UNIVERSITY OF ZAMBIA
SCHOOL OF ENGINEERING

MEC 3352 – STRENGTH OF MATERIALS II

Rotating Discs and Cylinders

SYMBOLS/UNITS

Tensile stresses are considered positive and compressive stresses are negative,

$P_1 = Internal\ Pressure\ \left(N/_{m^2}\right)$

$P_2 = Internal\ Pressure\ \left(N/_{m^2}\right)$

$\sigma_r = Radial\ Stress\ \left(N/_{m^2}\right)$

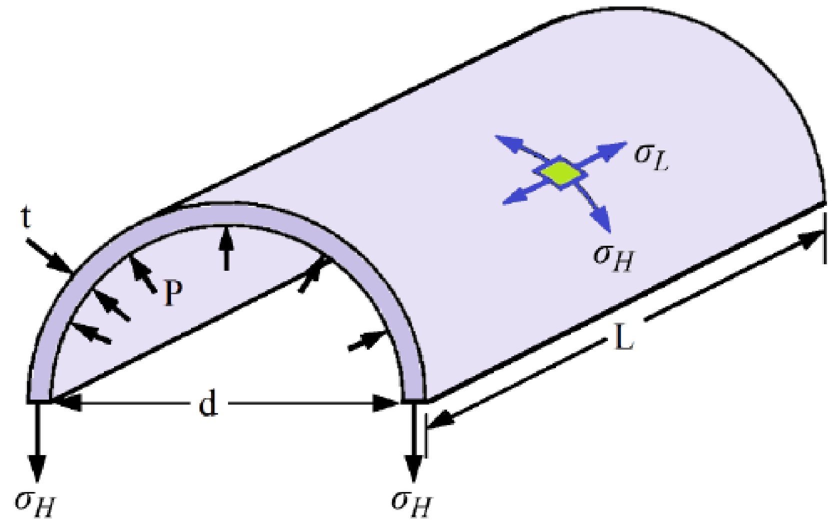
$\sigma_t = Tangential\ Stress\ \left(N/_{m^2}\right)$

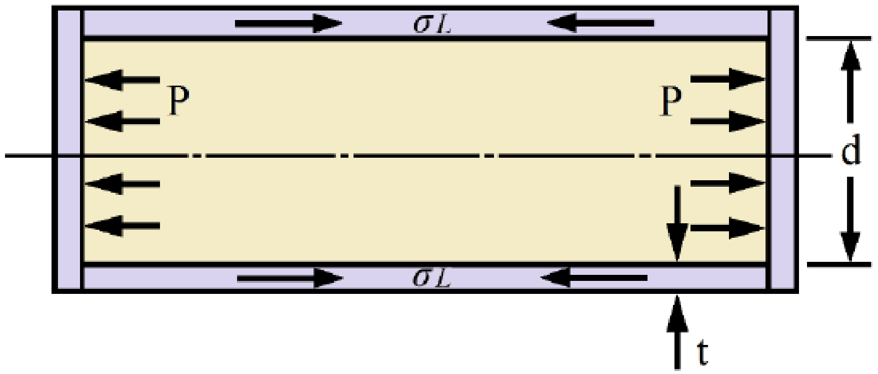
$\sigma_a = Axial/longitudinal\ Stress\ \left(N/_{m^2}\right)$

$E = Young's Modulus\ of\ Elasticity\ \left(N/_{m^2}\right)$

$\rho = Density \left(kg/_{m^3}\right)$

$\nu = Poissoin's\ ration$





SYMBOLS/UNITS

$r \ = Radius\ at\ point\ of\ analysis\ (m)$

$R_1 = Internal\ radius\ of\ the\ cylinder\ (m)$

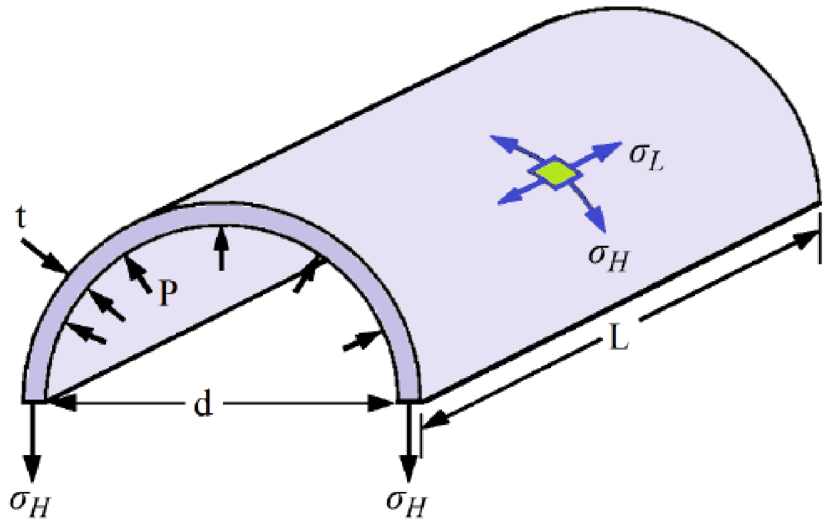
$R_2 = External\ radius\ of\ the\ cylinder\ (m)$

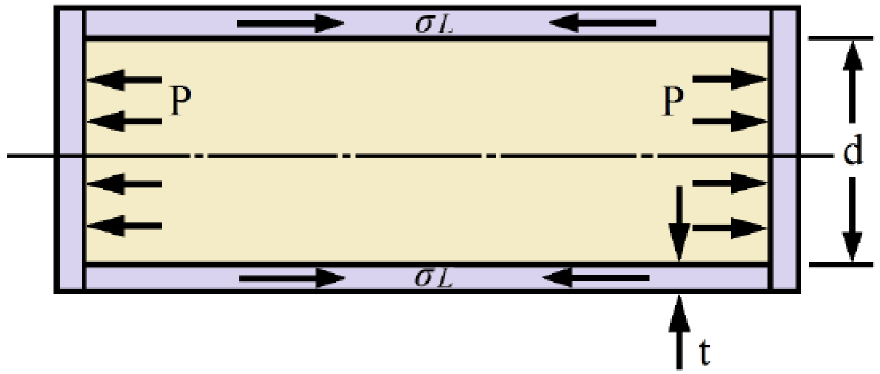
$\varepsilon_r = Radial\ Strain$

$\varepsilon_t = Tangential\ \ Strain$

$\varepsilon_a = Axial/longitudinal\ Strain$

$w = Radial\ deflection\ (m)$





INITIAL ASSUMPTIONS

For an infinitesimal cube acted upon by the stresses shown, ε_1, ε_2 and ε_3 are the strains associated with the stresses σ_1, σ_2 and σ_3. υ = Poisson's ratio.

The strains are given by the relations:

$$\varepsilon_1 = \sigma_1/E - \upsilon\sigma_2/E - \upsilon\sigma_3/E$$
$$\varepsilon_2 = \sigma_2/E - \upsilon\sigma_1/E - \upsilon\sigma_3/E$$
$$\varepsilon_3 = \sigma_3/E - \upsilon\sigma_1/E - \upsilon\sigma_2/E$$

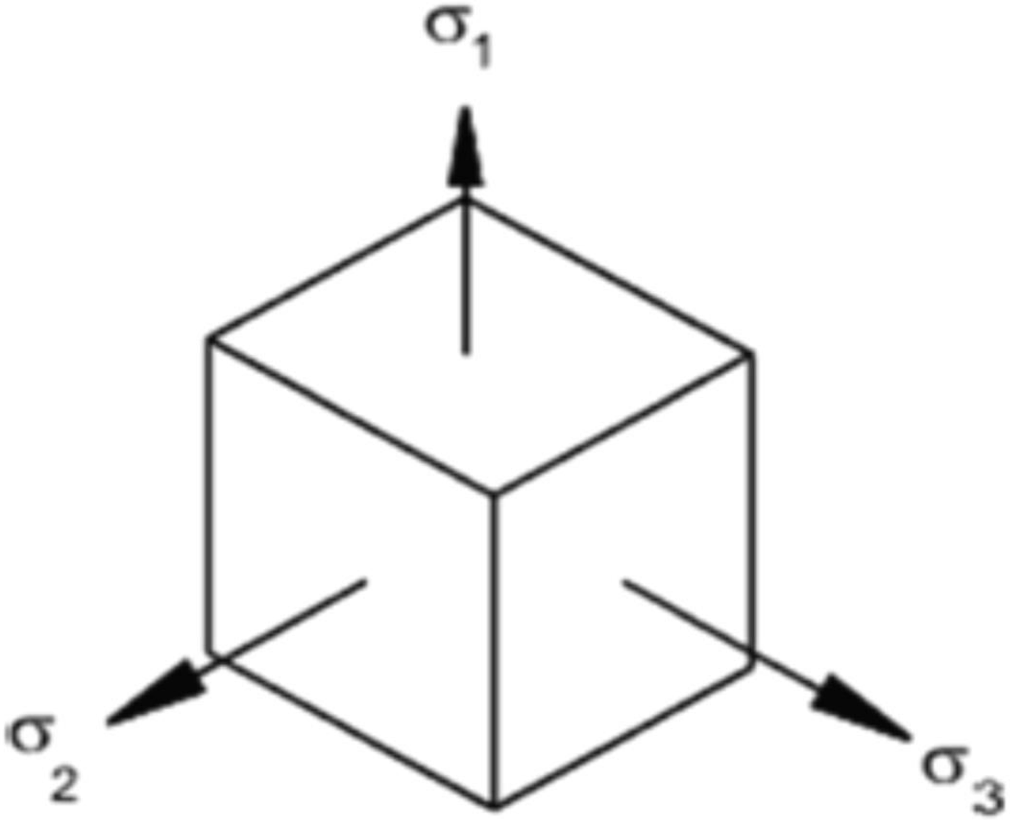


Figure 3.0 Stresses acting on an infinitesimal cube

THIN DISC

- Consider a “disc” or a “thin ring” subject to internal stresses resulting from the internal forces as a result of its rotational speed.
- Under the action of the internal forces only, the three principal stresses will be σ_r tensile radial stress, σ_t tensile tangential stress and σ_a an axial stress which is generally also tensile.
- The stress conditions occur throughout the section and vary primarily relative to the radius r.
- It is assumed that the axial stress σ_a is constant along the length of the section because the disc is thin compared to its diameter.
- The axial stress throughout the section is assumed zero. It is also assumed that the internal pressure is P_1 and the external pressure $P_2 = 0$.

THIN ROTATING DISC OR CYLINDER

- Consider a thin disc/ring or cylinder rotating at an angular speed ω rad/s.
- The rotating disc or cylinder is subjected to a radial pressure (centrifugal force ***p***) caused by the centrifugal effect on the rotating mass of the disc.

