MEC 3352 – STRENGTH OF MATERIALS II

Rotating Discs and Cylinders

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Brainstorming

1. What is a disk?
2. What is a cylinder?
3. What is the difference between the two?
4. Can a disk be considered a cylinder or a cylinder considered a disc?

ROTATING DISCS AND CYLINDERS

Introduction

These notes relate to the stresses and strains existing in rotating thick walled cylinders.

They are generally applicable to the design of flywheels. The primary assumption is that the cylinders are not subject to internal or external pressure.

A basic review of solid discs, rings and cylinders is carried out.

Symbols/Units

Tensile stresses are considered positive and compressive stresses are negative.

- p_1 = Internal pressure (N/m^2)
- p_2 = External pressure (N/m^2)
- σ_r = Radial stress (N/m^2)
- σ_t = Tangential (Hoop) stress (N/m^2)
- σ_a = Axial/longitudinal stress (N/m^2)
- E = Young's modulus (N/m^2)
- ρ = Density (kg/m^3)
- υ, ν = Poisson's ratio

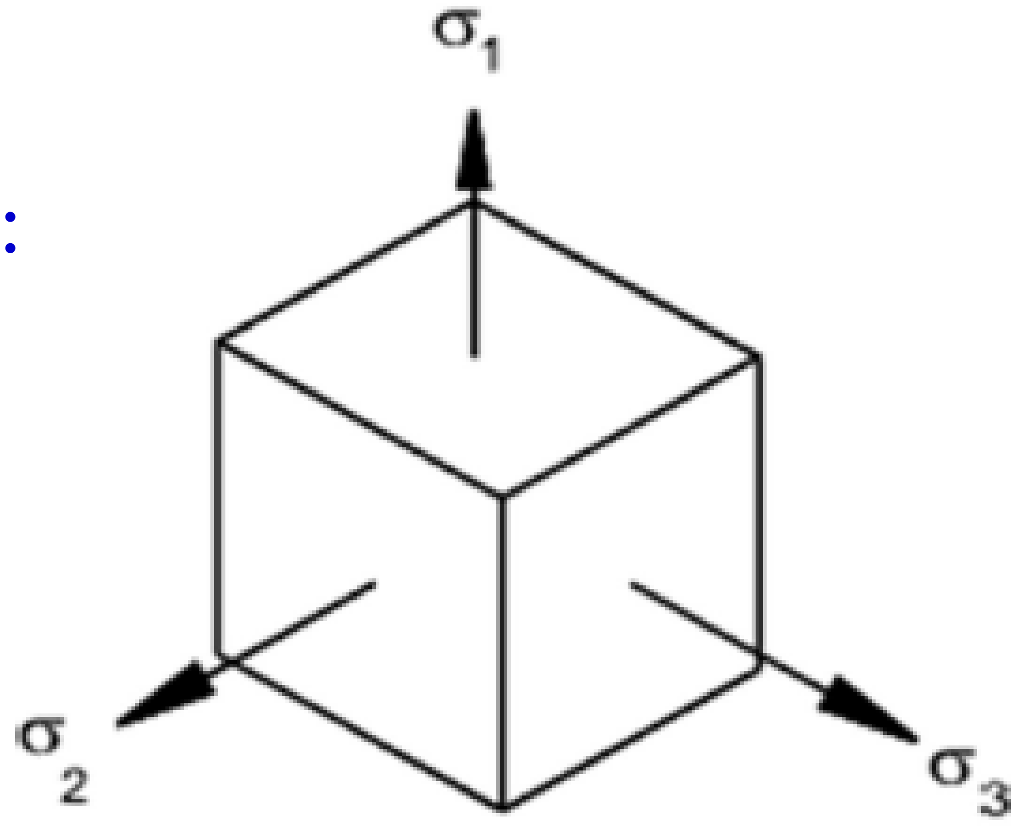
- r = Radius at point of analysis (m)
- R_1 = Internal radius (m)
- R_2 = External radius (m)
- ε_r = Radial strain
- ε_t = Tangential (Hoop) strain
- ε_a = Axial/longitudinal strain
- u = Radial deflection (m)

Initial Assumptions

For an infinitesimal cube acted upon by the stresses shown,
ε_1, ε_2 and ε_3 are the strains associated with the stresses σ_1, σ_2 and σ_3.
υ = Poisson's ratio.

