given that the yield stress is 36 MPa, determine whether or not the material will fail under the maximum shear stress theory.

Maximum-Shear-Stress Theory:

$$|\sigma_1 - \sigma_2| \leq \sigma_Y$$

$$|9.56 - (-28.66)| \overset{?}{\leq} 36$$

$$38.2 > 36$$

$$\left.\begin{array}{l} |\sigma_1| = \sigma_Y \\ |\sigma_2| = \sigma_Y \end{array}\right\} \quad \sigma_1, \sigma_2 \text{ have same signs}$$

$$|\sigma_1 - \sigma_2| = \sigma_Y\} \quad \sigma_1, \sigma_2 \text{ have opposite signs}$$

Thus, shear failure of the material will occur according to this theory.

Determine whether the shaft will fail or not under the maximum principal stress theory.

THEORIES OF FAILURE

(3) Strain Energy Theory
(Haigh's Theory).

- Based on the argument that strains are reversible up to the elastic limit.
- And that the energy absorbed by the material up to this point is a single-value function at failure, independent of the stress system causing it.
- In this theory, strain energy per unit volume causing failure is equal to the strain energy at the elastic limit in simple tension.

$$\left(\frac{1}{2E}\right)\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right)\right]=\left(\frac{\sigma^2}{2E}\right) \quad 2.3a$$

$$\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)=\sigma^2 \quad 2.3b$$

Graphical Representation: Maximum Strain Energy Theory

- For a 2D case, the failure criterion is $\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2 = \sigma_t^2 = \sigma^2$
- The above equation is an ellipse with the centre at the origin and axes inclined at 45°.

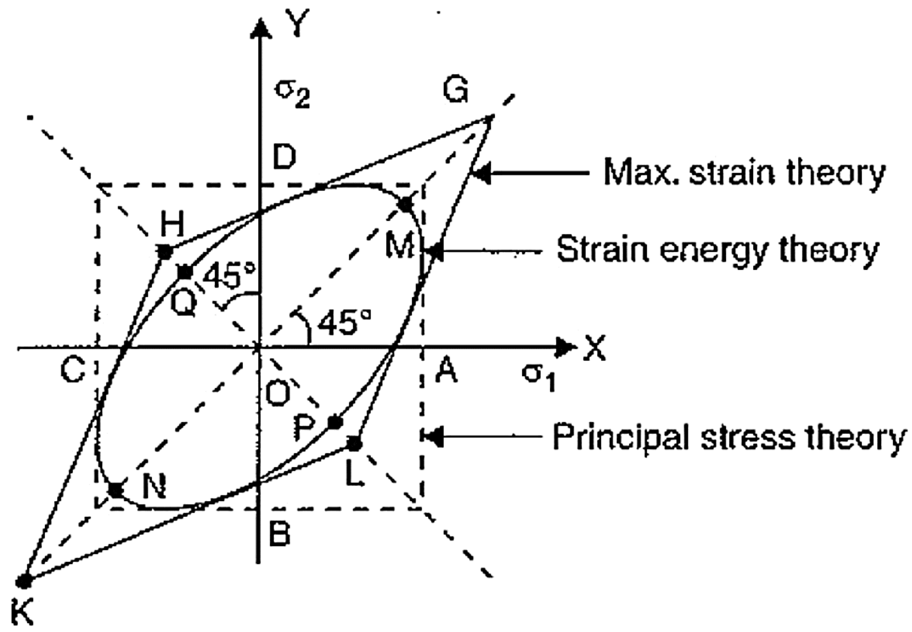

Figure 2.3: Graphical representation of Maximum Strain Energy theory

- We use the failure criterion to plot the ellipse inside the parallelogram GLKHG.
- Failure occurs if a point (σ_1, σ_2) lies outside the ellipse APBNCQDMA.

THEORIES OF FAILURE

(4) Shear Strain Energy Theory
(von Mises and Hencky's Theory)

At failure, the shear strain energy in the complex system are equal to the shear strain energy in simple tension, i.e.

$$\left(\frac{1}{12G}\right)\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]=\left(\frac{\sigma^2}{6G}\right) \qquad 2.4a$$

The shear strain energy failure theory equation reduces to:

$$[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]=2\sigma^2 \qquad 2.4b$$

The above equation reduces to:

$$\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2=\sigma_Y^2$$

Graphical Representation: Maximum Shear Strain Theory

- For a 2D case, the failure criterion is

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_Y^2$$

- The above equation is an ellipse with the centre at the origin and axes inclined at 45°.