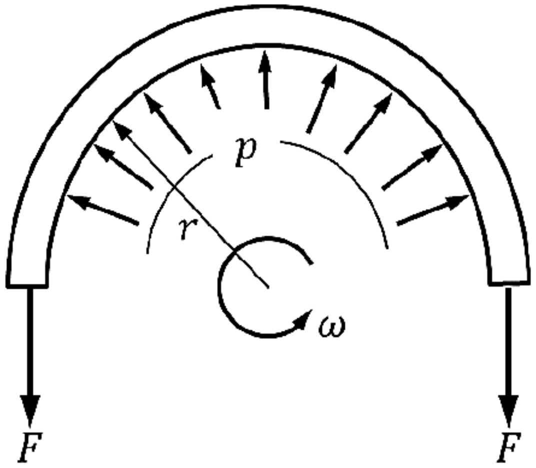


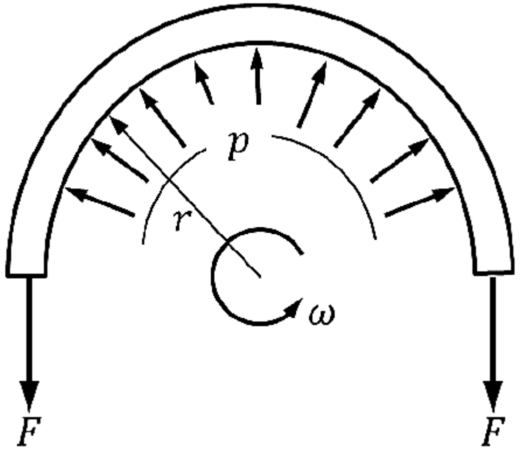
Figure 3.1: Section of a thin rotating disc

- Where we have the following:

$$p = internal\ pressure\ (N/m^2)$$
$$r = internal\ radius\ (m)$$
$$\omega = angular\ speed\ (rad/s)$$
$$F = Hoop\ tension\ force\ (N), due\ to\ rotation$$

THIN ROTATING DISC OR CYLINDER

- We now consider equilibrium of the half ring over unit thickness of the circumference:



$$2F = p \times d \times 1$$

$$2F = p \times 2r$$

$$F = p\,r$$

$$F = pr = (m\omega^2 r)r = m\omega^2 r^2$$

- The centrifugal effect on a unit length of the circumference is: $p = m\omega^2 r$
- We assume the cylinder wall is so thin such that the centrifugal effect is constant through the thickness of the wall.
- In this case, the hoop tension (F) is transmitted through the entire circumference and is resisted by the complete cross-sectional area.
- The hoop stress is given by:

$$\sigma_\theta = \frac{F}{A} = \frac{m\omega^2 r^2}{A}$$

THIN ROTATING DISC OR CYLINDER

- The hoop stress (σ_θ) is also known as the tangential or circumferential stress
- Note that A is the cross-sectional area of the ring.
- And since we are considering a unit length, the ratio (m/A) is the mass of the material per unit volume which is the density. Thus,

$$\sigma_\theta = \rho\omega^2 r^2 \qquad 3.1$$

- Consider the element of a disc (of unit thickness) shown in figure 3.2:

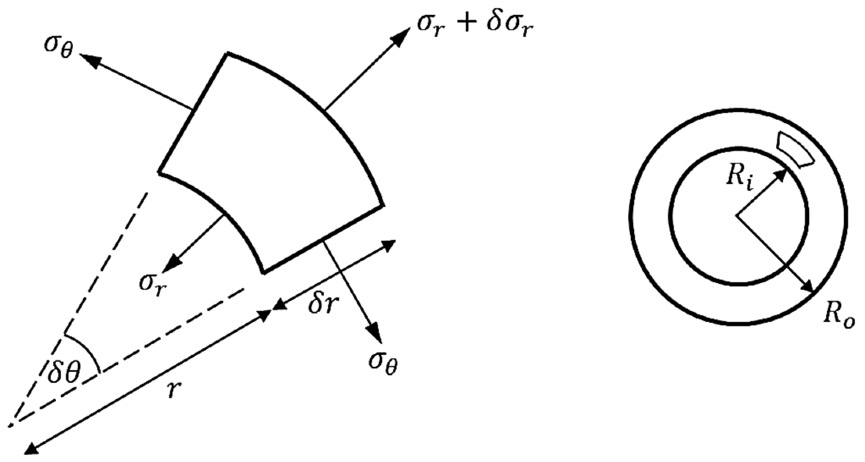


Figure 3.2: Element of a rotating disc

- The three principal stresses are:

$\sigma_r = radial\ stress$

$\sigma_\theta = Hoop\ stress\ (tangential, circumferential)$

$\sigma_l = Longitudinal\ (axial\ stress)$

- The axial (longitudinal stresses is normal to the plan

ROTATING DISC

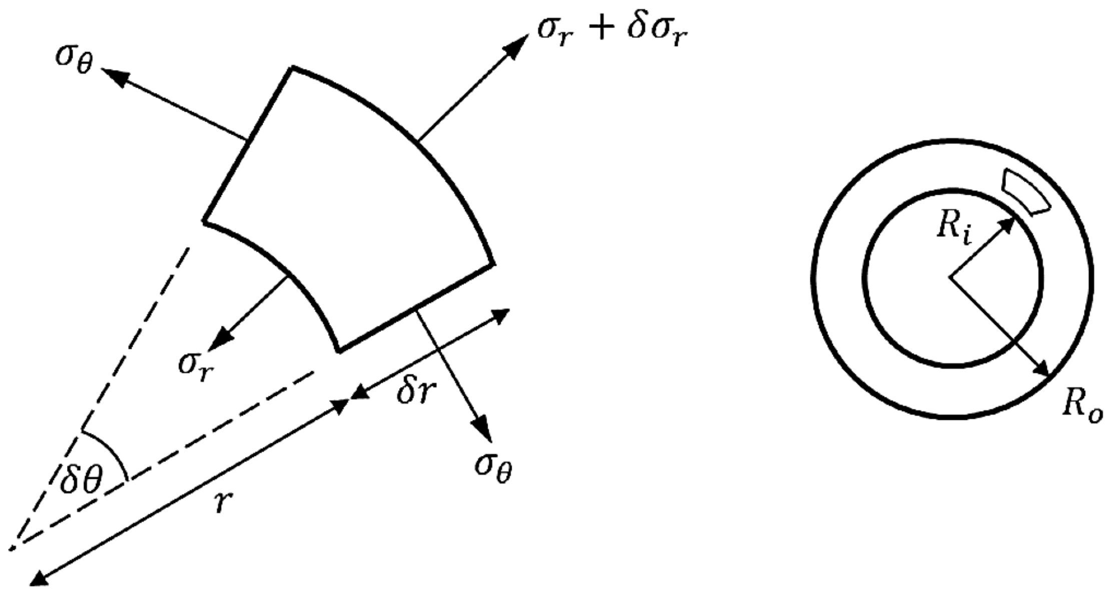


Figure 3.2: Element of a rotating disc

- At a radius r, and assuming unit thickness, the volume of the element is given by:

$$\delta v = r\delta\theta.\delta r.1 = r\delta\theta\delta r$$

- The mass of the element is given by:

$$\delta m = \rho\delta v = \rho r\delta\theta\delta r$$

- The centrifugal force acting on the element is given by:

$$F = m\omega^2 r$$

$$F = \rho r\delta\theta\delta r.\omega^2 r = \rho\omega^2 r^2\delta\theta\delta r$$

- Considering radial equilibrium of the element, we have forces acting on an element of unit thickness shown in figure 3.3:

ROTATING DISC

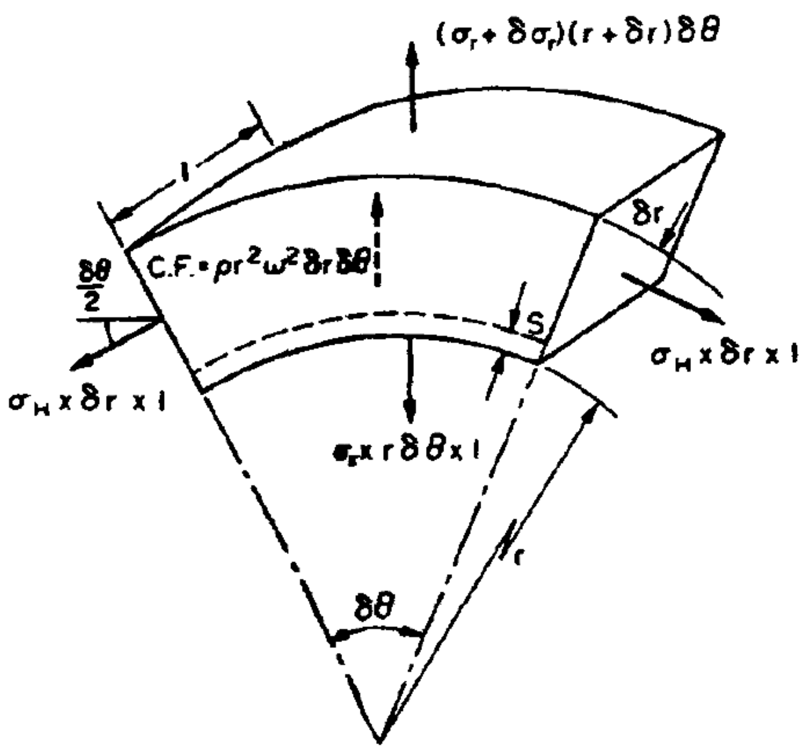


Figure 3.3: Element of a rotating disc

- For equilibrium of the element radially (resolve forces in radial direction

$$2\sigma_H \delta r \sin\frac{\delta\theta}{2} + \sigma_r r\delta\theta - (\sigma_r + \delta\sigma_r)(r+\delta r)\delta\theta = \rho r^2\omega^2\delta\theta\,\delta r$$

- For a small value of $\delta\theta$, $sin\left(\frac{\delta\theta}{2}\right) \approx \frac{\delta\theta}{2}$ radians

$$2\sigma_H\delta r\left(\frac{\delta\theta}{2}\right) + \sigma_r r\delta\theta - (\sigma_r + \delta\sigma_r)(r+\delta r)\delta\theta = \rho r^2\omega^2\delta\theta\delta r$$

$$\sigma_H\delta r + \sigma_r r - (\sigma_r + \delta\sigma_r)(r+\delta r) = \rho r^2\omega^2\delta r$$

$$\sigma_H\delta r + \sigma_r r - (\sigma_r r + \delta\sigma_r r + \sigma_r\delta r + \delta\sigma_r\delta r) = \rho r^2\omega^2\delta r$$

$$\sigma_H\delta r + \sigma_r r - \sigma_r r - \delta\sigma_r r - \sigma_r\delta r - \delta\sigma_r\delta r = \rho r^2\omega^2\delta r$$

