MEC 3352

STRENGTH OF MATERIALS II

THICK CYLINDERS

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Some Applications of Thick Cylinders

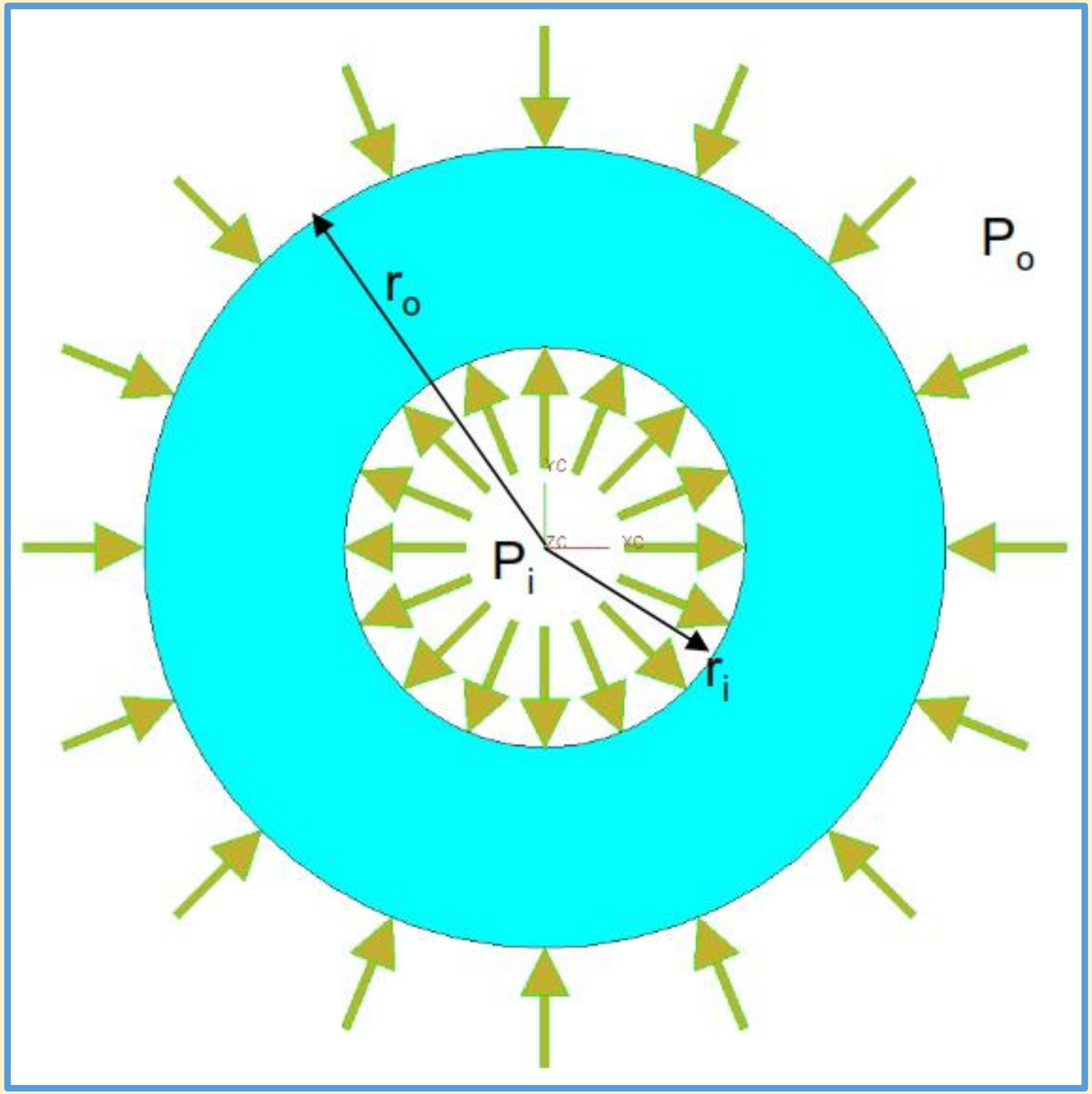
- Pressure vessels in compressors and other process applications
- Hydraulics
- Pneumatics
- Shock absorbers
- Motor vehicle bottle jacks
- Submarines
- Journal bearings
- Etc.

Generally they can withstand very high internal/external pressures though normally used for internal pressure applications.

Consider a thick walled cylinder with open ends:

It has:
- inner radius, r_i and
- outer radius, r_o.