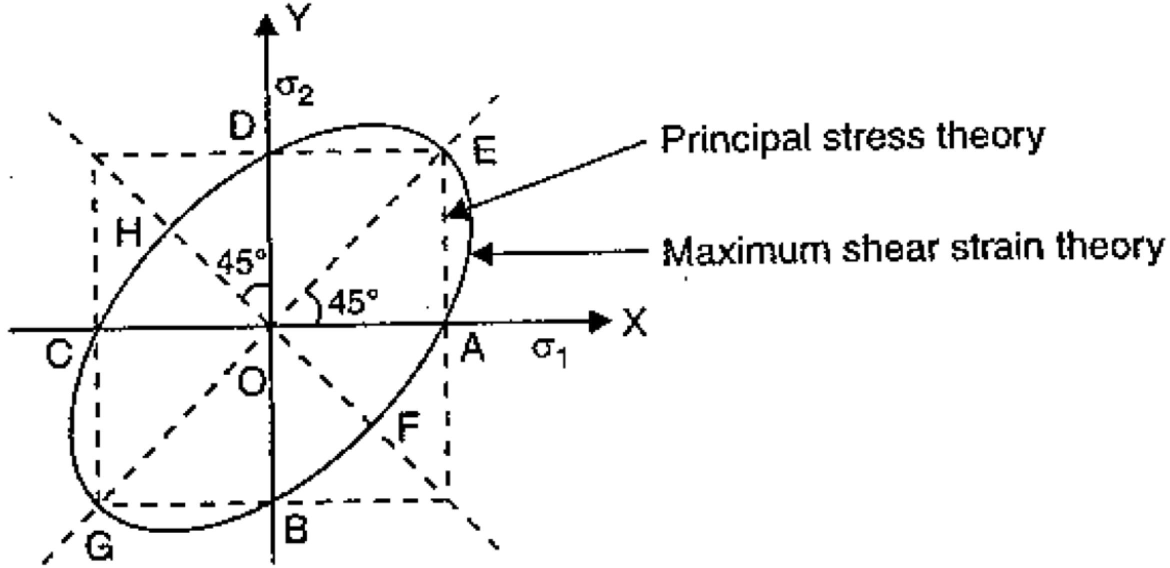

Figure 2.4: Graphical representation of Maximum Shear Strain theory

- We use the failure criterion to plot the ellipse AEDHCGBFA.
- Failure occurs if a point (σ_1, σ_2) lies outside the ellipse AEDHCGBFA.

THEORIES OF FAILURE

(5) Maximum Principal Strain Theory (St. Venant's Theory)

Failure occurs in a material when the maximum principal strain reaches the strain due to yield stress in simple tension

If ε_1 is the maximum strain in the complex system, then according to this theory, it is equal to the strain in simple tension at the elastic limit, i.e.:

$$\varepsilon_1 = \left(\frac{1}{E}\right)\left(\sigma_1 - \nu\sigma_2 - \nu\sigma_3\right) = \left(\frac{\sigma}{E}\right) \qquad 2.5a$$

$$\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \sigma \qquad 2.5b$$

Graphical Representation: Maximum Principal Strain Theory

- For a 2D case failure criterion for this theory is

$$\sigma_1 - \nu\sigma_2 = \sigma_t = \sigma_Y$$
$$\sigma_2 - \nu\sigma_1 = \sigma_c = \sigma_Y$$