ROTATING DISC

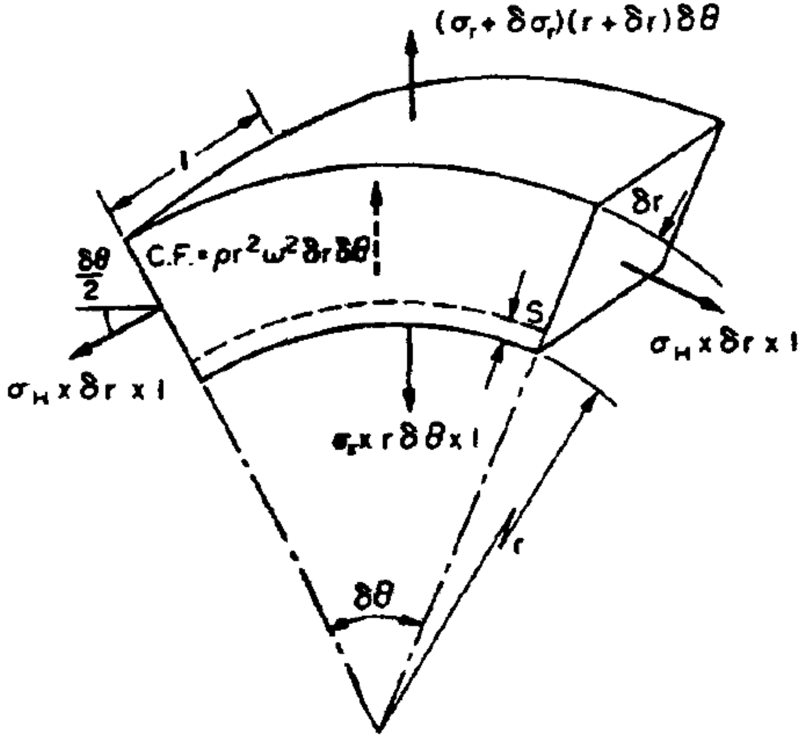


Figure 3.3: Element of a rotating disc

$$\sigma_H\delta r - \delta\sigma_r r - \sigma_r\delta r - \delta\sigma_r\delta r = \rho r^2\omega^2\delta r$$

$$\sigma_H - r\left(\frac{\delta\sigma_r}{\delta r}\right) - \sigma_r - \delta\sigma_r = \rho r^2\omega^2$$

- The $\lim_{\delta r\to 0}\left(\frac{\delta\sigma_r}{\delta r}\right) = \frac{d\sigma_r}{dr}$, as $\boldsymbol{\delta\sigma_r} \to$ **0. Thus,** the radial equilibrium equation reduces to:

$$\sigma_H - \sigma_r - r\frac{d\sigma_r}{dr} = \rho r^2\omega^2 \qquad \textbf{3.2}$$

- If there is a radial movement or “shift” of the element by an amount ***s*** as the disc rotates, the radial strain is given by:

$$\varepsilon_r = \frac{ds}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_H) \qquad \textbf{3.3}$$

ROTATING DISC

- The circumferential strain is given by:

$$\frac{s}{r} = \frac{1}{E}(\sigma_H - \nu\sigma_r)$$

$$s = \frac{1}{E}(\sigma_H - \nu\sigma_r)r$$

- Differentiating:

$$\frac{ds}{dr} = \frac{1}{E}(\sigma_H - \nu\sigma_r) + \frac{r}{E}\left[\frac{d\sigma_H}{dr} - \nu\frac{d\sigma_r}{dr}\right] \quad 3.4$$

$$\varepsilon_r = \frac{ds}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_H) \quad 3.3$$

- **Equating eqns. (3.3) and (3.4) and simplifying,**

$$(\sigma_H - \sigma_r)(1+\nu) + r\frac{d\sigma_H}{dr} - \nu\frac{d\sigma_r}{dr} = 0 \quad 3.5$$

- From eqn. (3.2)

$$\sigma_H - \sigma_r = \rho r^2\omega^2 + r\frac{d\sigma_r}{dr} \quad 3.2$$

$$\therefore \quad \left(r\frac{d\sigma_r}{dr} + \rho r^2\omega^2\right)(1+\nu) + r\frac{d\sigma_H}{dr} - \nu\frac{d\sigma_r}{dr} = 0$$

ROTATING DISC

$$\therefore \quad \left(r\frac{d\sigma_r}{dr} + \rho r^2\omega^2\right)(1+\nu) + r\frac{d\sigma_H}{dr} - \nu\frac{d\sigma_r}{dr} = 0$$

- Re-arranging and Simplifying,

$$\frac{d}{dr}(\sigma_H + \sigma_r) = -\rho r\omega^2(1+\nu)$$

- Integrating,

$$\sigma_H + \sigma_r = -\frac{\rho r^2\omega^2}{2}(1+\nu) + 2A$$

- Where ***2A*** is just some convenient constant of integration.
- Subtracting eqn. (3.2) from the above eqn.

$$\sigma_H - \sigma_r - r\frac{d\sigma_r}{dr} = \rho r^2\omega^2 \qquad 3.2$$

$$2\sigma_r + r\frac{d\sigma_r}{dr} = -\frac{\rho r^2\omega^2}{2}(3+\nu) + 2A$$

- But

$$2\sigma_r + r\frac{d\sigma_r}{dr} = \frac{d}{dr}[(r^2\sigma_r)].\frac{1}{r}$$

ROTATING DISC

$$2\sigma_r + r\frac{d\sigma_r}{dr} = \frac{d}{dr}\left[\left(r^2\sigma_r\right)\right].\frac{1}{r}$$

$$\therefore \qquad \frac{d\left(r^2\sigma_r\right)}{dr} = r\left[-\frac{\rho r^2\omega^2}{2}(3+\nu) + 2A\right]$$

- Integrating

$$r^2\sigma_r = -\frac{\rho r^4\omega^2}{8}(3+\nu) + \frac{2Ar^2}{2} - B$$

- Again, **- *B*** is just another convenient constant of integration,

$$\sigma_r = A - \frac{B}{r^2} - (3+\nu)\frac{\rho r^2\omega^2}{8} \qquad 3.6$$

- From equation 3.5

$$(\sigma_H - \nu\sigma_r)(1+\nu) + r\frac{d\sigma_H}{dr} - \nu\frac{d\sigma_r}{dr} = 0 \qquad 3.5$$

- We can get the following equation:

$$\sigma_H = A + \frac{B}{r^2} - (1+3\nu)\frac{\rho r^2\omega^2}{8} \qquad 3.7$$

SOLID DISC

- For a solid disc the stress at the centre is obtained when ***r*** = 0.
- With ***r*** equal to zero equations **3.6** and **3.7** will yield infinite stresses whatever the speed of rotation unless ***B*** is also zero, i.e. ***B*** = 0
- Thus, $B/r^2 = 0$ gives the only finite solution.
- At the outside radius ***R***, the radial stress must be zero since there are no external forces to provide the necessary balance of equilibrium if σ_r were not zero.
- Therefore, from eqn. (3.6),

$$\sigma_r = A - \frac{B}{r^2} - (3+\nu)\frac{\rho r^2 \omega^2}{8} \qquad 3.6$$

$$0 = A - 0 - (3+\nu)\frac{\rho r^2 \omega^2}{8}$$

$$A = (3+\nu)\frac{\rho r^2 \omega^2}{8}$$

- Substituting ***A*** **into** eqns. (3.6) and (3.7) the hoop and radial stresses at any radius ***r*** in a solid disc are given by:

SOLID DISC

- Substituting in eqns. (3.6) and (3.7) the hoop and radial stresses at any radius ***r*** in a solid disc are given by:

$$\sigma_r = (3+\nu)\frac{\rho R^2\omega^2}{8} - (3+\nu)\frac{\rho r^2\omega^2}{8}$$

$$\sigma_r = (3+\nu)\frac{\rho\omega^2}{8}\left[R^2 - r^2\right] \qquad 3.8$$

$$\sigma_H = (3+\nu)\frac{\rho\omega^2 R^2}{8} - (1+3\nu)\frac{\rho\omega^2 r^2}{8}$$

$$\sigma_H = \frac{\rho\omega^2}{8}\left[(3+\nu)R^2 - (1+3\nu)r^2\right] \qquad 3.9$$

SOLID DISC

MAXIMUM STRESSES

- At the ***centre*** of the disc, where ***r*** = 0, equations **3.8** and **3.9** yield **equal values of hoop and radial stress**.
- The maximum hoop and radial stresses are at the centre of the disc (see figure 3.4)
- The maximum stress is given by:

$$\sigma_{max} = (3+\nu)\frac{\rho\omega^2R^2}{8} \qquad 3.10$$

- At the ***outside*** of the disc, at ***r*** = ***R,*** the equations give:

$$\sigma_r = 0$$

$$\sigma_H = (1-\nu)\frac{\rho\omega^2R^2}{4}$$

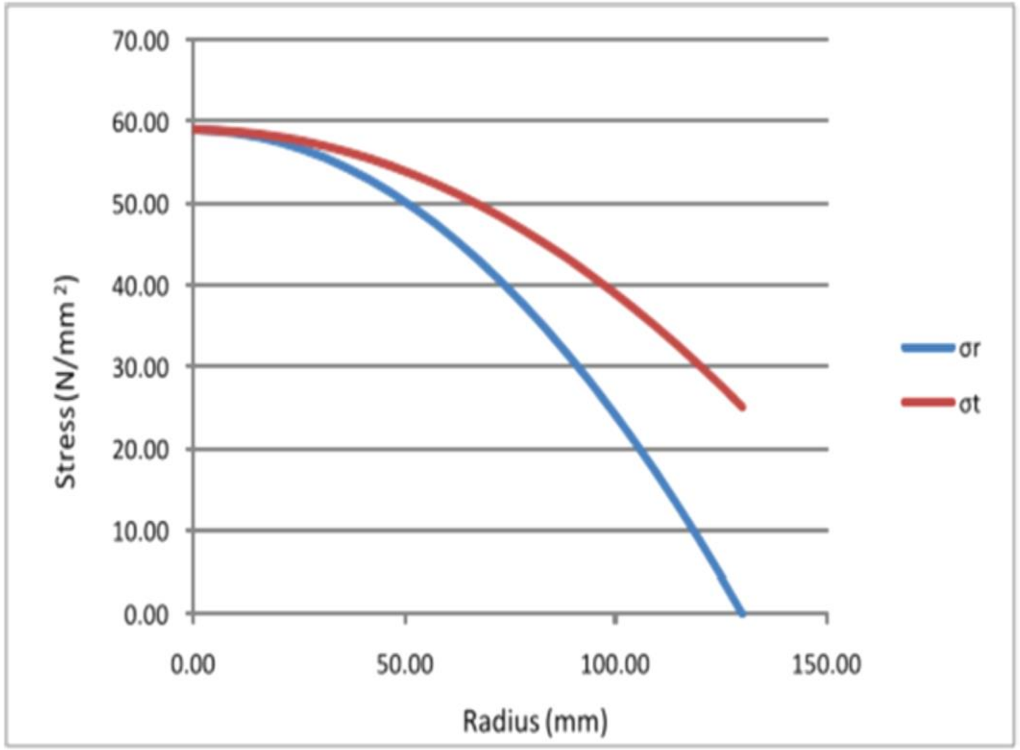


Figure 3.4: Variation of radial and tangential (hoop) stress in a solid disc

ROTATING DISC WITH A CENTRAL HOLE

- The general equations for the stresses in a rotating hollow disc can be obtained in the same way as those **for** the solid disc.

$$\sigma_r = A - \frac{B}{r^2} - (3+\nu)\frac{\rho r^2 \omega^2}{8} \qquad 3.6$$

$$\sigma_H = A + \frac{B}{r^2} - (1+3\nu)\frac{\rho r^2 \omega^2}{8} \qquad 3.7$$

- In the case of a disc with a hole at the centre, we use different boundary conditions to evaluate the constants ***A*** and ***B*** since, in this case, ***B*** is not zero.
- For the case of rotation only, the required boundary conditions are zero radial stress at both the inside and outside radius.

