The University of Zambia
School of Engineering
Department of Mechanical Engineering

MEC 3352 – STRENGTH OF MATERIALS II

Thick Circular Cylinders

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Thin shell theory

- Satisfactory when the ratio of the shell thickness to shell radius is less than 1/30.
- Assumes that the circumferential (hoop stress) and longitudinal stress are constant over the thickness; radial stress is negligible.
- When the thickness: radius ratio is greater than this 1/30, errors start to occur and thick shell theory should be used.

Thick shell theory

- When the thickness to shell radius is greater than 1/30, we have a thick cylinder
- The hoop stress in thick cylinders vary over the thickness
- Thick shells appear in the form of gun barrels, nuclear reactor pressure vessels, and deep diving submersibles

Stress and Strain in Thick Circular Cylinders

We assume all the stresses and strains are tensile and positive. At any radius, ***r*** **:**

$\sigma_\theta = hoop\ stress$

$\sigma_r = radial\ stress$

$\sigma_z = Longitudinal\ stress$

$\varepsilon_\theta = hoop\ strain$

$\varepsilon_r = radial\ strain$

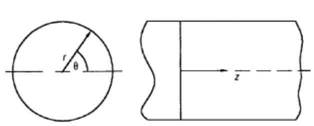


Fig.3.0: Thick Cylinder

$\varepsilon_z = longitudinal\ Strain\ assumed\ to\ be\ constant$

$w = radial\ deflection$

Stress and Strain in Thick Circular Cylinders

Longitudinal Stress σ_z

Consider a cross-section of a thick cylinder with closed ends subjected to an internal pressure P_1 and an external pressure P_2.

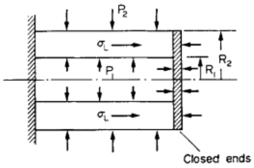


Fig. 3.1: Cylinder longitudinal section

For horizontal equilibrium:

$$P_1 * \pi R_1^2 - P_2 * \pi R_2^2 = \sigma_z * \pi(R_2^2 - R_1^2)$$

$$\sigma_z = \frac{P_1 R_1^2 - P_2 R_2^2}{R_2^2 - R_1^2} \qquad 3.0$$

We see that the longitudinal stress set up in the cylinder walls is constant for the given internal and external pressures

Derivation of the hoop and radial stress equations

From Figure 3.2, it can be seen that at any radius *r*,

$$\varepsilon_\theta = \frac{2\pi(r+w) - 2\pi r}{2\pi r}$$

$$\varepsilon_\theta = \frac{w}{r} \qquad 3.1$$

Similarly,

$$\varepsilon_r = \frac{\delta w}{\delta r} = \frac{dw}{dr} \qquad 3.2$$

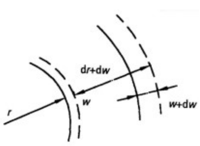


Fig.3.2: Thick Cylinder Deformation at any radius r

From the standard stress-strain relationships

$$E\varepsilon_z = \sigma_z - \nu(\sigma_\theta + \sigma_r) = \text{a constant}$$

$$E\varepsilon_\theta = E\frac{w}{r} = \sigma_\theta - \nu(\sigma_z + \sigma_r) \qquad 3.3$$

$$E\varepsilon_r = E\frac{dw}{dr} = \sigma_r - \nu(\sigma_\theta + \sigma_z) \qquad 3.4$$

Derivation of the hoop and radial stress equations

Multiplying equation (3.3) by r,

$$Ew = \sigma_\theta * r - \nu\sigma_z * r - \sigma_r * r \qquad 3.5$$

Differentiating equation (3.5) with respect to r, we get

$$E\frac{dw}{dr} = \sigma_\theta - \nu\sigma_z - \nu\sigma_r + r\left[\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_z}{dr} - \nu\frac{d\sigma_r}{dr}\right] \qquad 3.6$$

Subtracting equation (3.4) from equation (3.6),

$$(\sigma_\theta - \sigma_r)(1+\nu) + r\frac{d\sigma_\theta}{dr} - \nu r\frac{d\sigma_z}{dr} - \nu r\frac{d\sigma_r}{dr} = 0 \qquad 3.7$$

As ε_z, is constant

$$\sigma_z - \nu\sigma_\theta - \nu\sigma_r = \text{constant} \qquad 3.8$$

Derivation of the hoop and radial stress equations

Differentiating equation (3.8)with respect to r,

$$\frac{d\sigma_z}{dr} - \nu\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr} = 0$$

$$\frac{d\sigma_z}{dr} = \nu\left(\frac{d\sigma_\theta}{dr} + \frac{d\sigma_r}{dr}\right) \qquad 3.9$$