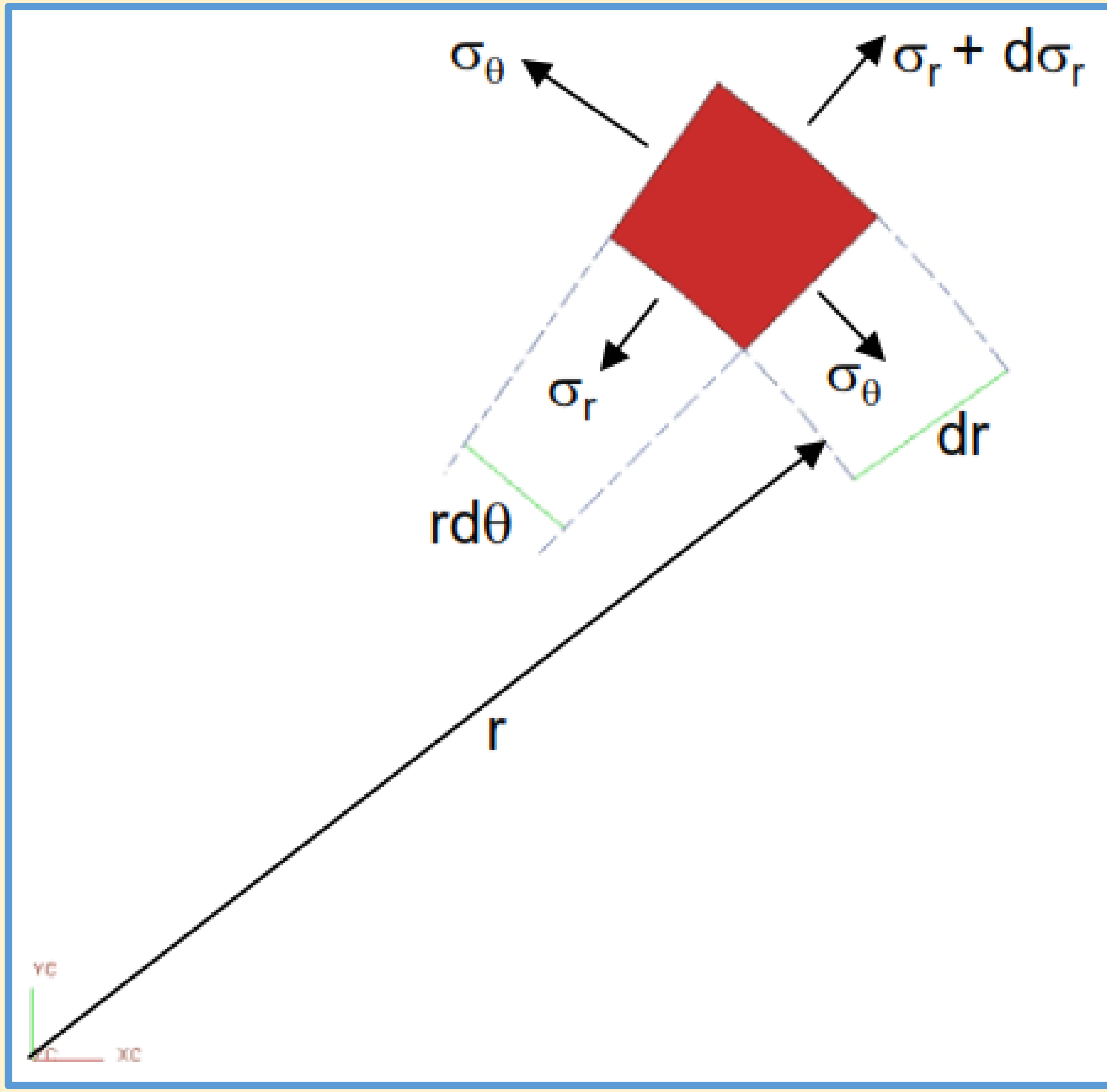
And loaded by:
- internal pressure, P_i
- external pressure, P_o



Now consider and element at radius r and defined by an angle increment $d\theta$ and a radial increment dr.

By circular symmetry,

- stresses σ_θ and σ_r are functions of r only, not of θ and
- the shear stress on the element must be zero.



For an element of unit thickness (assuming no body forces), radial force equilibrium gives:

$$(\sigma_r + d\sigma_r)(r + dr)d\theta = \sigma_r r d\theta + \sigma_\theta d\theta dr$$

Ignoring second order terms:

$$\boxed{\frac{d\sigma_r}{dr} + \frac{\sigma_r + \sigma_\theta}{r} = 0}$$

or

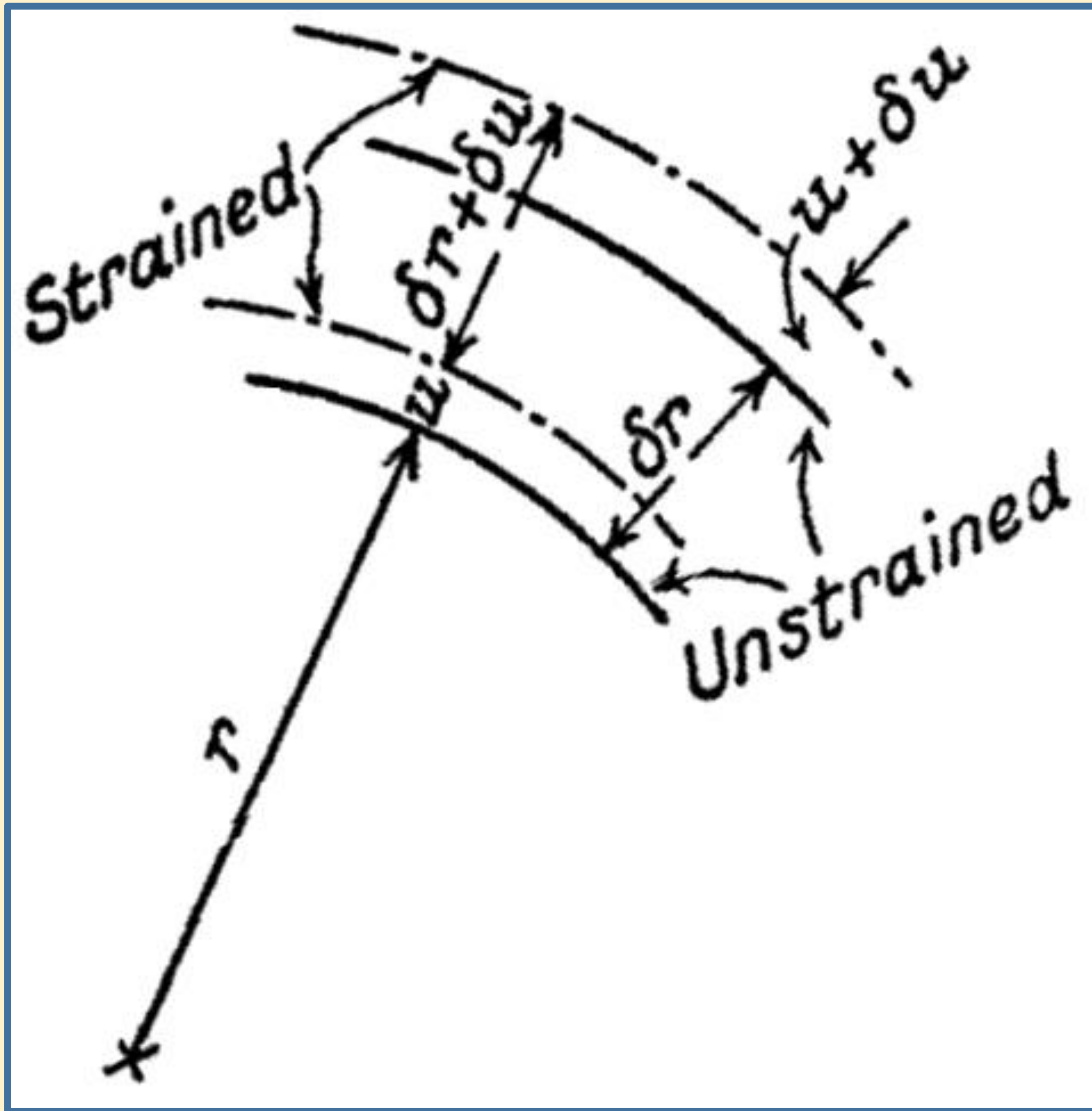
$$\boxed{\sigma_\theta + \sigma_r + r\frac{d\sigma_r}{dr} = 0}$$

(1)

The inner radius dilates and increases by $\boldsymbol{u}$ from the unstrained state to the strained.

The outer radius dilates and increases by $\boldsymbol{u} + \boldsymbol{du}$ from the unstrained state to the strained.