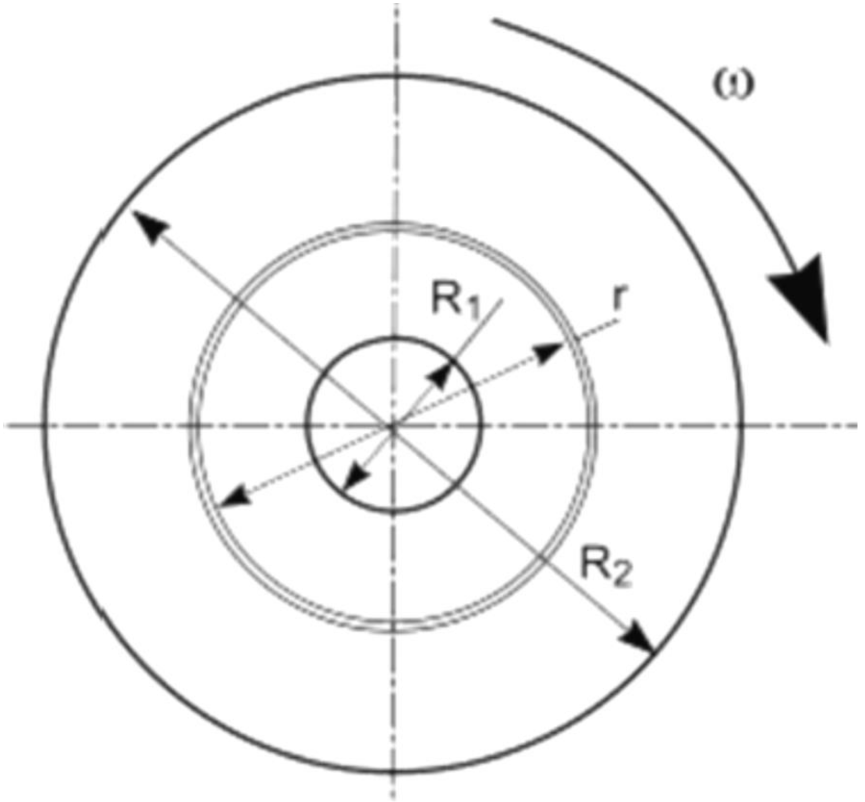


Figure 3.5: Rotating disc with central hole

ROTATING DISC WITH A CENTRAL HOLE

- At $r = R_1$, $\sigma_r = 0$

$$0 = A - \frac{B}{R_1^2} - (3+\nu)\frac{\rho\omega^2 R_1^2}{8}$$

- At $r = R_2$, $\sigma_r = 0$

$$0 = A - \frac{B}{R_2^2} - (3+\nu)\frac{\rho\omega^2 R_2^2}{8}$$

- Subtracting and simplifying, appropriately, we get:

$$B = (3+\nu)\frac{\rho\omega^2 R_1^2 R_2^2}{8}$$

$$A = (3+\nu)\frac{\rho\omega^2 (R_1^2 + R_2^2)}{8}$$

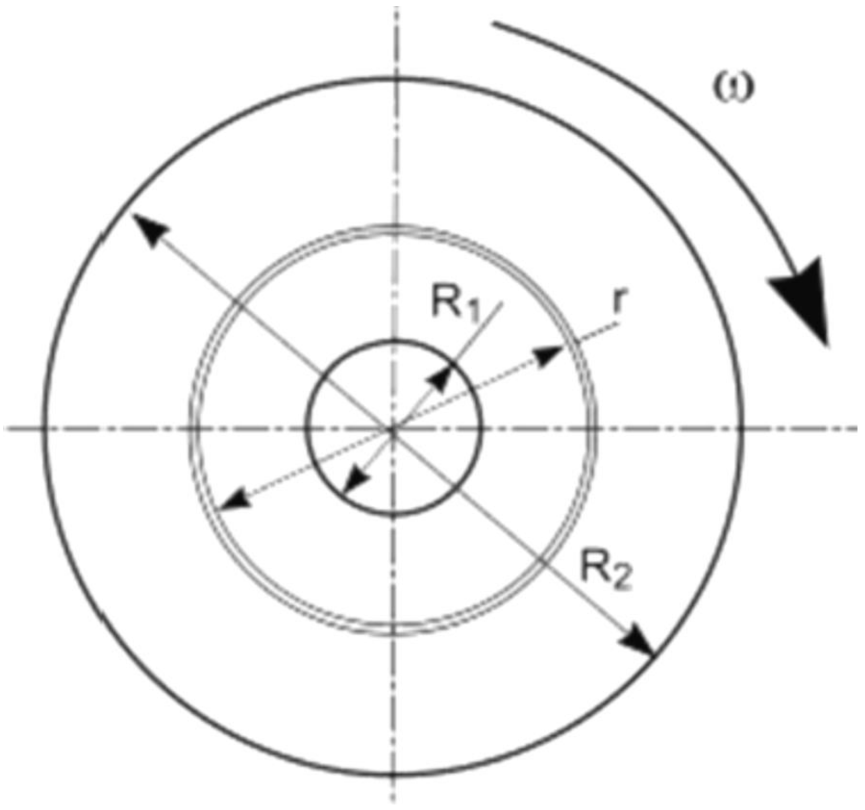


Figure 3.5: Rotating disc with central hole

ROTATING DISC WITH A CENTRAL HOLE

- Substituting ***A*** and ***B*** into eqns. (3.6) and (3.7):

$$\sigma_r = A - \frac{B}{r^2} - (3+\nu)\frac{\rho r^2 \omega^2}{8} \qquad 3.6$$

$$\sigma_H = A + \frac{B}{r^2} - (1+3\nu)\frac{\rho r^2 \omega^2}{8} \qquad 3.7$$

$$B = (3+\nu)\frac{\rho \omega^2 R_1^2 R_2^2}{8}$$

$$A = (3+\nu)\frac{\rho \omega^2 \left(R_1^2 + R_2^2\right)}{8}$$

- The final equations for the stresses, after relevant manipulations are

$$\sigma_r = (3+\nu)\frac{\rho\omega^2}{8}\left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2\right] \qquad 3.11$$

$$\sigma_H = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_1^2 + R_2^2 + \frac{R_1^2 R_2^2}{r^2}\right) - (1+3\nu)r^2\right] \qquad 3.12$$

ROTATING DISC WITH A CENTRAL HOLE

MAXIMUM HOOP STRESS

- The maximum hoop stress occurs at the inside radius where ***r = R₁***. (See figure 3.6)

$$\sigma_{H_{max}} = \frac{\rho\omega^2}{8}\left[(3+\nu)(R_1^2+R_2^2+R_2^2)-(1+3\nu)R_1^2\right]$$

$$= \frac{\rho\omega^2}{4}\left[(3+\nu)R_2^2+(1-\nu)R_1^2\right] \qquad 3.13$$

- The minimum hoop stress is at the outside surface of the disc when $r = R_2$.

$$\sigma_{H_{min}} = \frac{\rho\omega^2}{4}\left[(3+\nu)R_1^2+(1-\nu)R_2^2\right] \qquad 3.14$$

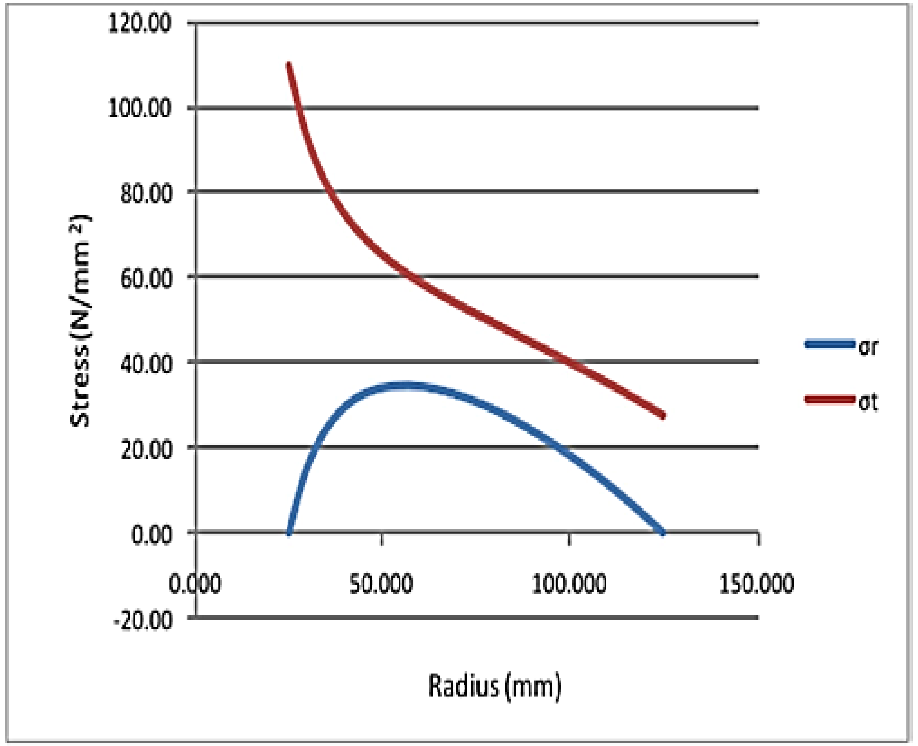


Figure 3.6: Variation of stresses in a disc with a central hole

ROTATING DISC WITH A CENTRAL HOLE

$$\sigma_{H_{max}} = \frac{\rho\omega^2}{4}\left[(3+\nu)R_2^2 - (1-\nu)R_1^2\right] \qquad 3.13$$

- **As** the value of the inside radius approaches zero the maximum hoop stress value approaches

$$\sigma_{H_{max}} = \frac{\rho\omega^2}{4}(3+\nu)R_2^2$$

- Note that this is twice the value obtained at the centre of a solid disc rotating at the same speed.
- We see that drilling of even a very small hole at the centre of a solid disc will double the maximum hoop stress set up due to rotation.
- At the outside of the disc when r = R_2.

$$\sigma_{H_{min}} = \frac{\rho\omega^2}{4}\left[(3+\nu)R_1^2 + (1-\nu)R_2^2\right] \qquad 3.14$$