Substituting equation (3.9) into equation (3.7),

$$(\sigma_\theta - \sigma_r)(1+\nu) + r(1-\nu^2)\frac{d\sigma_\theta}{dr} - \nu r(1+\nu)\frac{d\sigma_r}{dr} = 0 \qquad 3.10$$

Dividing equation (3.10) by $(1 + \nu)$, we get

$$\sigma_\theta - \sigma_r + r(1+\nu)\frac{d\sigma_\theta}{dr} - \nu r\frac{d\sigma_r}{dr} = 0 \qquad 3.11$$

Derivation of the hoop and radial stress equations

Consider the radial equilibrium of the shell element, shown in Figure 3.3,

$$(\sigma_r + d\sigma_r)(r + dr) * 1 - \sigma_r * rd\theta * 1 = 2\sigma_\theta * dr * 1 * \sin\left(\frac{d\theta}{2}\right) \qquad 3.12$$

For small angles $sin\left(\frac{d\theta}{2}\right) \cong \frac{\theta}{2}\ rad$

Neglecting higher order terms in the above, we get

$$\sigma_\theta - \sigma_r - r\frac{d\sigma_r}{dr} = 0 \qquad 3.13$$

Subtracting equation (3.1 1) from equation (3.12)

$$\frac{d\sigma_\theta}{dr} + \frac{d\sigma_r}{dr} = 0 \qquad \frac{d}{dr}(\sigma_\theta + \sigma_r) = 0 \qquad 3.14$$

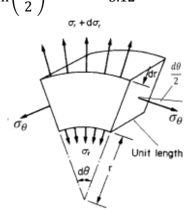


Fig. 3.3 : Shell element

Derivation of the hoop and radial stress equations

From equation (3.14), we get

$$\sigma_\theta + \sigma_r = constant = 2A \quad \Rightarrow \sigma_\theta = 2A - \sigma_r \qquad 3.15$$

Note that we let the constant of integration be 2A (say)

Substituting (3.15) into equation (3.13**)**,

$$2A - \sigma_r - \sigma_r - r\frac{d\sigma_r}{dr} = 0$$

Multiplying through by ***r*** and rearranging,

$$2r\sigma_r + r^2\frac{d\sigma_r}{dr} - 2Ar = 0$$

$$\frac{d}{dr}(\sigma_r r^2 - Ar^2) = 0$$

$$\sigma_r r^2 - Ar^2 = constant = -B\ (say)$$

Derivation of the hoop and radial stress equations

Integrating and simplifying, we get

$$\sigma_r = A - \frac{B}{r^2} \qquad 3.16$$

Substituting equation (3.16) into (3.15) and simplifying, we get

$$\sigma_\theta = A + \frac{B}{r^2} \qquad 3.17$$

Equations (3.16) and (3.17) are called **Lamé's** Equations

The Lame equations when plotted on stress and $1/r^2$ axes produce straight lines, as shown in Figure 3.4 a

The two lines may be modified to a single a straight line, where (σ_r), lies to the left and (σ_θ), to the right, as shown by Figure 3.4 b.

Graphical Representation of **Lamé's** Equations

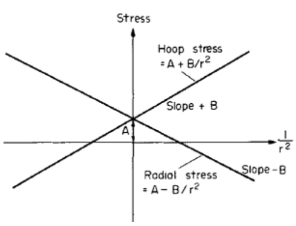


Fig. 3.4 (a): Graphical representation of **Lame** equations - **Lame** line.

- Both lines have exactly the same intercept A and the same magnitude of slope ***B,*** the only difference being the sign of their slopes.
- The two are therefore combined by plotting hoop stress values to the left of the σ axis (again against $^1/_{r^2}$) instead of to the right to give the single line shown in Fig. 3.4(b)**.**
- In most questions one value of σ_r, and one value of σ_θ, or alternatively two values of σ_r, are given. In both cases the single line can then be drawn.

Graphical Representation of **Lamé's** Equations

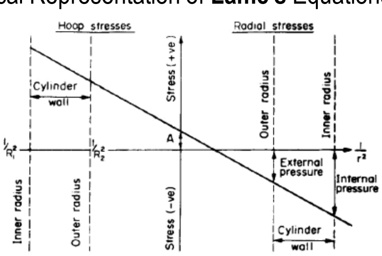


Fig. 3.4 (b) Lame line solution for cylinder with internal and external pressures