These strains are given by the relations:

$$\varepsilon_1 = \sigma_1/E - \upsilon\sigma_2/E - \upsilon\sigma_3/E$$
$$\varepsilon_2 = \sigma_2/E - \upsilon\sigma_1/E - \upsilon\sigma_3/E$$
$$\varepsilon_3 = \sigma_3/E - \upsilon\sigma_1/E - \upsilon\sigma_2/E$$



1) Thick Disc Basics

- Consider a “disc”/“thin ring” subject to internal stresses resulting from the internal forces as a result of its rotational speed.

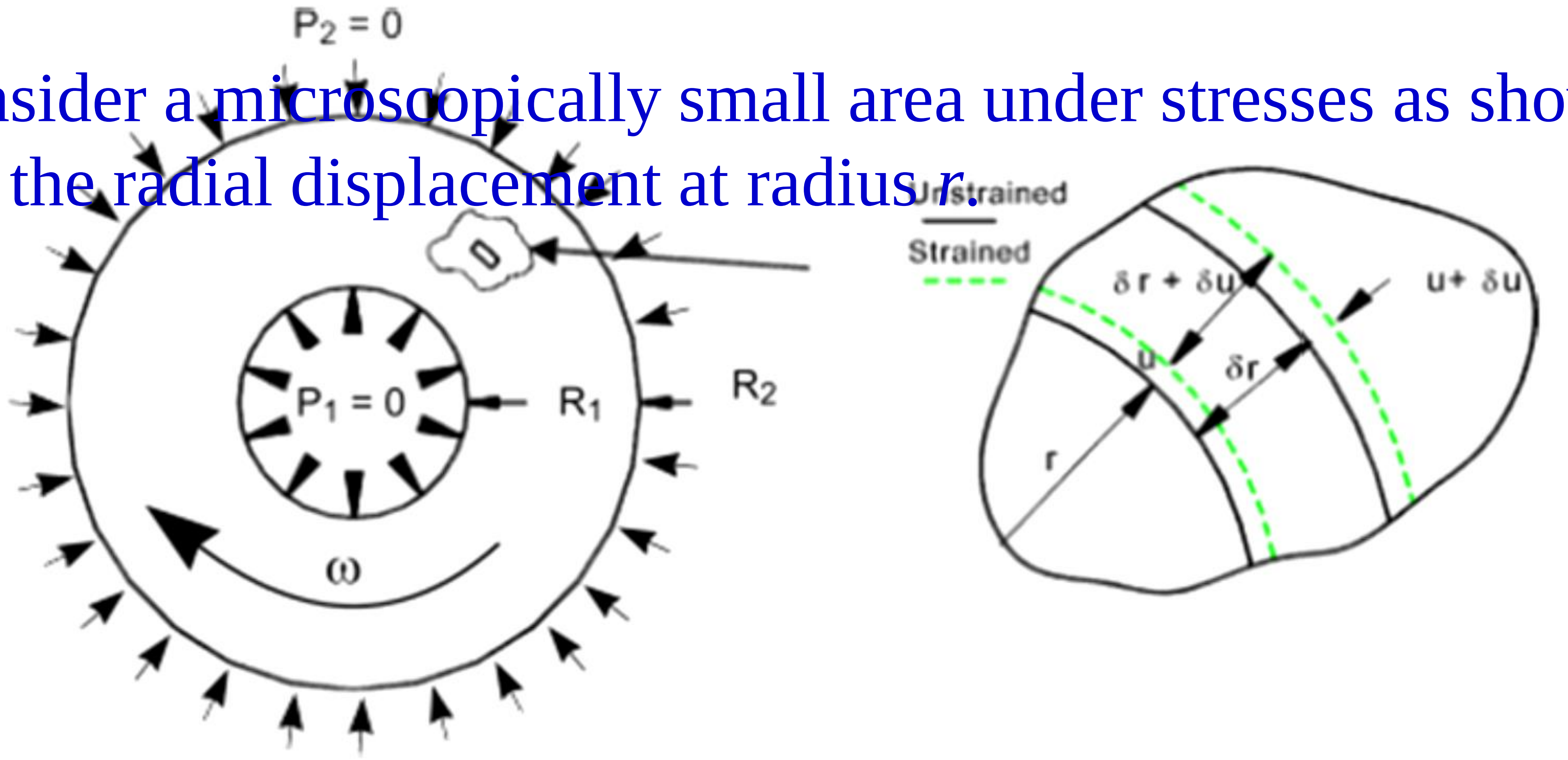
- Under the action of the internal forces only, the three principal stresses will be σ_r tensile radial stress, σ_t tensile tangential stress and σ_a an axial stress which is generally also tensile.