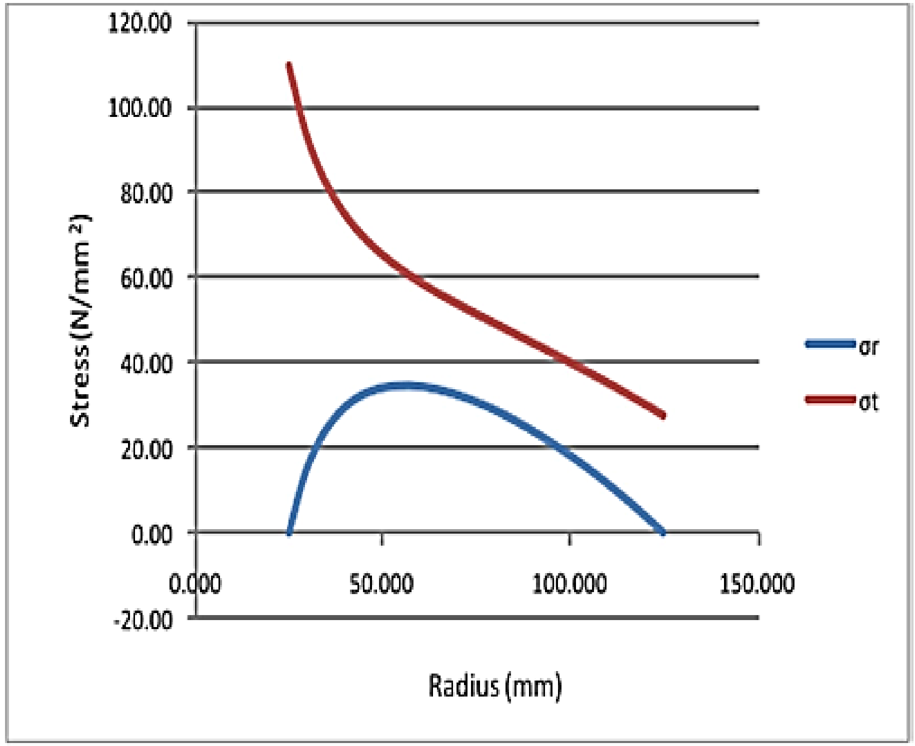


Figure 3.6: Variation of stresses in a disc with a central hole

ROTATING DISC WITH A CENTRAL HOLE

MAXIMUM RADIAL STRESS

- The *maximum radial stress* is found by consideration of the equation

$$\sigma_r = (3+\nu)\frac{\rho\omega^2}{8}\left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2\right] \qquad \mathbf{3.11}$$

- This will be a maximum when $\dfrac{d\sigma_r}{dr} = 0,$

$$0 = \frac{d}{dr}\left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2\right]$$

$$0 = R_1^2 R_2^2 \frac{2}{r^3} - 2r$$

$$r^4 = R_1^2 R_2^2$$

$$r = \sqrt{(R_1 R_2)}$$

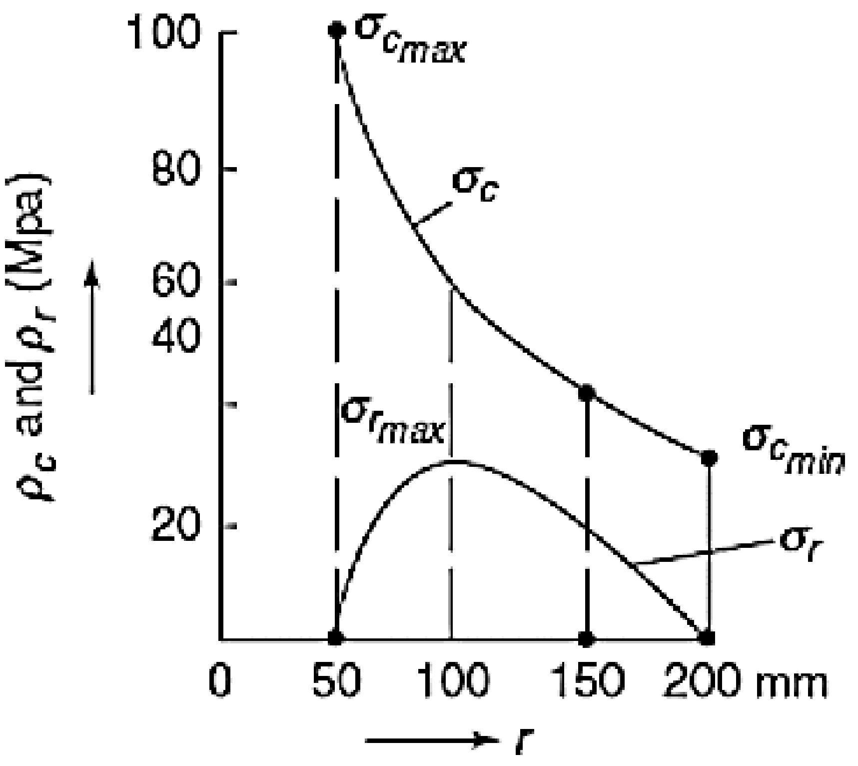


Figure 3.6: Variation of stresses in a disc with a central hole – Typical example

ROTATING DISC WITH A CENTRAL HOLE

MAXIMUM RADIAL STRESS

- Substituting for ***r*** into eqn. (3.11).

$$\sigma_{r_{\max}} = (3+\nu)\frac{\rho\omega^2}{8}\left[R_1^2 + R_2^2 - R_1R_2 - R_1R_2\right]$$

$$= (3+\nu)\frac{\rho\omega^2}{8}\left[R_2 - R_1\right]^2 \qquad 3.15$$

- We see from the stress distribution in figure 3.7 that radial stress is zero at inside and outside surfaces.
- We also see that tangential (hoop stress is maximum at the inside surface and minimum at the outside surface.

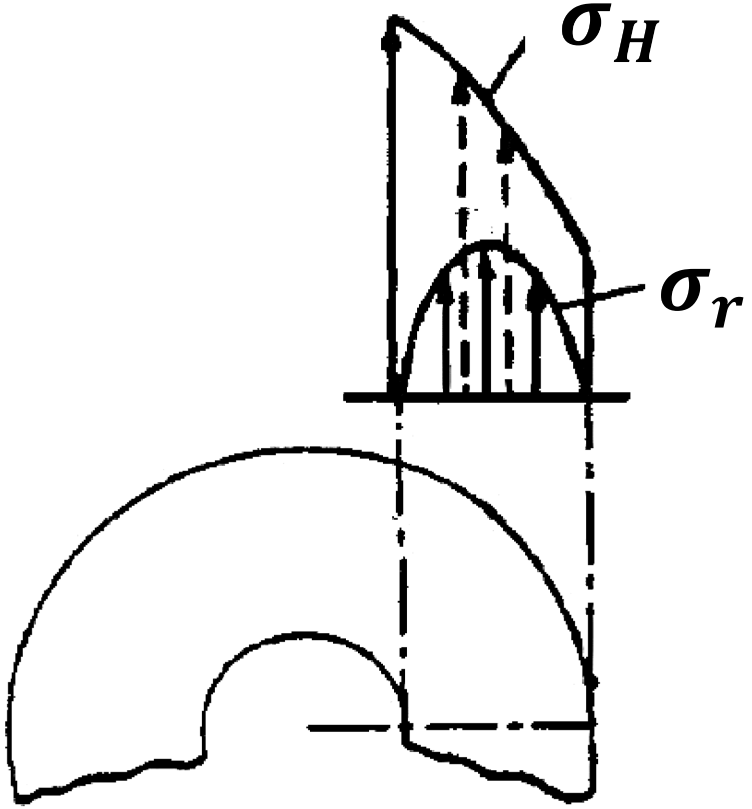


Figure 3.7: Distribution of radial and hoop stresses in a hollow disc

ROTATING DISC WITH A CENTRAL HOLE

Example 3.1

A steel ring of outer diameter 300 mm and internal diameter 200 mm is shrunk onto a solid steel shaft. The interference is arranged such that the radial pressure between the mating surfaces will not fall below 30 MN/m^2 whilst the assembly rotates in service. If the maximum circumferential stress on the inside surface of the ring is limited to 240 MN/m^2, determine the maximum speed at which the assembly can be rotated. It may be assumed that no relative slip occurs between the shaft and the ring. For steel, $\boldsymbol{\rho}$ = 7470 kg/m3, $\boldsymbol{\nu}$ = 0.3, **E** = 208 GN/m^2.

Solution 3.1

$$\sigma_r = A - \frac{B}{r^2} - \frac{(3+\nu)}{8}\rho\omega^2 r^2 \qquad (1)$$

when $r = 0.15, \qquad \sigma_r = 0$

$$0 = A - \frac{B}{0.15^2} - \frac{3.3}{8}\rho\omega^2(0.15)^2 \qquad (2)$$

ROTATING DISC WITH A CENTRAL HOLE

Solution 3.1

when $r = 0.1$, $\sigma_r = -30\ \text{MN/m}^2$

$$-30 \times 10^6 = A - \frac{B}{0.1^2} - \frac{3.3}{8}\rho\omega^2(0.1)^2 \qquad (3)$$

$$0 = A - \frac{B}{0.15^2} - \frac{3.3}{8}\rho\omega^2(0.15)^2 \qquad (2)$$

(2)–(3),

$$30 \times 10^6 = B(100 - 44.4) - \frac{3.3}{8}\rho\omega^2(0.0225 - 0.01)$$

$$B = \frac{30 \times 10^6}{55.6} + 3.3 \times \frac{0.0125 \times 7470}{8 \times 55.6}\omega^2$$

$$B = 0.54 \times 10^6 + 0.693\omega^2$$

and from (3),

$$A = 100(0.54 \times 10^6 + 0.693\omega^2) + \frac{3.3 \times 7470 \times 0.01\omega^2}{8} - 30 \times 10^6$$

$$= 54 \times 10^6 + 69.3\omega^2 + 30.8\omega^2 - 30 \times 10^6$$

ROTATING DISC WITH A CENTRAL HOLE

Solution 3.1

$$= 54 \times 10^6 + 69.3\omega^2 + 30.8\omega^2 - 30 \times 10^6$$

$$= 24 \times 10^6 + 100.1\omega^2$$

The maximum hoop stress at the inside radius **is** limited to 240 MN/m^2

$$\sigma_H = A + \frac{B}{r^2} - \frac{(1+3\nu)}{8}\rho\omega^2 r^2$$

$$240 \times 10^6 = (24 \times 10^6 + 100.1\omega^2) + \frac{(0.54 \times 10^6 + 0.693\omega^2)}{0.1^2} - \frac{1.9}{8} \times 7470 \times 0.01\omega^2$$

$$240 \times 10^6 = 78 \times 10^6 + 169.3\omega^2 - 17.7\omega^2$$

$$\therefore \quad 151.7\omega^2 = 162 \times 10^6$$

$$\omega^2 = \frac{162 \times 10^6}{151.7} = 1.067 \times 10^6$$

$$\omega = 1033 \text{ rad/s} \quad = \mathbf{9860\ rev/min}$$

ROTATING DISC WITH A CENTRAL HOLE

Example 3.2

A steel rotor disc which is part of a turbine assembly has a uniform thickness of 40 mm. The disc has an outer diameter of 600 mm and a central hole of 100 mm diameter. If there are 200 blades each of mass 0.153 kg pitched evenly around the periphery of the disc at an effective radius of 320 mm, determine the rotational speed at which yielding of the disc first occurs according to the maximum shear stress criterion of elastic failure.

For steel, **E** = 200 GN/m², $\boldsymbol{\nu}$ = 0.3, $\boldsymbol{\rho}$ = 7470 kg/m³ and the yield stress $\boldsymbol{\sigma_y}$ in simple tension = 500 MN/m².