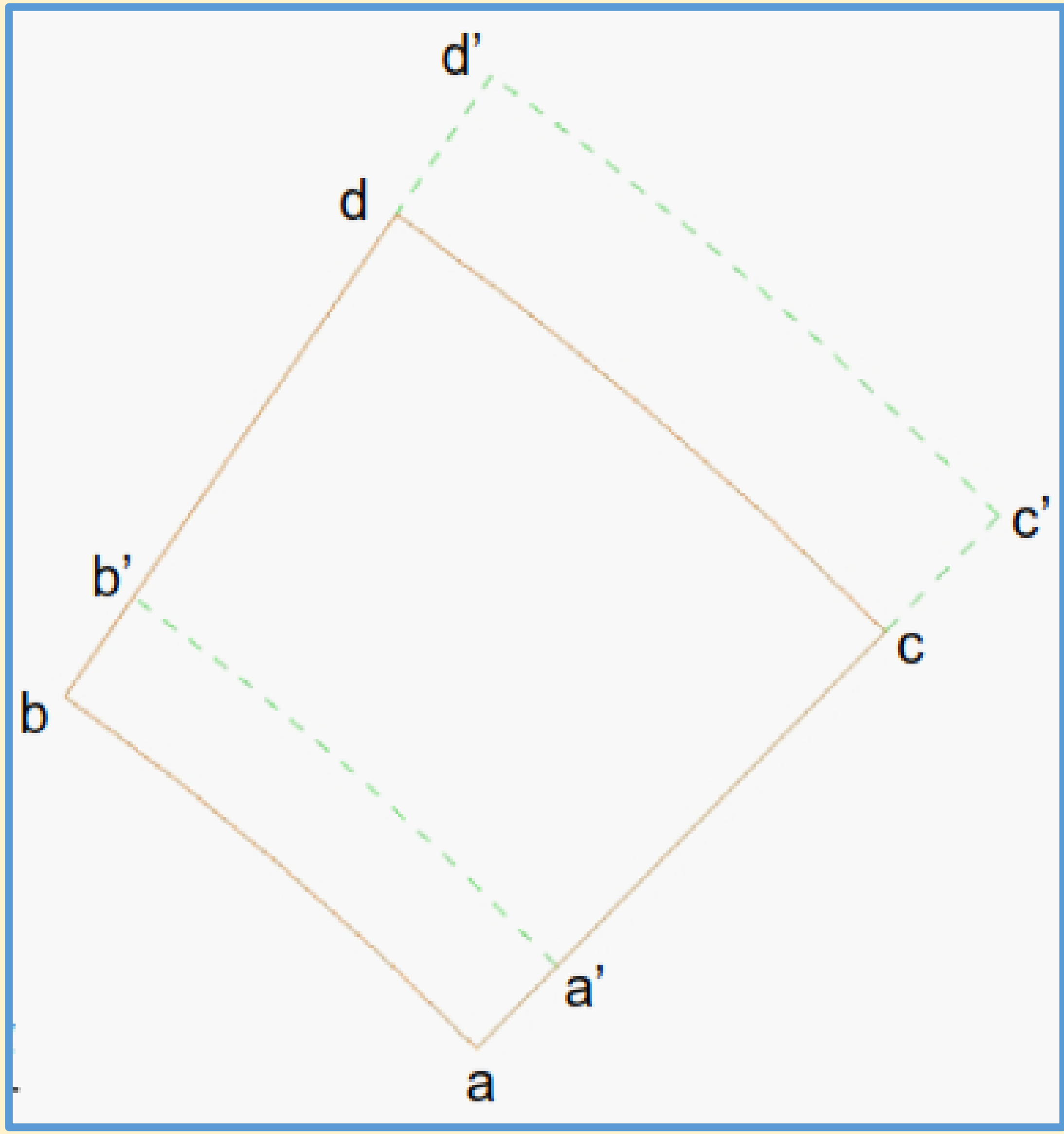
The thickness of the element changes by $\boldsymbol{du}$ to $\boldsymbol{dr} + \boldsymbol{du}$.



Now consider strains in the element.

By symmetry

- there is no $\boldsymbol{\theta}$ displacement.
- there is only a radial displacement
- Point a is displaced radially by $\boldsymbol{u}$ given by lines $\boldsymbol{aa'}$ and $\boldsymbol{bb'}$.
- Point c is displaced radially by $(\boldsymbol{u}+\boldsymbol{du})$ given by lines $\boldsymbol{cc'}$ and $\boldsymbol{dd'}$.



As the original radial length of the element is dr (line ac), the radial strain is:

$$\varepsilon_r = \frac{u + du - u}{dr} = \frac{du}{dr}$$

Line ab has length $rd\theta$ and line $a'b'$has length $(r+u)d\theta.$ Thus the tangential strain is:

$$\varepsilon_\theta = \frac{(r+u)d\theta - rd\theta}{rd\theta} = \frac{u}{r}$$

As the ends are open, $\sigma_z = \sigma_3 = 0$ and we thus have plane stress conditions.

From Hooke's law we get:

$$\varepsilon_r = \frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) \qquad \text{and} \qquad \varepsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

Solving for the stresses gives:

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \qquad \text{and} \qquad \sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

Substituting into equation above yields:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

Which has the solution :

$$u = C_1 r + \frac{C_2}{r}$$

Giving the stresses as:

$$\boldsymbol{\sigma_r = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{r^2}\right)\right]} \tag{2}$$

and

$$\boldsymbol{\sigma_\theta = \frac{E}{1-\nu^2}\left[C_1(1+\nu) + C_2\left(\frac{1-\nu}{r^2}\right)\right]} \tag{3}$$

The boundary conditions are:

$$\sigma_r(r_i) = -P_i \quad \text{and} \quad \sigma_r(r_o) = -P_o$$

This yields the integration constants:

$$C_1 = \frac{1-\nu}{E}\left(\frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2}\right)$$

and

$$C_2 = \frac{1-\nu}{E}\left(\frac{r_i^2 r_o^2 (P_i - P_o)}{r_o^2 - r_i^2}\right)$$

Giving the stresses as functions of the radius:

$$\boldsymbol{\sigma_r = \frac{P_i r_i^2 - P_o r_o^2}{(r_o^2 - r_i^2)} - \frac{(P_i - P_o) r_i^2 r_o^2}{(r_o^2 - r_i^2) r^2}} \qquad (4)$$

and

$$\boldsymbol{\sigma_\theta = \frac{P_i r_i^2 - P_o r_o^2}{(r_o^2 - r_i^2)} + \frac{(P_i - P_o) r_i^2 r_o^2}{(r_o^2 - r_i^2) r^2}} \qquad (5)$$

Eqs. (4) & (5) are known as **Lamé's Equations**.