- The stress conditions occur throughout the section and vary primarily relative to the radius r.

- It is assumed that the axial stress σ_a is constant along the length of the section and because the disc is thin compared to its diameter, the axial stress throughout the section is assumed zero.

- It is also assumed that the internal pressure P_1 and the external pressure $P_2 = 0$.

Consider a microscopically small area under stresses as shown.
u is the radial displacement at radius *r*.



The circumferential (Hoop) strain due to the internal pressure is:

The circumferential (Hoop) strain due to the internal pressure is:

$$\varepsilon_t = \frac{Increase\ in\ circumference}{Original\ circumference} = \frac{2\pi(r+u)-2\pi}{2\pi} = \frac{u}{r}$$

At the outer radius of the small sectional area $(r + \delta r)$, the radius will increase by $(u + \delta u)$. The resulting radial strain as $\delta r \rightarrow 0$ is

$$\varepsilon_r = \frac{Increase\ in\ \delta r}{\delta r} = \frac{u+\delta u-u}{\delta r} = \frac{\delta u}{\delta r}$$

Referring to the stress/strain relationship as stated above. The following equations are derived:

Basis of equations:

σ_r is equivalent to σ_1

σ_t is equivalent to σ_2

σ_a is equivalent to σ_3

Derived Equations:

Strictly, the following equations apply:

Eq. 1) $E\varepsilon_a = \sigma_a - \upsilon\sigma_t - \upsilon\sigma_r$

Eq. 2) $E\varepsilon_t = E\frac{u}{r} = \sigma_t - \upsilon\sigma_a - \upsilon\sigma_r$

Eq. 3) $E\varepsilon_r = E\frac{du}{dr} = \sigma_r - \upsilon\sigma_t - \upsilon\sigma_a$

However because of the assumption that $\sigma_a = 0$ the equations are modified as follow.

Eq. 1) $E\varepsilon_a = 0 - \upsilon\sigma_t - \upsilon\sigma_r$

Eq. 2) $E\varepsilon_t = E\frac{u}{r} = \sigma_t - \upsilon\sigma_r$

Eq. 3) $E\varepsilon_r = E\frac{du}{dr} = \sigma_r - \upsilon\sigma_t$

Multiplying 2) x r

$$Eu = r(\sigma_t - \upsilon\sigma_r)$$

Differentiating

$$Edu/dr = \sigma_t - \upsilon\sigma_r + r.[d\sigma_t/dr - \upsilon.(d\sigma_r/dr)] = \sigma_r - \upsilon\sigma_t \text{..from 3)}$$

Simplifying by collecting terms:

Eq. 4 $$(\sigma_t - \sigma_r)(1 + v) + r\left(\frac{\sigma_t}{dr}\right) - vr\left(\frac{\sigma_r}{dr}\right) = 0$$

Now considering the radial equilibrium of the element of the section. Forces based on unit length of cylinder:

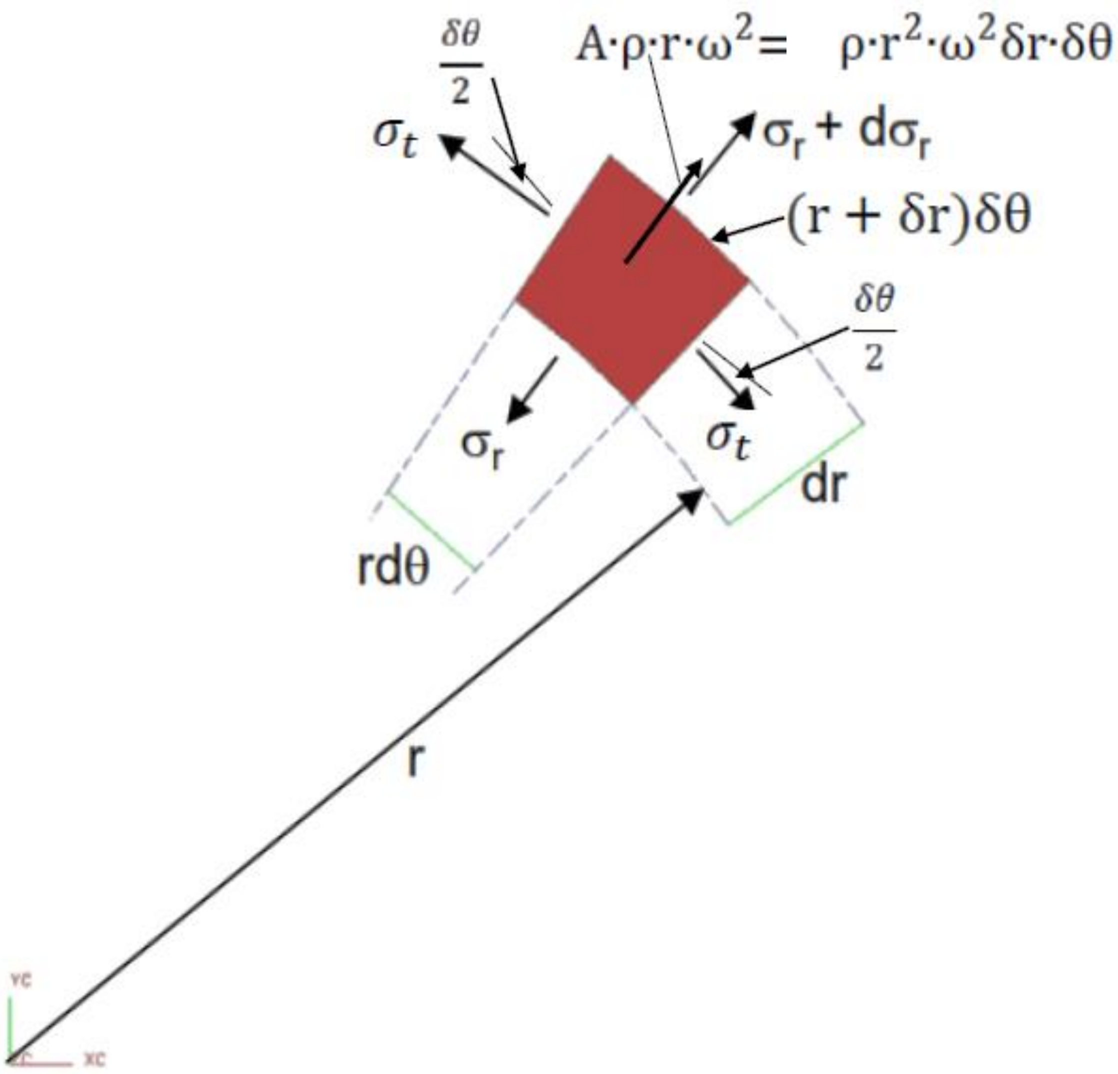
Given a small element of unit width, length = $r\delta\theta$ and thickness = δr;

Centrifugal force = $m\omega^2 r = \rho r\delta\theta\delta r\omega^2 r$

$$= \rho r^2\omega^2\delta r\delta\theta$$

$$2\cdot\sigma_t\cdot\delta r\cdot\sin\left(\frac{\delta\theta}{2}\right)+\sigma_r\delta\theta$$

$$-(\sigma_r+\delta\sigma_r)(r+\delta r)\delta\theta = \rho r^2\omega^2\delta r\delta\theta$$



In the limit this reduces to

Eq 5) $\sigma_t - \sigma_r - r d\sigma_r/dr = \rho r^2\omega^2$

Substitute for $\sigma_t - \sigma_r$ into Equation 4 results in

$$(rd\sigma_r/dr + \rho r^2\omega^2).(1 + \upsilon) + r.(d\sigma_t/dr)) - \upsilon.r.(d\sigma_r/dr) = 0$$

Therefore

$$d\sigma_t/dr + d\sigma_r/dr = -\rho r\omega^2(1 + \upsilon)$$

Integrating

Eq 6) $\sigma_t + \sigma_r = -\rho r^2\omega^2(1 + \upsilon)/2 + 2A$

Subtract equation 5...

$$2.\sigma_r + r.d\sigma_r/dr = -\rho r^2\omega^2(3 + \upsilon)/2 + 2A$$

This is the same as

$$(1/r).d(\sigma_r.r^2)dr = -\rho r^2\omega^2(3+\upsilon)/2 + 2A$$

Integrating

$$\sigma_r.r^2 = -(\rho r^4\omega^2(3+\upsilon)/8 + Ar^2 + B$$

Eq. 7 $\boldsymbol{\sigma_r = A + \frac{B}{r^2} - (3+\upsilon)\rho r^2\omega^2/8}$ (dividing by r^2)