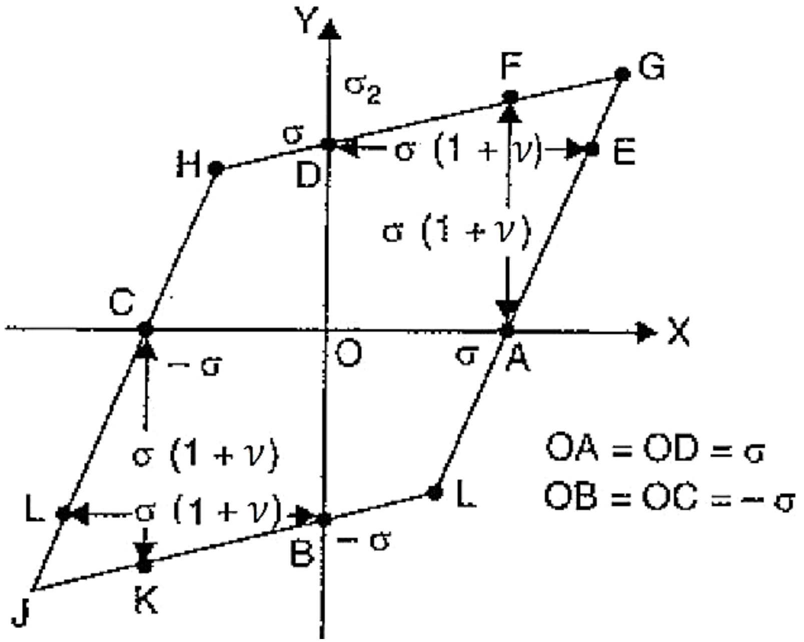

Figure 2.5: Graphical representation of Maximum Principal Strain theory

- We use the failure criterion to plot the parallelogram HGLJ.
- Failure occurs if a point (σ_1, σ_2) lies outside HGLJ

IMPORTANT NOTES ON THEORIES OF FAILURE:

1. From experiments done on various stress systems such as tubes under internal pressure, end loads and torsion; also on different materials, NO conclusive evidence has been produced in favour of any one theory.
2. Cause of failure depends not only on the properties of the material but also on the stress system to which it is subjected, and it may not be possible to embody the results for all cases in one comprehensive formula.

3. In General, it has been observed that:
   - For brittle materials (cast iron) – the maximum principal stress theory should be used.
   - For ductile materials – the maximum shear stress or strain energy theories give a good approximation, though the shear strain energy theory is to be preferred, particularly when the mean principal stress is compressive.
   - The maximum strain energy theory should not be used in general, as it only gives results in particular cases.

- Since the shear stress or strain energy theories depend only on stress differences, they are independent of the value of the mean stress and imply that a material will not fail under a “hydrostatic” stress system (i.e. $\sigma_1 = \sigma_2 = \sigma_3$).
- In practice the effect of such a stress system, if tensile, is to produce a brittle type of fracture in a normally ductile material.
- Conversely, a tri-axial compressive system will produce a ductile type fracture in a normally brittle material. In general the tendency to ductility is increased as the ratio of maximum shear stress to maximum tensile stress under load is increased.

Examples

Question 1

The Principal stresses at a point in an elastic material are 100 N/mm^2 (tensile) 80 N/mm^2 tensile and 50 N/mm^2 Compressive. If the stress at the elastic limit is 200 MPa, determine whether the failure of material will occur according to the maximum principal stress theory. If failure does not occur, what is the factor of safety.

Question 2

The Principal stresses at a point in an elastic material are 200 N/mm^2 (tensile) 100 N/mm^2 tensile and 50 N/mm^2 Compressive. If the stress at the elastic limit is 200 N/mm^2, determine whether the failure of material will occur according to the maximum principal strain theory. Take Poisson's ratio to be 0.3.

Examples

Question 3

The Principal stresses at a point in an elastic material are 200 N/mm^2 (tensile) 100 N/mm^2 tensile and 50 N/mm^2 Compressive. If the stress at the elastic limit is 200 N/mm^2, determine whether the failure of material will occur or not, according to the maximum shear stress theory. Take Poisson's ratio to be 0.3.

Question 4

The Principal stresses at a point in an elastic material are 200 N/mm^2 (tensile) 100 N/mm^2 tensile and 50 N/mm^2 Compressive. If the stress at the elastic limit is 200 N/mm^2, determine whether the failure of material will occur or not, according to the maximum shear strain energy theory. Take Poisson's ratio to be 0.3.

Questions (Assignment)

Question 1

A hollow mild steel shaft having 100mm external diameter and 50mm internal diameter is subjected to a twisting moment of 8kNm and a bending moment of 2.5kNm. Calculate the principal stresses and find the direct stress, which acting alone would produce the same

i) Maximum elastic strain energy

ii) Maximum elastic shear strain energy

As that produced by the principal stresses acting together. Take Poisson's ratio to be 0.25.

Questions (Assignment)

Question 2

In a 2D stress system, the direct stresses on two mutually perpendicular planes are $120 MN/m^2$ and σ MN/m^2. These planes also carry a shear stress of $40 MN/m^2$. If the factor of safety on the elastic limit is 3, compute:

i) The value of σ when the shear strain energy is minimum
ii) The elastic limit of the material in simple tension

GRAZIE