Lamé's equations may also be written in terms of the diameters d_i and d_o as:

$$\boldsymbol{\sigma_r = \frac{P_i d_i^2 - P_o d_o^2}{(d_o^2 - d_i^2)} - \frac{(P_i - P_o) d_i^2 d_o^2}{(d_o^2 - d_i^2) d^2}} \tag{4a}$$

and

$$\boldsymbol{\sigma_\theta = \frac{P_i d_i^2 - P_o d_o^2}{(d_o^2 - d_i^2)} + \frac{(P_i - P_o) d_i^2 d_o^2}{(d_o^2 - d_i^2) d^2}} \tag{5a}$$

From Eqs. 2 & 3 we see that the sum of radial and tangential stresses is constant, regardless of radius:

$$\boldsymbol{\sigma_r + \sigma_\theta = \frac{2EC_1}{(1-\nu)} = C = \text{Constant}} \qquad (5)$$

Hence the longitudinal strain is also constant since:

$$\boldsymbol{\varepsilon_z = -\frac{\nu}{E}(\sigma_r + \sigma_\theta) = C = \text{Constant}} \qquad (6)$$

Hence we get

$$\boldsymbol{\sigma_z = E\varepsilon_z = \text{Constant} = C} \qquad (7)$$

If the ends of the cylinder are open and free we have $F_z = 0$, hence:

$$\left.\begin{aligned} &\boldsymbol{\int_{r_i}^{r_o} \sigma_z \cdot 2\pi r dr = \sigma_z \pi\left(r_o^2 - r_i^2\right) = 0} \\ &\text{or} \\ &\boldsymbol{\sigma_z = C = 0} \quad \text{as we assumed since } \pi\left(r_o^2 - r_i^2\right) \neq 0. \end{aligned}\right\} \tag{8}$$

If the cylinder has closed ends, the axial stress can be found separately using only force equilibrium considerations as was done for the thin walled cylinder.

The result is then simply superimposed on the above equations.

The pressure P_i acts on area given by $\pi {r_i}^2$

The pressure P_o acts on area given by $\pi {r_o}^2$

The axial stress σ_z acts on an area given by $\pi({r_o}^2 - {r_i}^2)$

Force equilibrium then gives:

$$\sigma_z = \left(\frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2}\right)$$

Summary of equations to determine the stresses for thick walled cylindrical pressure vessels.

- Generally, vessels are subjected to both internal and external pressures.
- Most vessels also have closed ends – this results in an axial stress component.

Principal stresses at radius r:

$$\sigma_1 = \sigma_r = -\mathrm{K} - C/r^2 \quad \text{and} \quad \sigma_2 = \sigma_\theta = -\mathrm{K} + C/r^2$$

And, if the ends are closed,

$$\sigma_3 = \sigma_{axial} = -K$$

Where:

$$\boldsymbol{C} = (\boldsymbol{P_o} - \boldsymbol{P_i})\left(\frac{\boldsymbol{r_i^2 r_o^2}}{\boldsymbol{r_o^2} - \boldsymbol{r_i^2}}\right) \quad \text{and} \quad \boldsymbol{K} = \frac{\left(\boldsymbol{P_o r_o^2} - \boldsymbol{P_i r_i^2}\right)}{\boldsymbol{r_o^2} - \boldsymbol{r_i^2}}$$

Consider Particular Scenarios:

(a) Internal Pressure only (P_o = 0):

Lamé's equations become:

$$\sigma_\theta = \frac{P_i r_i^2}{r_o^2 - r_i^2}\left[1 + \frac{r_o^2}{r^2}\right]$$

and

$$\sigma_r = \frac{P_i r_i^2}{r_o^2 - r_i^2}\left[1 - \frac{r_o^2}{r^2}\right]$$

And axial stress:

$$\sigma_z = \frac{-P_i r_i^2}{r_o^2 - r_i^2}$$

At inside surface, $r = r_i$:

$$\boldsymbol{\sigma_\theta = P_i \left[\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2}\right]}$$

$$\boldsymbol{\sigma_r = -P_i}$$

$$\boldsymbol{\sigma_z = \frac{P_i r_i^2}{r_o^2 - r_i^2}}$$

At outside surface, $r = r_o$:

$$\boldsymbol{\sigma_\theta = \frac{2P_i r_i^2}{r_o^2 - r_i^2}}$$

$$\boldsymbol{\sigma_r = 0}$$

$$\boldsymbol{\sigma_z = \frac{P_i r_i^2}{r_o^2 - r_i^2}}$$

(b) External Pressure only ($P_i = 0$):