Combining Eq. 7 with Eq. 6 (i.e. substituting σ_r from Eq. 7 into Eq.6):

Eq. 8 $\boldsymbol{\sigma_t = A - \frac{B}{r^2} - (1+3\upsilon)\rho r^2\omega^2/8}$

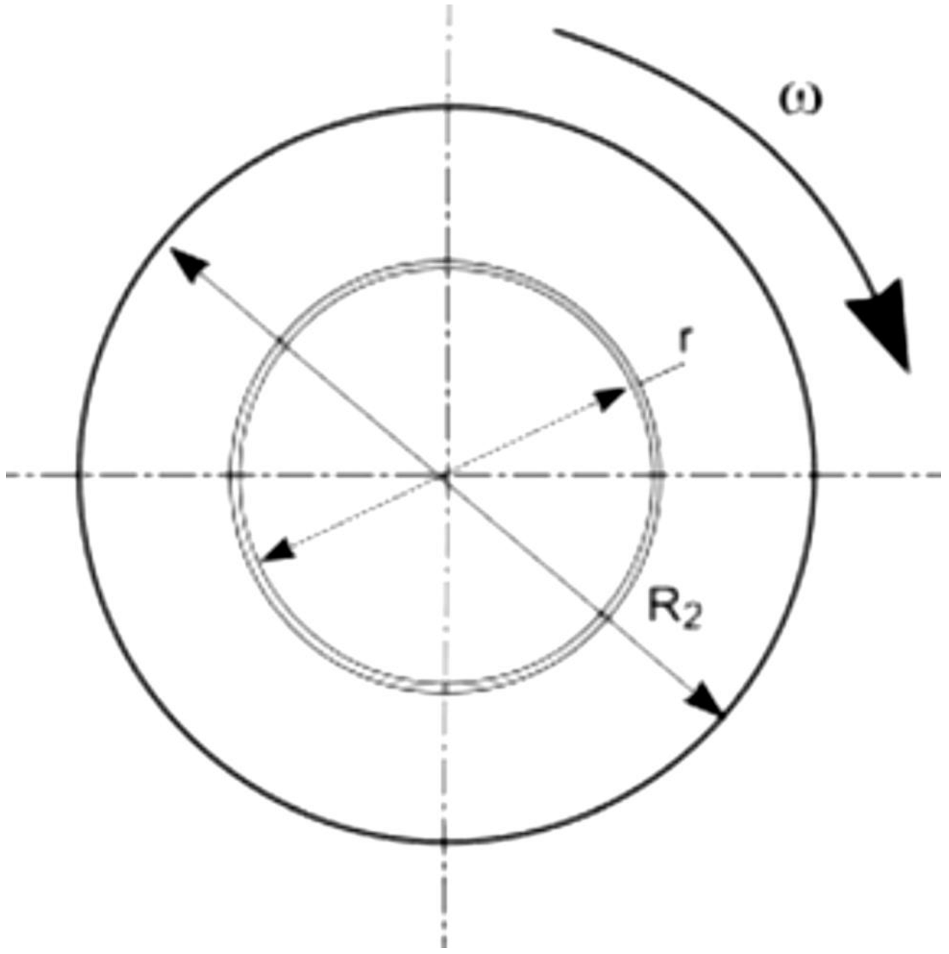
1) Solid Disk

At the centre the B/r^2 term implies infinite stresses which are clearly not credible and therefore B must equal 0.

At $r = R_2$ on the outside edge of the disk. The radial stress is equal to the surface stress which is equal to 0.