Solution 3.2

$$\text{Total mass of blades} = 200 \times 0.153 = 30.6 \text{ kg}$$

$$\text{Effective radius} = 320 \text{ mm}$$

$$\text{centrifugal force on the blades} = m\omega^2 r = 30.6 \times \omega^2 \times 0.32$$

$$\text{the area of the disc rim} = \pi d t = \pi \times 0.6 \times 0.004 = 0.024\pi \text{m}^2$$

ROTATING DISC WITH A CENTRAL HOLE

Solution 3.2

- The centrifugal force acting on the area ($\pi dt = 0.024\pi\ m^2$) produces an effective radial stress acting on the outside surface of the disc
- We assume the blades produce a uniform loading around the periphery of the disc.
- Thus, the radial stress on the outside surface is:

$$= \frac{30.6 \times \omega^2 \times 0.32}{0.024\pi} = 130\omega^2 \text{ N/m}^2 \quad \text{(tensile)}$$

$$\sigma_r = A - \frac{B}{r^2} - \frac{(3+\nu)}{8}\rho\omega^2 r^2 \qquad (1)$$

$$\sigma_H = A + \frac{B}{r^2} - \frac{(1+3\nu)}{8}\rho\omega^2 r^2 \qquad (2)$$

When $r = 0.05$, $\sigma_r = 0$

$$0 = A - 400B - \frac{3.3}{8}\rho\omega^2 (0.05)^2 \qquad (3)$$

ROTATING DISC WITH A CENTRAL HOLE

Solution 3.2

$$0 = A - 400B - \frac{3.3}{8}\rho\omega^2(0.05)^2 \qquad (3)$$

When $r = 0.3$, $\qquad \sigma_r = +130\omega^2$

$$130\omega^2 = A - 11.1B - \frac{3.3}{8}\rho\omega^2(0.3)^2 \qquad (4)$$

(4)–(3), $$130\omega^2 = 388.9B - \frac{3.3}{8}\rho\omega^2(9 - 0.25)10^{-2}$$

$$130\omega^2 = 388.9B - 270\omega^2$$

$$B = \frac{(130 + 270)}{388.9}\omega^2 = 1.03\omega^2$$

- Substituting ***B*** in equation **(3)**,

$$A = 412\omega^2 + \frac{3.3}{8} \times 7470(0.05)^2\omega^2$$

ROTATING DISC WITH A CENTRAL HOLE

Solution 3.2

$$A = 412\omega^2 + \frac{3.3}{8} \times 7470(0.05)^2\omega^2$$

$$= 419.7\omega^2 = 420\omega^2$$

- Substituting in ***A*** and ***B*** into equations (2) and (1), the stress conditions at the inside surface are

$$\sigma_H = 420\omega^2 + 412\omega^2 - 4.43\omega^2 = 827\omega^2$$

$$\sigma_r = 0$$

- and at the outside surface

$$\sigma_H = 420\omega^2 + 11.42\omega^2 - 159\omega^2 = 272\omega^2$$

$$\sigma_r = 130\omega^2$$

- The most severe stress conditions occur at the inside radius where the maximum shear stress is greatest:

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{827\omega^2 - 0}{2}$$

ROTATING DISC WITH A CENTRAL HOLE

Solution 3.2

- The maximum shear stress theory of elastic failure states that failure is assumed to occur when this stress equals the value of **shear stress** at the yield point in simple tension,

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_y - 0}{2} = \frac{\sigma_y}{2}$$

- Thus, for failure to occur according to this theory,

$$\frac{\sigma_y}{2} = \frac{827\omega^2}{2}$$

$$827\omega^2 = \sigma_y = 500 \times 10^6$$

$$\omega^2 = \frac{500}{827} \times 10^6 = 0.604 \times 10^6$$

$$\omega = 780 \text{ rad/s} = \mathbf{7450\ rev/min}$$

ROTATING CYLINDERS (SOLID SHAFTS) – THICK DISC

- In the case of rotating thick cylinders the longitudinal stress $\boldsymbol{\sigma_L}$ must be taken into account
- The longitudinal strain (ε_L) is assumed to be constant.
- Thus, writing the equations for the strain in three mutually perpendicular directions, we have the following:

$$\varepsilon_L = \frac{1}{E}(\sigma_L - \nu\sigma_H - \nu\sigma_r) \qquad 3.16$$

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_H - \nu\sigma_L) = \frac{ds}{dr} \qquad 3.17$$

$$\varepsilon_H = \frac{1}{E}(\sigma_H - \nu\sigma_r - \nu\sigma_L) = \frac{s}{r} \qquad 3.18$$

- From eqn. (3.18)

$$Es = r[\sigma_H - \nu(\sigma_r + \sigma_L)]$$

ROTATING THICK CYLINDERS (SOLID SHAFTS)

- From eqn. (3.18)

$$Es = r[\sigma_H - \nu(\sigma_r + \sigma_L)]$$

- Differentiating this equation, we get

$$E\frac{ds}{dr} = r\left[\frac{d\sigma_H}{dr} - \nu\frac{d\sigma_r}{dr} - \nu\frac{d\sigma_L}{dr}\right] + 1\,[\sigma_H - \nu\sigma_r - \nu\sigma_L]$$

- Substituting for $E(ds/dr)$ into eqn. **(3.17)**,

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_H - \nu\sigma_L) = \frac{ds}{dr} \qquad \textbf{3.17}$$

$$\sigma_r - \nu\sigma_H - \nu\sigma_L = r\left[\frac{d\sigma_H}{dr} - \nu\frac{d\sigma_r}{dr} - \nu\frac{d\sigma_L}{dr}\right] + \sigma_H - \nu\sigma_r - \nu\sigma_L$$

$$0 = (\sigma_H - \sigma_r)(1 + \nu) + r\frac{d\sigma_H}{dr} - \nu r\frac{d\sigma_r}{dr} - \nu r\frac{d\sigma_L}{dr}$$

ROTATING THICK CYLINDERS (SOLID SHAFTS)

- Recall that ε_L is constant, differentiating eqn. (3.16),

$$\varepsilon_L = \frac{1}{E}(\sigma_L - \nu\sigma_H - \nu\sigma_r) \qquad 3.16$$

- We get

$$0 = \frac{1}{E}\left(\frac{d\sigma_L}{dr} - \nu\frac{d\sigma_H}{dr} - \nu\frac{d\sigma_r}{dr}\right)$$

$$\frac{d\sigma_L}{dr} = \nu\left(\frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr}\right)$$

- Thus the equation:

$$0 = (\sigma_H - \sigma_r)(1+\nu) + r\frac{d\sigma_H}{dr} - \nu r\frac{d\sigma_r}{dr} - \nu r\frac{d\sigma_L}{dr}$$

- Becomes:

$$0 = (\sigma_H - \sigma_r)(1+\nu) + r(1-\nu^2)\frac{d\sigma_H}{dr} - \nu r(1+\nu)\frac{d\sigma_r}{dr}$$

ROTATING THICK CYLINDERS (SOLID SHAFTS)

- Dividing through by $(1 + \nu)$ we get

$$0 = (\sigma_H - \sigma_r) + r(1 - \nu)\frac{d\sigma_H}{dr} - \nu r\frac{d\sigma_r}{dr}$$

- Remember the general equilibrium equation:

$$\sigma_H - \sigma_r - r\frac{d\sigma_r}{dr} = \rho\omega^2 r^2 \qquad 3.2$$

- Substituting for: $(\sigma_H - \sigma_r)$,

$$0 = \rho\omega^2 r^2 + r\frac{d\sigma_r}{dr} + r(1 - \nu)\frac{d\sigma_H}{dr} - \nu r\frac{d\sigma_r}{dr}$$

$$0 = \rho\omega^2 r^2 + r(1 - \nu)\left[\frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr}\right]$$

$$\therefore \qquad \frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr} = -\frac{\rho\omega^2 r}{(1 - \nu)}$$

ROTATING THICK CYLINDERS (SOLID SHAFTS)

- Integrating

$$\frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr} = -\frac{\rho\omega^2 r}{(1-\nu)}$$

- We get:

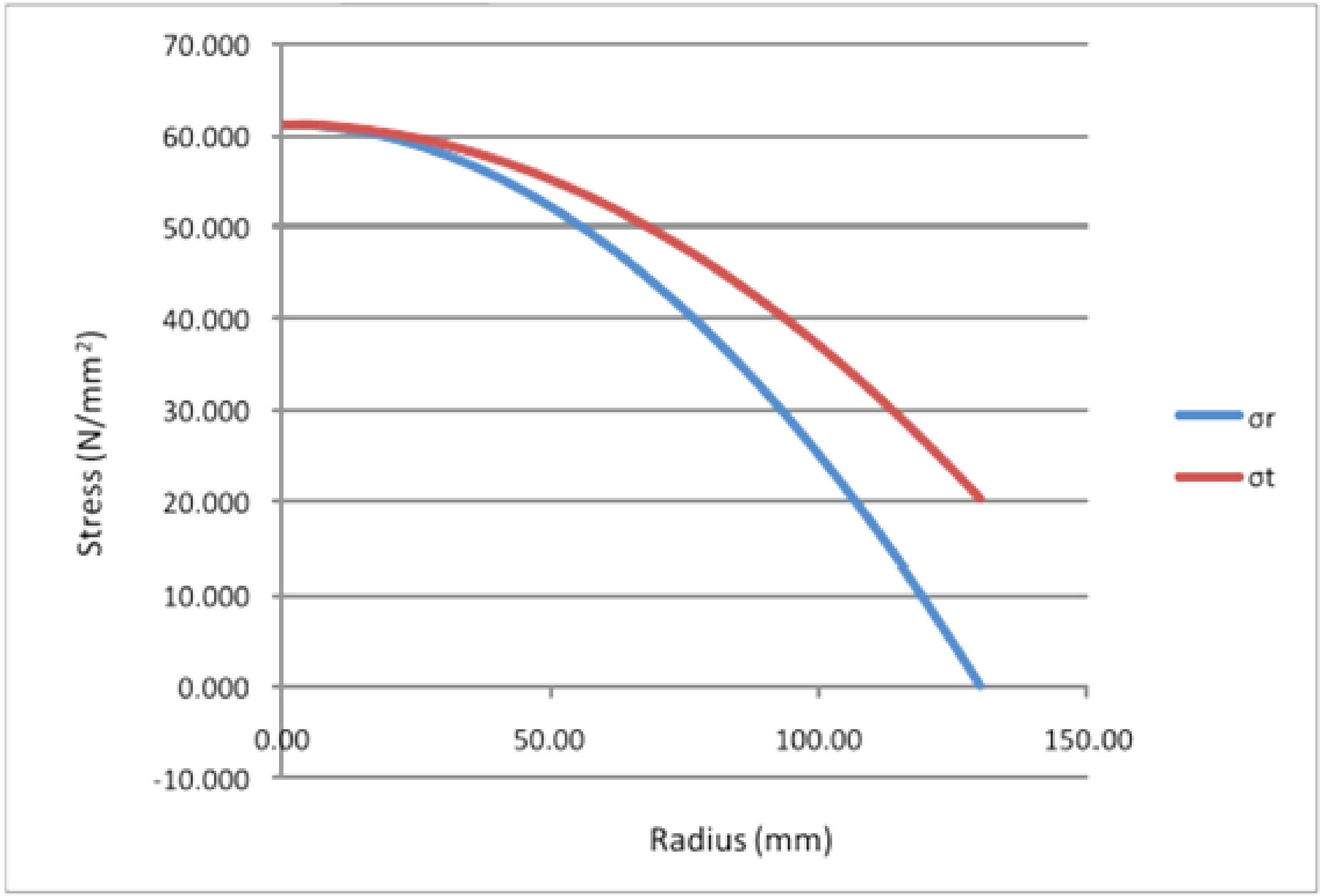
$$\sigma_H + \sigma_r = -\frac{\rho\omega^2 r^2}{2(1-\nu)} + 2A$$