Lamé's equations become:

$$\sigma_\theta = \frac{-P_o r_o^2}{r_o^2 - r_i^2}\left[1 + \frac{r_i^2}{r^2}\right]$$

$$\sigma_r = \frac{-P_o r_o^2}{r_o^2 - r_i^2}\left[1 - \frac{r_i^2}{r^2}\right]$$

$$\sigma_z = \frac{-P_o r_o^2}{r_o^2 - r_i^2}$$

At inside surface, $r = r_i$:

$$\sigma_\theta = \frac{-2P_o r_o^2}{r_o^2 - r_i^2}$$

$$\sigma_r = 0$$

$$\sigma_z = \frac{-P_o r_o^2}{r_o^2 - r_i^2}$$

At outside surface, $r = r_o$:

$$\sigma_\theta = -P_o \left[\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2}\right]$$

$$\sigma_r = -P_o$$

$$\sigma_z = \frac{-P_o r_o^2}{r_o^2 - r_i^2}$$

The stress variation with diameter is as shown in the figure, the two curves being “parallel” since (from (5):

$$\sigma_\theta + \sigma_r = 2a$$

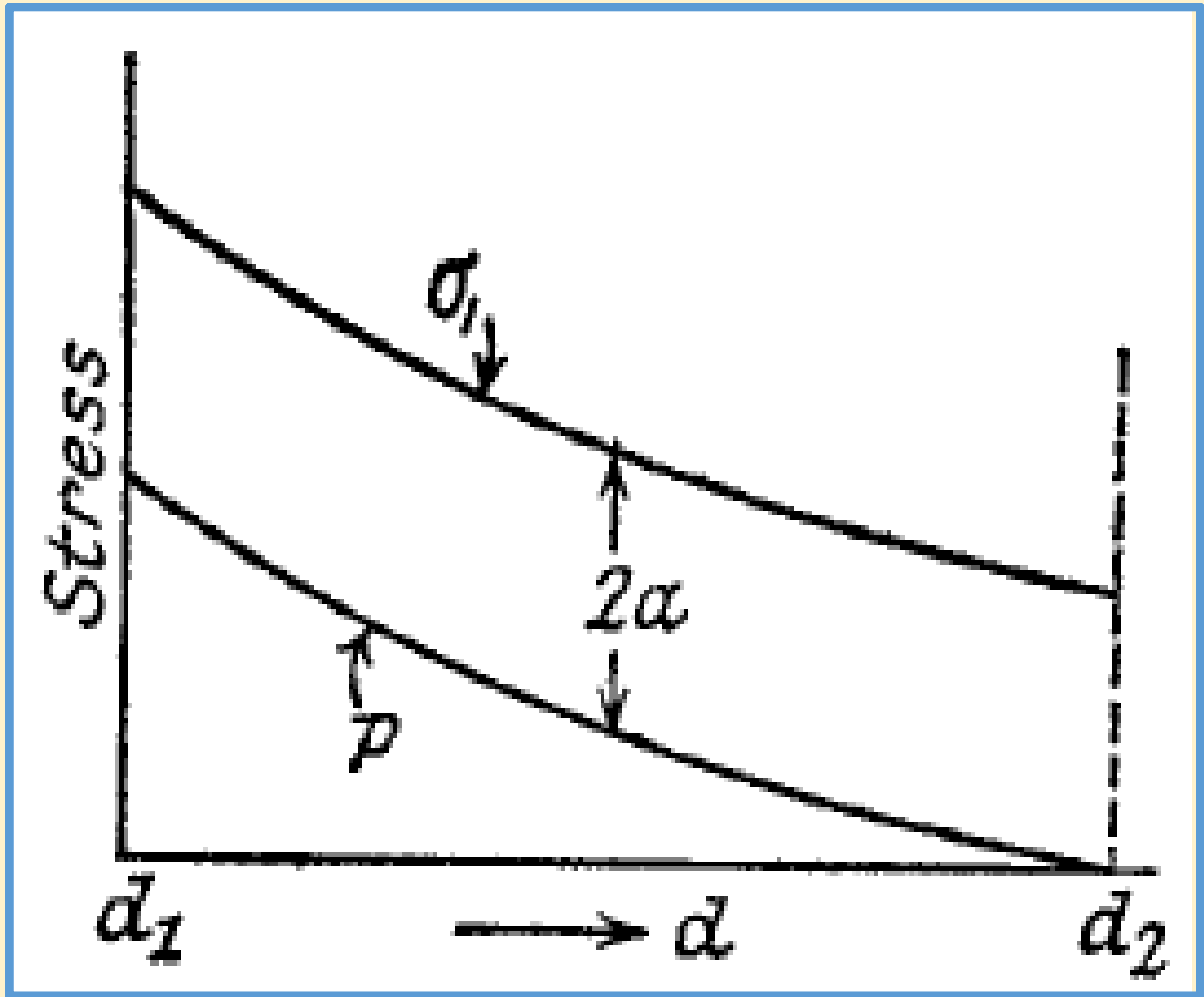
where:

$$a = \frac{EC_1}{(1-\nu)}$$

$$\sigma_{r_i} = -P_i$$

$$\sigma_{r_o} = -P_o$$

$$C_1 = \frac{1-\nu}{E}\left(\frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2}\right)$$



The maximum hoop stress is at $r = r_i$

$$\hat{\boldsymbol{\sigma}}_{\boldsymbol{\theta}} = \boldsymbol{P}_{\boldsymbol{i}}\left[\frac{\boldsymbol{r}_{\boldsymbol{o}}^{2} + \boldsymbol{r}_{\boldsymbol{i}}^{2}}{\boldsymbol{r}_{\boldsymbol{o}}^{2} - \boldsymbol{r}_{\boldsymbol{i}}^{2}}\right]$$

The maximum shear stress is:

$$\hat{\boldsymbol{\tau}}_{\mathbf{Max}} = \frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{\sigma}_{\boldsymbol{\theta}} + \boldsymbol{\sigma}_{\boldsymbol{r}_{\boldsymbol{i}}}\right) = \boldsymbol{P}_{\boldsymbol{i}}\left[\frac{\boldsymbol{r}_{\boldsymbol{o}}^{2}}{\left(\boldsymbol{r}_{\boldsymbol{o}}^{2} - \boldsymbol{r}_{\boldsymbol{i}}^{2}\right)}\right]$$

The longitudinal stress σ_z has already been shown to be constant.

But for cylinder with closed ends, σ_z is obtained from the equilibrium equation for any transverse section:

$$\sigma_z \pi\left(r_o^2 - r_i^2\right) = \pi r_i^2 P_i$$

Therefore:

$$\sigma_z = P_i \left[\frac{r_i^2}{r_o^2 - r_i^2}\right]$$

Error in "thin cylinder" formula:

In thin cylinder theory, for a cylinder of diameter d (radius r) and thickness t being acted upon by an internal pressure P_i, the hoop stress is given as

$$\boldsymbol{\sigma_\theta = \frac{P_i d}{2t} = \frac{P_i r}{t}}$$

But for a thick cylinder of thickness t, we can re-write an earlier expression for hoop stress as:

$$\boldsymbol{\sigma_\theta = P_i \left[\frac{\left(d_i^2 + 2t\right)^2 + d_i^2}{\left(d_i^2 + 2t\right)^2 - d_i^2}\right] = \frac{2\left(\frac{d_i}{t}\right)^2 + 4\left(\frac{d_i}{t}\right) + 4}{4\left(\frac{d_i}{t}\right) + 4} P_i}$$

Yielding: $\sigma_\theta = \dfrac{2(d_i/t)^2 + 4(d_i/t) + 4}{4(d_i/t) + 4} P_i$

Assuming $\dfrac{d_i}{t} = 10$ or $r_i = 5t$;

$\boldsymbol{\sigma_\theta = 5.55 P_i}$; which is 11% higher that the mean value given by $\sigma_\theta = P_i d/2t$.

Assuming $\dfrac{d_i}{t} = 20$ or $r_i = 10t$;

$\boldsymbol{\sigma_\theta = 10.5 P_i}$; which is 5% higher than $\sigma_\theta = P_i d/2t$.

Note that:

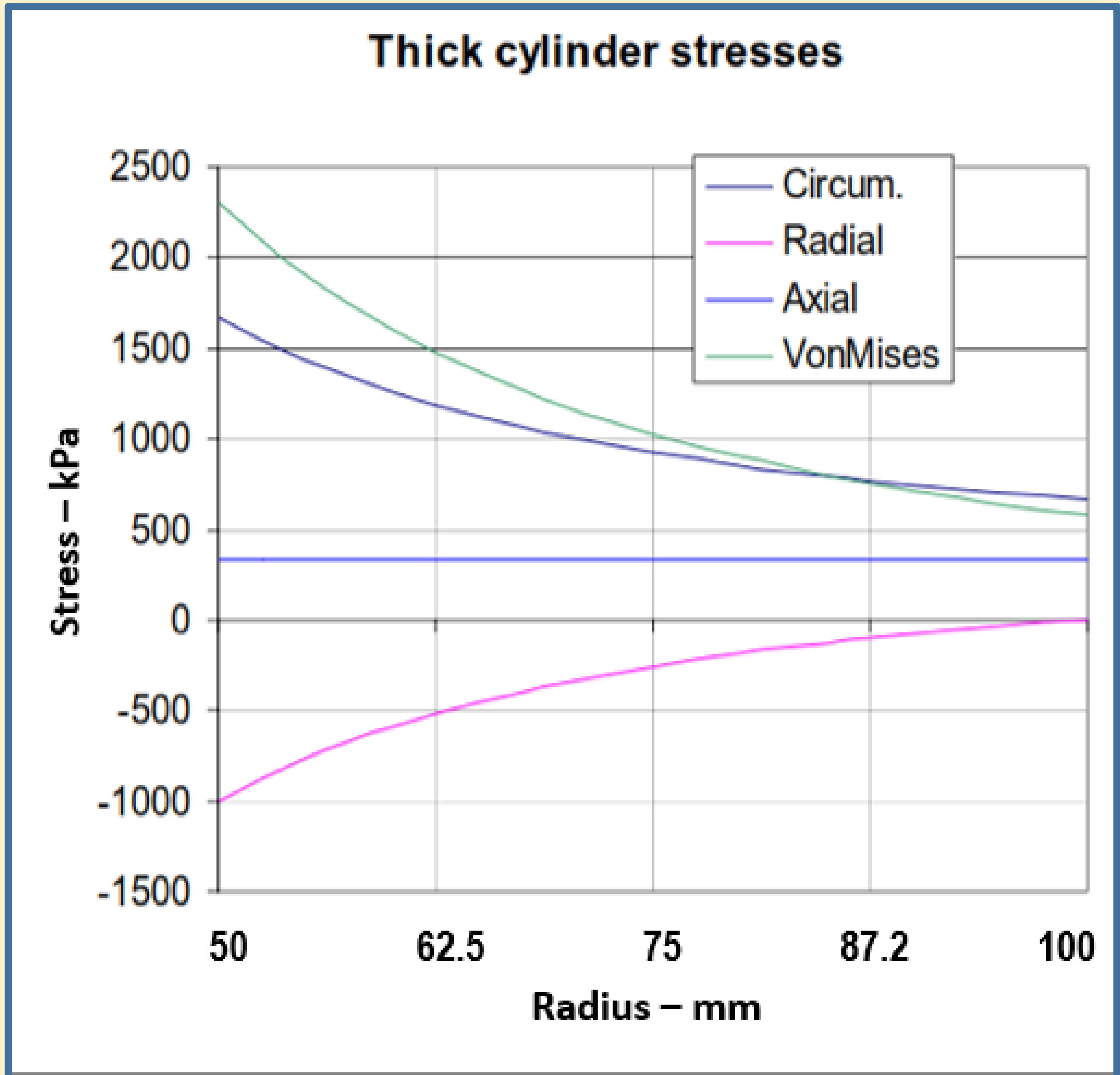
1. *Clearly as* $\dfrac{d_i}{t} \longrightarrow \infty$, *the hoop stresses by the two formulae tend to be equal.*

2. *If the mean diameter is used in the thin cylinder formula, the error is practically eliminated! (The student to prove this.)*

Note that:

- In all cases the greatest magnitude of direct stress is the tangential (von Mises) stress at the in-side surface.

- The maximum magnitude of shear stress also occurs at the inside surface because that is where the greatest stress difference occurs.



Example of cylinder with P_i = 1000 kPa, r_i = 50 mm and r_o = 100 mm.

(c) Press and Shrink Fits:

When a press or shrink fit is used between 2 cylinders of the same material, an interface pressure p_i is developed at the junction of the cylinders.

If this pressure is calculated, the stresses in the cylinders can be found using the above equations.

The interface pressure is:

$$\boldsymbol{p_i = \frac{E\delta}{b}\left[\frac{(c^2 - b^2)(b^2 - a^2)}{2b^2(c^2 - a^2)}\right]}$$

where:

$E =$ Young's Modulus

$\delta =$ radial interference between the two cylinders

$a =$ inner radius of the inner cylinder

$b =$ outer radius of inner cylinder and inner radius of outer cylinder

$c =$ outer radius of outer cylinder

It is assumed that δ is very small compared to the radius b and that there are no axial stresses.

Thus we have

$$\boldsymbol{\delta = b_{\mathbf{inner}} - b_{\mathbf{outer}}}$$

Note that this small difference in the radii is ignored in the values of b in the above equation for P_i.

The End

But wait, wait, wait!!!!

Problem:

Prove that if the mean diameter is used in the thin cylinder formula, the error between the thick cylinder and thin cylinder formulae is practically eliminated.

Do this for $\frac{d_i}{t} = 30$ *and* $\frac{d_i}{t} = 40$ *in the thick cylinder*

Practice Examples from Ryder:

Solve the following:

(1) Example 7, p. 271

(2) Example 9, pp. 272-3

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ME 3352: Strength of Materials II

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