Therefore at R_2

$$\sigma_r = 0 = A - (3 + \upsilon)\rho R_2^2\omega^2/8$$



Therefore $A = (3 + \upsilon)\rho R_2^2\omega^2/8$

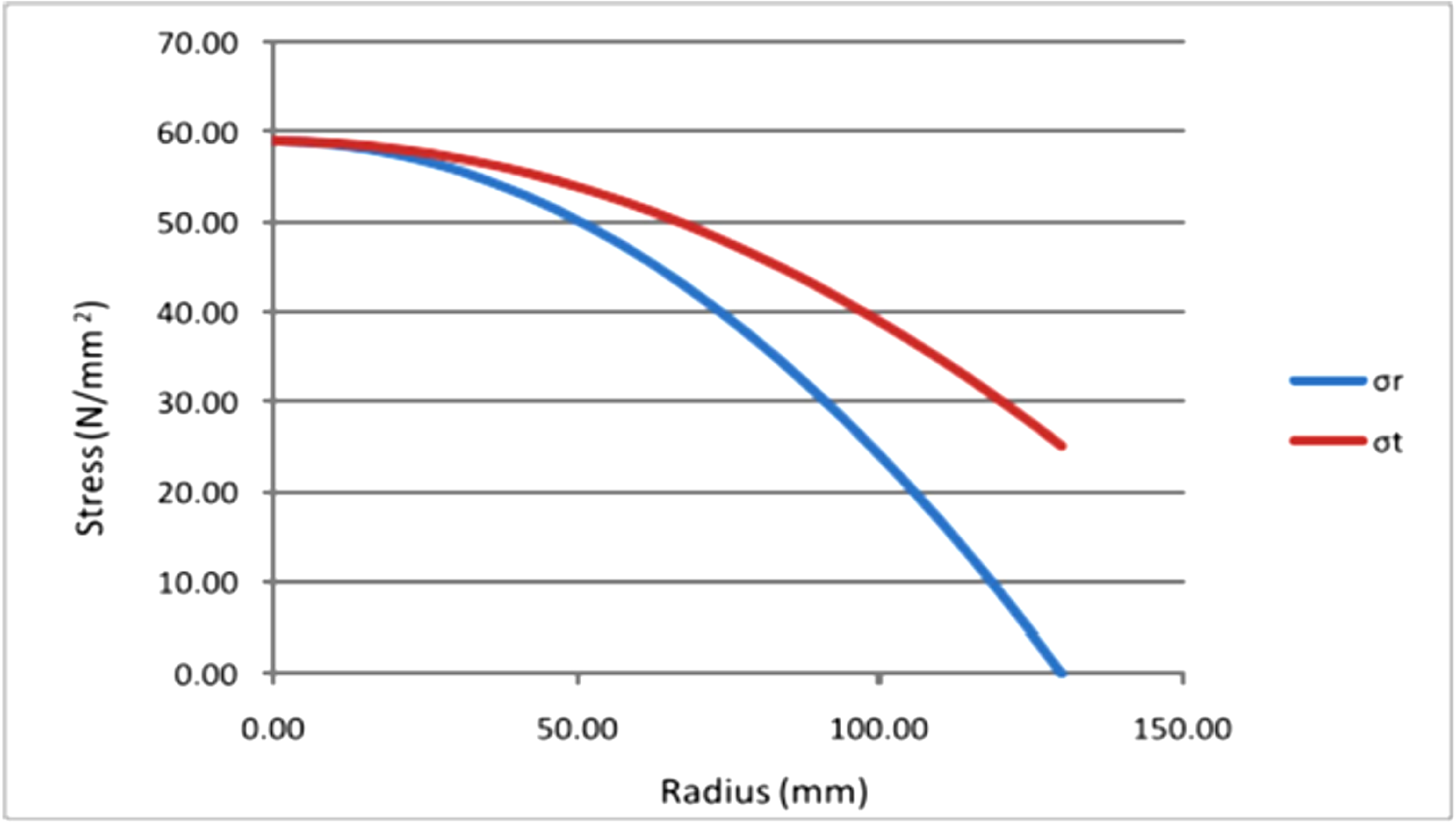
$B = 0$

$\sigma_t = (\rho\omega^2/8)[(3 + \upsilon)R_2^2 - (1 + 3\upsilon)r^2]$

$\sigma_r = (\rho\omega^2/8)[(3 + \upsilon)(R_2^2 - r^2)]$

The maximum stress is at the centre

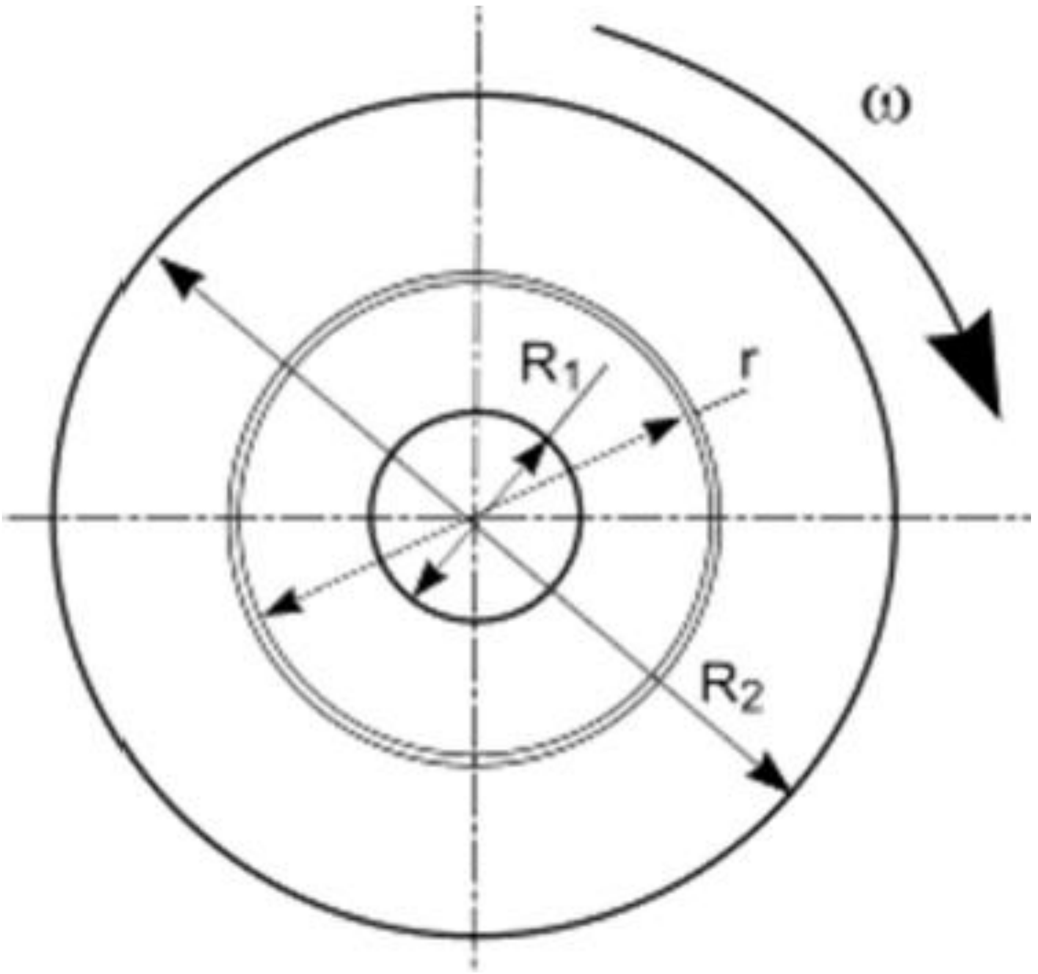
$\sigma_{t\_max} = \sigma_{r\_max} = (\rho\omega^2/8).(3 + \upsilon)R_2^2$



250mm diameter disc at 10,000 rpm

2) Disk with a hole

At the outside edge $r = R_2$ and at the hole radius $r = R_1$ the radial surface stress is assumed to be 0



Therefore

$$\sigma_r = 0 = A + B/R_2^2 - (3 + \upsilon)\rho R_2^2\omega^2/8$$
$$\sigma_r = 0 = A + B/R_1^2 - (3 + \upsilon)\rho R_1^2\omega^2/8$$

Solving

$$B = -(3 + \upsilon)\rho\omega^2/8.(R_1{}^2.R_2{}^2)$$
$$A = (3 + \upsilon)\rho\omega^2/8.(R_1{}^2 + R_2{}^2)$$

$$\sigma_r = (3 + \upsilon)\rho\omega^2/8)(R_1{}^2 + R_2{}^2 - R_1{}^2.R_2{}^2/r^2 - r^2)$$ and
$$\sigma_t = \rho\omega^2/8)[(3 + \upsilon)(R_1{}^2 + R_2{}^2 + R_1{}^2.R_2{}^2/r^2) - (1 + 3\upsilon)r^2)]$$