- **Thus hoop and radial stresses in rotating thick cylinders can be obtained from the equations for rotating discs provided that Poisson's ratio ν is replaced by $\frac{\nu}{1-\nu}$**
- For example, the stress at the centre of a rotating solid shaft will be given by modifying the equation (3.10) for a solid disc as follows

$$\sigma_{max} = (3+\nu)\frac{\rho\omega^2 R^2}{8} \qquad 3.10$$

$$\sigma_{max} = \left(3 + \frac{\nu}{1-\nu}\right)\frac{\rho\omega^2 R^2}{8}$$

ROTATING THICK CYLINDERS (SOLID SHAFTS)



250mm dia solid cylinder at 10000rpm

Figure 3.9: Variation of hoop and radial stress in a rotating solid cylinder

ROTATING DISC OF UNIFORM STRENGTH

- In applications such as turbine blades rotating at high speeds it is often desirable to design for constant stress conditions
- We design such rotating parts for uniform stress under the action of the high centrifugal forces to which they are subjected.
- The condition of equal stress can only be achieved, as in the case of uniform strength cantilevers, by varying the thickness of the disc.

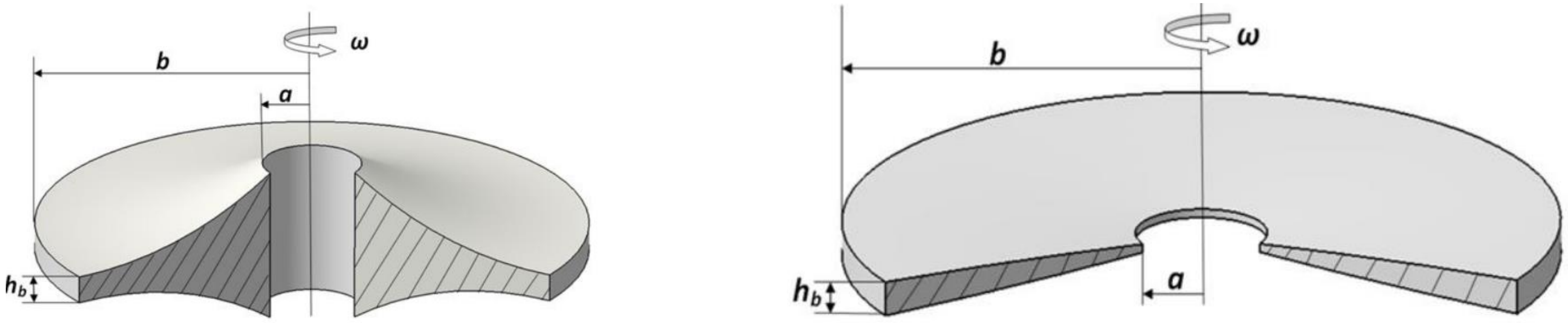


Figure 3.8: Variable thickness – Uniform stress rotating Discs

- We now Consider an element of a disc subjected to equal uniform stress.
- In such a case, the hoop and radial stresses are equal i.e., $\sigma_H = \sigma_r = \sigma$

ROTATING DISC OF UNIFORM STRENGTH

- Consider an element of a disc subjected to equal hoop and radial stresses: $\sigma_H = \sigma_r = \sigma$

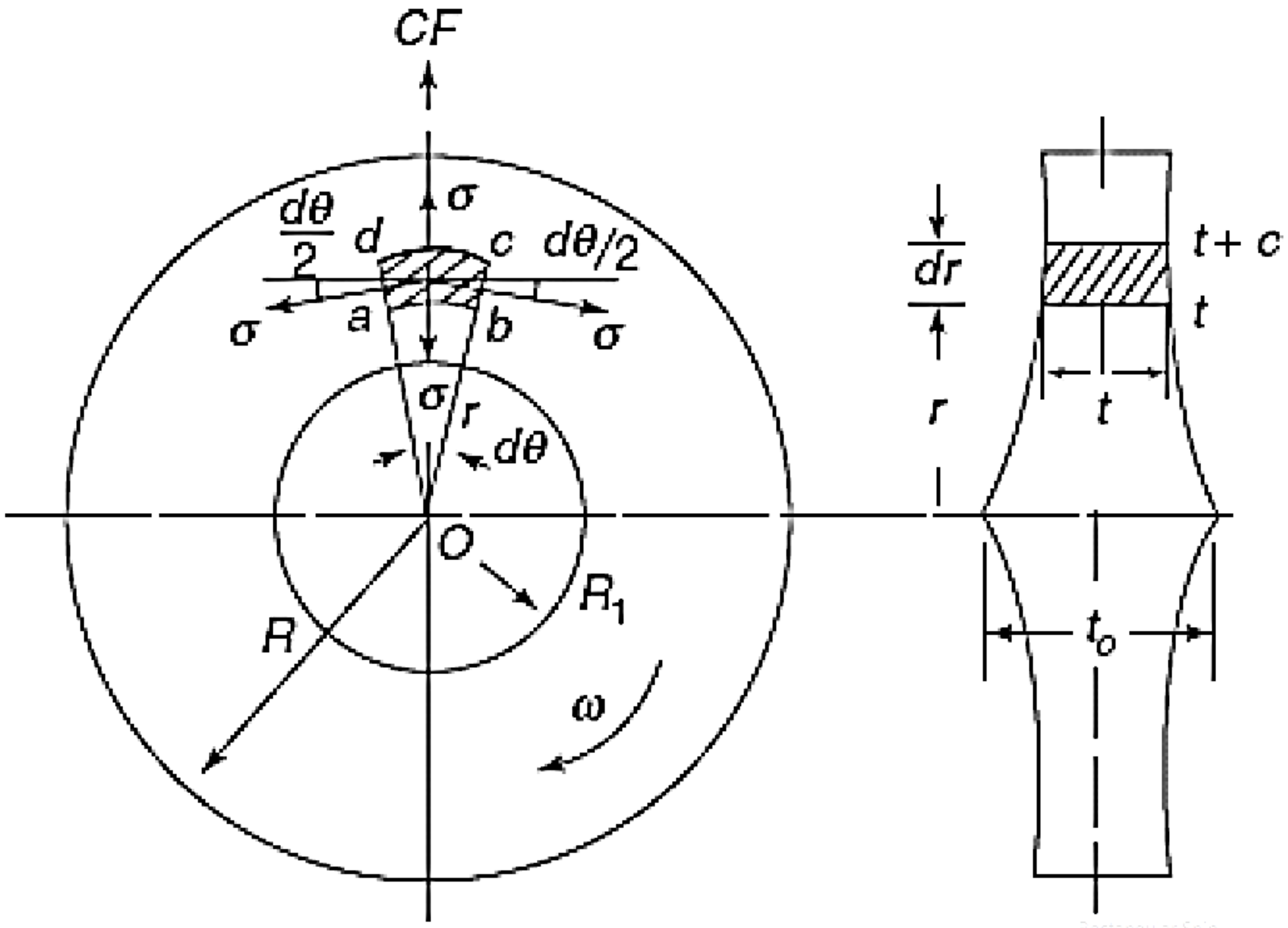
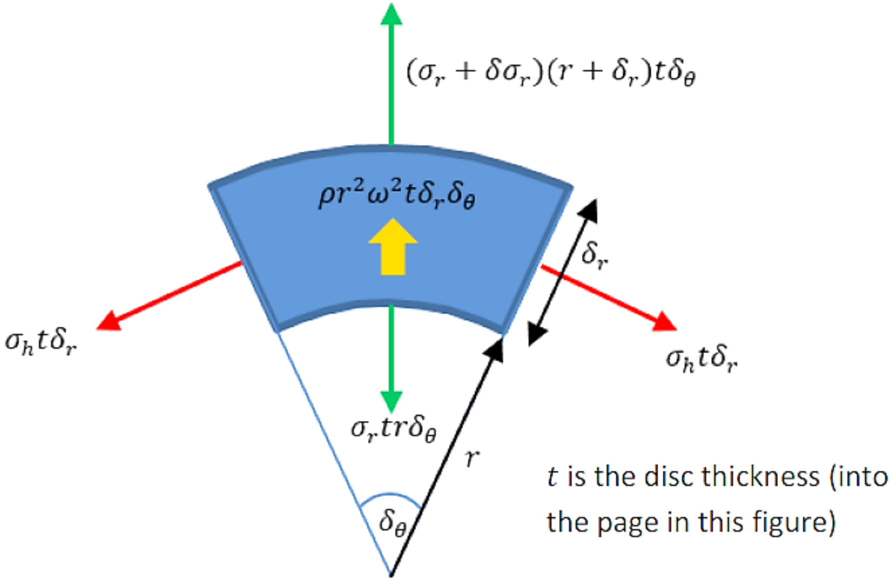


Figure 3.9 (a): Element of rotating disc of variable thickness

ROTATING DISC OF UNIFORM STRENGTH

- Consider an element of a disc subjected to equal hoop and radial stresses: $\boldsymbol{\sigma_H = \sigma_r = \sigma}$



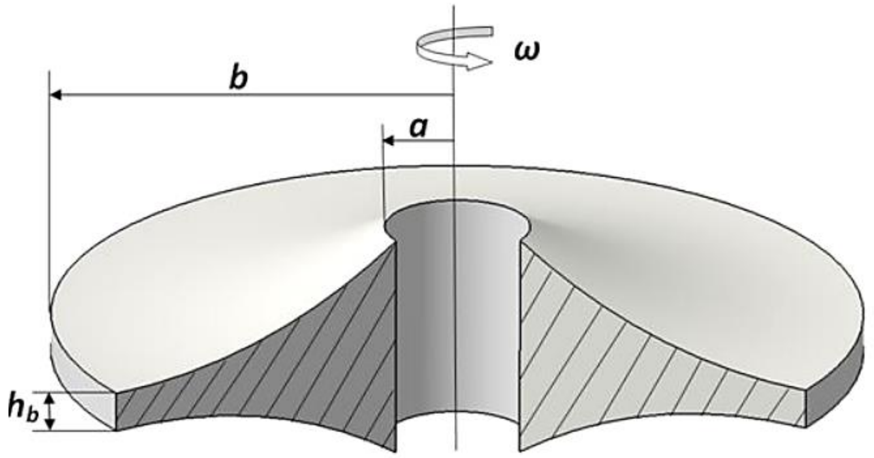


Figure 3.9 (b): Element of rotating disc of variable thickness

- Let t be the thickness at a radius r, and $(t + \delta t)$ the thickness at a radius $(r + \delta r)$
- The mass of the element shown in figure 3.9 will be: $m = \rho r\delta\theta\delta r \cdot t$
- And the centrifugal force acting on the rotating element will be: $\rho r^2\omega^2 t\delta\theta\delta r$