Graphical Representation of **Lamé's** Equations

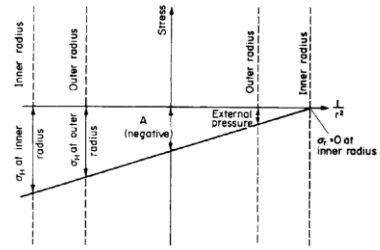


Fig. 3.4 (c) Lame line solution for cylinder subjected to external pressure only

Graphical Representation of **Lamé's** Equations

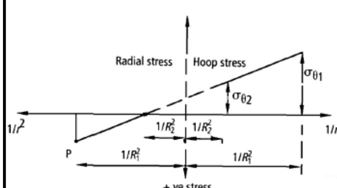


Fig. 3.4 (d): Lame line for the case of internal pressure

For the case shown in Figure 3.4 (d), (σ_r),is compressive and (σ_θ), tensile, where:

$\sigma_{\theta 1}$= internal hoop stress, which can be seen to be the maximum stress
$\sigma_{\theta 2}$= external hoop stress
P = internal cylinder pressure

Note that for all graphs in figures 3.4 (a – d) the value of the longitudinal stress σ_z is given by the intercept A on the σ axis.

Maximum Shear Stress in Thick Cylinders

The stresses on an element at any point in the cylinder wall are principal stresses.
Thus, the maximum shear stress at any point will be given by equation 3.18 as follows:

$$\tau_{max} = \frac{\sigma_\theta - \sigma_r}{2}$$

$$\tau_{max} = \frac{1}{2}\left[\left(A + \frac{B}{r^2}\right) - \left(A - \frac{B}{r^2}\right)\right]$$

$$\tau_{max} = \frac{1}{2}\left(2 * \frac{B}{r^2}\right)$$

$$\tau_{max} = \frac{B}{r^2} \qquad 3.18$$

The greatest value of shear stress normally occurs at the inside radius where r = R_1.

Stresses in Compound Thick Cylinders

- Consider a cylinder shrinked over another cylinder
- The inner cylinder is in initial compression, whereas the outer cylinder is in initial tension
- When the compound cylinder is subjected to internal fluid pressure both the inner and outer cylinders will be subjected to hoop tensile stress.
- The net effect of initial stresses due to shrinkage and those due to internal fluid pressure make the resulting stresses relatively uniform.
- In compound cylinders, a much smaller total fluctuation of hoop stress is obtained. A similar effect is obtained if a cylinder is wound with wire or steel tape under tension.

Stresses in Compound Thick Cylinders

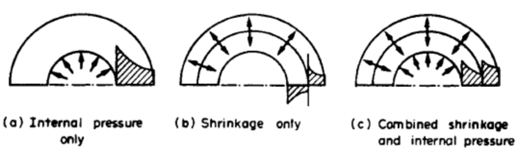


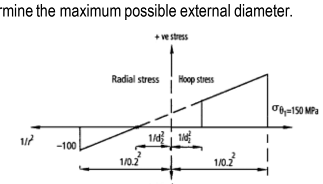
Fig. 3.5: Compound cylinders-combined internal pressure and shrinkage effects

Examples

Question 1

A thick-walled circular cylinder of internal diameter 0.2 m is subjected to an internal pressure of 100 MPa. If the maximum permissible stress in the cylinder is limited to 150 MPa, determine the maximum possible external diameter.

Solution

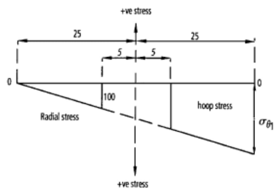


Examples

Question 2

If the cylinder in the previous problem were subjected to an external pressure **of** 100 MPa and an internal pressure of zero, what would be the maximum magnitude of stress.

Solution

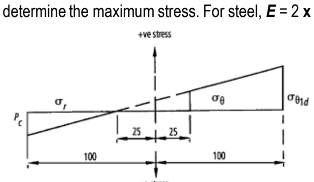


Examples

Question 3

A steel disc of external diameter 0.2 m and internal diameter 0.1 m is shrunk onto a solid steel shaft of external diameter 0.1 m, where all the dimensions are nominal. If the interference fit, based on diameters, between the shaft and the disc at the common surface is 0.2 mm, determine the maximum stress. For steel, ***E*** = 2 **x** 10^{11}N/m^2, v = 0.3

Solution



Questions

Question 1

Determine the maximum and minimum hoop stress across the section of pipe of 400mm internal diameter and 100mm thick, the pipe contains a fluid at a pressure of 8N/mm^2. Also sketch the radial pressure distribution and hoop stress distribution across the section.

Question 2

Find the thickness of metal necessary for a cylindrical shell of internal diameter 160mm to withstand an internal pressure of 8N/mm^2. The maximum hoop stress in the section is not to exceed 35N/mm^2.

Grazie