The maximum tangential stress σ_t is at the inside hole surface and equals

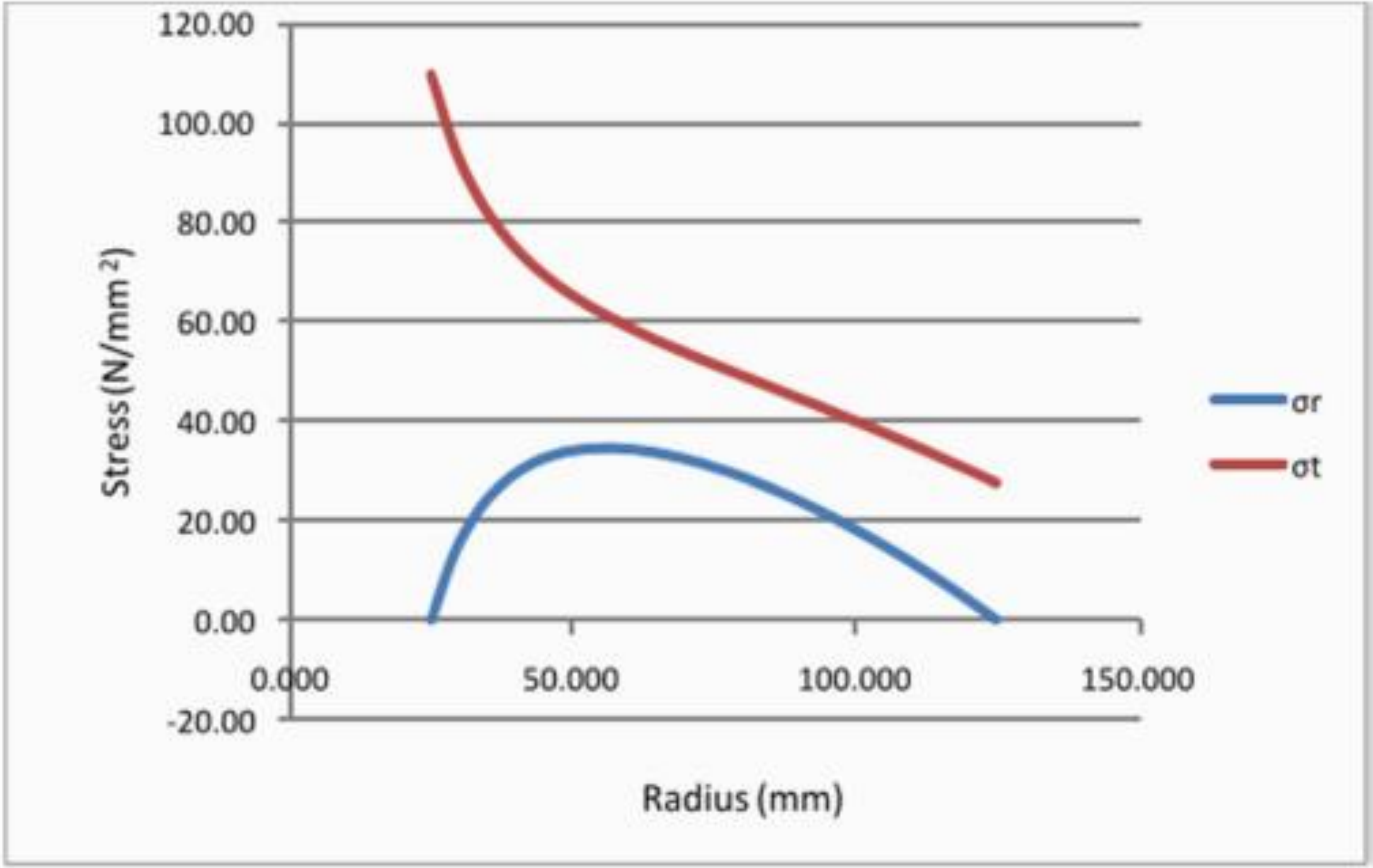
$$\sigma_{t\_max} = \rho\omega^2/4)[(1 - \upsilon)R_1{}^2 + (3 + \upsilon)R_2{}^2)]$$

The maximum tangential stress σ_t is at the inside hole surface and equals

$$\sigma_{t\_max} = \rho\omega^2/4)[(1 - \upsilon)R_1^2 + (3 + \upsilon)R_2^2)]$$

The maximum radial stress σ_r is at $r = \mathrm{sqrt}(R_1.R_2)$ and equals.

$$\sigma_{r\_max} = (3 + \upsilon)(\rho\omega^2/8)(R_2 - R_1)^2$$



250mm OD x 50mm ID ring at 10,000 rpm

To be continued.

Enjoy ME 3352.

ME 3352: Strength of Materials II

For it is, this far, the best there can be among courses!!!