ROTATING DISC OF UNIFORM STRENGTH

- The equilibrium equation in vertical direction is then

$$2\sigma\delta r\cdot t\cdot\sin\left(\frac{\delta\theta}{2}\right)+\sigma\delta\theta\cdot t=\sigma(r+\delta r)\delta\theta(t+\delta t)+\rho r^2\omega^2\delta\theta\cdot\sigma r\cdot t$$

- Considering that :

$\sin\left(\frac{\delta\theta}{2}\right)\approx\frac{\delta\theta}{2}$, since $\frac{\delta\theta}{2}$ is small

$$2\sigma\delta r\cdot t\cdot\left(\frac{\delta\theta}{2}\right)+\sigma\delta\theta\cdot t=\sigma(r+\delta r)\delta\theta(t+\delta t)+\rho r^2\omega^2\delta\theta\cdot\sigma r\cdot t$$

$$\sigma\delta r\cdot t\cdot\delta\theta+\sigma\delta\theta\cdot t=\sigma(r+\delta r)\delta\theta(t+\delta t)+\rho r^2\omega^2\delta\theta\cdot\sigma r\cdot t$$

$$\sigma\delta r\cdot t\cdot\delta\theta+\sigma\delta\theta\cdot t=\sigma(rt+t\delta r+r\delta t+\delta r\delta t)\delta\theta+\rho r^2\omega^2\delta\theta\cdot\sigma r\cdot t$$

$$\sigma\delta rt+\sigma t=\sigma(rt+t\delta r+r\delta t+\delta r\delta t)+\rho r^2\omega^2\sigma r\cdot t$$

- Neglecting higher order terms:

$$\sigma\delta rt+\sigma t=\sigma(rt+t\delta r+r\delta t)+\rho r^2\omega^2\sigma r\cdot t$$

ROTATING DISC OF UNIFORM STRENGTH

- Considering the limit , we see that

$$\sigma t \cdot dr = \sigma r \cdot dt + \sigma t \cdot dr + \rho r^2 \omega^2 t \cdot dr$$

- Simplifying:

$$\frac{dt}{dr} = -\frac{\rho r \omega^2 t}{\sigma}$$

- Re-arranging:

$$\frac{dt}{t} = -\frac{\rho r \omega^2}{\sigma} dr$$

- Integrating:

$$\ln t = -\frac{\rho r \omega^2 t}{2\sigma} + C$$

or

$$t = e^{-\frac{\rho r \omega^2 + C}{2\sigma}} = e^C \cdot e^{-\frac{\rho r \omega^2}{2\sigma}} \qquad t = Ae^{-\frac{\rho r \omega^2}{2\sigma}}$$

- At $r = 0$,

$$t = Ae^{-\frac{\rho r \omega^2}{2\sigma}} = Ae^0 \qquad A_{(r=0)} = t_{(r=0)} = t_0$$

- Therefore in general:

$$t = t_0 e^{-\frac{\rho r \omega^2}{2\sigma}}$$

ROTATING DISC OF UNIFORM STRENGTH

Example 3.3

The cross-section of a turbine rotor disc is designed for uniform strength under rotational conditions. The disc is keyed to a 60 mm diameter shaft at which point its thickness is a maximum. It then tapers to a minimum thickness of 10 mm at the outer radius of 250 mm where the blades are attached. If the design stress of the shaft is 250 MN/m^2 at the design speed of 12000 rev/min, compute the required maximum thickness. For steel $\boldsymbol{\rho}$ = 7470 kg/m^3.

Solutions 3.3

The thickness of a uniform strength disc is given by

$$t = t_0 e^{(-\rho\omega^2 r^2)/(2\sigma)} \qquad (1)$$

where t_0 is the thickness at $r = 0$.

Now at $r = 0.25$,

$$\frac{\rho\omega^2 r^2}{2\sigma} = \frac{7470}{2 \times 250 \times 10^6}\left(12\,000 \times \frac{2\pi}{60}\right)^2 \times 0.25^2 = 1.47$$

ROTATING DISC OF UNIFORM STRENGTH

Solutions 3.3

and at $r = 0.03$,

$$\frac{\rho\omega^2 r^2}{2\sigma} = \frac{7470}{2 \times 250 \times 10^6}\left(12\,000 \times \frac{2\pi}{60}\right)^2 \times 0.03^2$$

$$= 1.47 \times \frac{9 \times 10^{-4}}{625 \times 10^{-4}} = 0.0212$$

But at $r = 0.25$, $t = 10$ mm

Substituting in equation (1), $0.01 = t_0 e^{-1.47} = 0.2299\ t_0$

$$t_0 = \frac{0.01}{0.2299} = 0.0435\text{m} = 43.5 \text{ mm}$$

Therefore at $r = 0.03$

$$t = 0.0435e^{-0.0212} = 0.0435 \times 0.98$$

$$= 0.0426 \text{ m} = \mathbf{42.6\ mm}$$

Example 3.4

A turbine rotor is 600 mm in diameter at the blade ring, and is keyed to a 50 mm diameter shaft. Given that the minimum disc thickness is 9.5 mm, compute the thickness at the shaft if the disc is designed for a uniform stress of 200 MPa when the assembly is rotating at 10,000 rpm. Density = 7,700 kg/m^3.

Solution 3.4

In general for a uniform strength rotating disc; $t = t_0 e^{-\rho r^2 \omega^2 / 2\sigma}$

At $r = 0.3\ m$ $\qquad t = 9.5 = Ae^{-\rho (0.3)^2 \omega^2 / 2\sigma} = Ae^{-\rho (0.09) \omega^2 / 2\sigma}$

At $r = 0.025\ m$ $\qquad t = 9.5 = Ae^{-\rho (0.025)^2 \omega^2 / 2\sigma} = Ae^{-\rho (0.000625)\, \omega^2 / 2\sigma}$

where: $\qquad t = 9.5 e^{-\rho (0.0894) \omega^2 / 2\sigma}$

$$\frac{\rho \omega^2 (0.0894)}{2\sigma} = 7700 \left(\frac{10,000\pi}{30} \right)^2 \frac{0.0894}{2 \times 200 \times 10^6} = 1.89$$

then: $\qquad t = 9.5 e^{1.89} \qquad t_{\text{at shaft}} = 63\ \text{mm}$

MEC 3352 - QUIZ 002

a) A thin uniform disc of inner radius 50 mm and outer diameter 400 mm is rotating at 6000 rpm about its axis. Giventhe density, ρ = 7800kg/m3, and the Poisson's ratio, ν = 0.3. Compute the;

   i) maximum hoop stress
   ii) minimum circumferential stress
   iii) maximum radial stress

b) Draw the distribution of hoop and radial stresses along the radius of the disc.

Questions

Question 3.1

Determine from first principles the hoop stress at the inside and outside radius of a thin steel disc of *300* mm diameter having a central hole of 100 mm diameter, if the disc is made to rotate at *5000* rpm. Determine the position and magnitude of the maximum radial stress.

$\rho = 7470\ \text{kg/m}^3;\quad \nu = 0.3;\quad E = 207\ \text{GN/m}^2$ [38.9. 12.3 MN/m^2; 87 mm rad; 8.4 MN/m^2.]

Question 3.2

A solid steel disc 300 mm diameter and of small constant thickness has a steel ring of outer diameter 450 mm and the same thickness shrunk onto it. If the interference pressure is reduced to zero at a rotational speed of *3000* rpm, calculate:

(a) the radial pressure at the interface when stationary:

(b) the difference in diameters of the mating surfaces of the disc and ring before assembly

$\rho = 7470\ \text{kg/m}^3;\quad \nu = 0.3;\quad E = 207\ \text{GN/m}^2$ [18.55 MN/m2; 0.045 mm]

Questions

Question 3.3

The "bursting" speed of a cast-iron flywheel rim ***3m*** mean diameter, is 850 rpm. Neglecting the effects of the spokes and the boss. and assuming that the flywheel rim can be considered as **a** thin rotating hoop, determine the ultimate tensile strength of the cast iron. Cast iron has a density of 7.3 **Mg/m^3.**A flywheel rim is to be made **of** the-same material and is required to rotate at 400 rpm. Determine the maximum permissible mean diameter using **a** factor of safety of 8.

$$\rho = 7470\ \mathrm{kg/m^3};\quad \nu = 0.3;\quad E = 207\ \mathrm{GN/m^2}$$

Question 3.4

A forged steel drum 0.524 m outside diameter and 19 mm wall thickness, has to be mounted in a machine and spun about its longitudinal axis. The centrifugal (hoop) stress induced in the cylindrical shell is not to exceed 83 MN/m^2. Determine the maximum speed (in rpm) **at** which the drum can be rotated. For steel, the density = 7.8 Mg/m3.

$$\rho = 7470\ \mathrm{kg/m^3};\quad \nu = 0.3;\quad E = 207\ \mathrm{GN/m^2}$$

[3630.]

Question 3.4

A steel disc of a turbine is to be designed so that the radial and circumferential stresses are to be the same throughout the thickness and radius of disc and is equal to 80 MPa, when running at 3500 rpm. If the axial thickness at the centre is 20 mm, what is the thickness at the radius of 500 mm?

Question 3.5

A solid long cylinder of diameter 600 mm is rotating at 3000 rpm. Calculate

(i) maximum and minimum hoop stresses and

(ii) maximum radial stress.

Given ρ = 0.07644 N/cm^3, g = 9.8 m/s^2, ν = 0.3

Question 3.6

Determine the intensities of principal stresses in a flat steel disc of uniform thickness, having a diameter of 1m and rotating at 2400 r.p.m. What will be the stress if the disc has a central hole of 0.2m diameter? Take Poisson's ratio to be 1/3, and the density of this steel as 7850kg/m^3.

Question 3.7

A disc of uniform thickness has inner and outer radii of 100mm and 400mm respectively and is rotating at 2400 r.p.m about its axis. The density of the material is 7800kg/m^3 and the Poisson’s ratio is 0.3. Using a suitable scale and intervals, draw on the same set of axes, the graph of stress against radius for both circumferential and radial stress.

Assignment 2 [Questions 3.6 & 3.7] : Due 25th September 2024

Question 3.8

A steel ring of outer diameter 300mm and internal diameter 200mm is shrunk onto a solid steel shaft. The interface is such that the radial pressure between the mating surfaces remains above 30MN/m^2 at all times whilst the assembly rotates in practice. The circumferential stress on the inside surface of the ring must not exceed 240MN/m2. Determine the maximum speed at which the assembly can rotate. Take ρ =7500kg/m3, ν =0.3 and E = 210GPa.

Grazie Signore