The University of Zambia
School of Engineering
Department of Mechanical Engineering

MEC 3352 – STRENGTH OF MATERIALS II

Thick Circular Cylinders

THIN SHELL THEORY

- The thin shell theory is satisfactory when the ratio of the shell thickness to shell radius is less than 1/20.
- It assumes that the circumferential (hoop stress) and longitudinal stress are constant over the thickness; radial stress is negligible.
- When the thickness to radius ratio is greater than 1/20, errors in the computations set in, and in such a case the thick shell theory should be used.

THICK SHELL THEORY

- When the thickness to shell radius is greater than 1/20, we have a thick cylinder
- The hoop stress in thick cylinders vary over the thickness
- Thick shells examples include:
    - i) gun barrels,
    - ii) nuclear reactor pressure vessels, and
    - iii) deep diving submersibles vessels.

STRESS AND STRAIN IN THICK CIRCULAR CYLINDERS

We assume all the stresses and strains are tensile and positive. At any radius, ***r* :**

$\sigma_\theta = hoop\ stress$

$\sigma_r = radial\ stress$

$\sigma_z = Longitudinal\ stress$

$\varepsilon_\theta = hoop\ strain$

$\varepsilon_r = radial\ strain$

$\varepsilon_z = longitudinal\ Strain\ assumed\ to\ be\ constant$

$w = radial\ deflection$

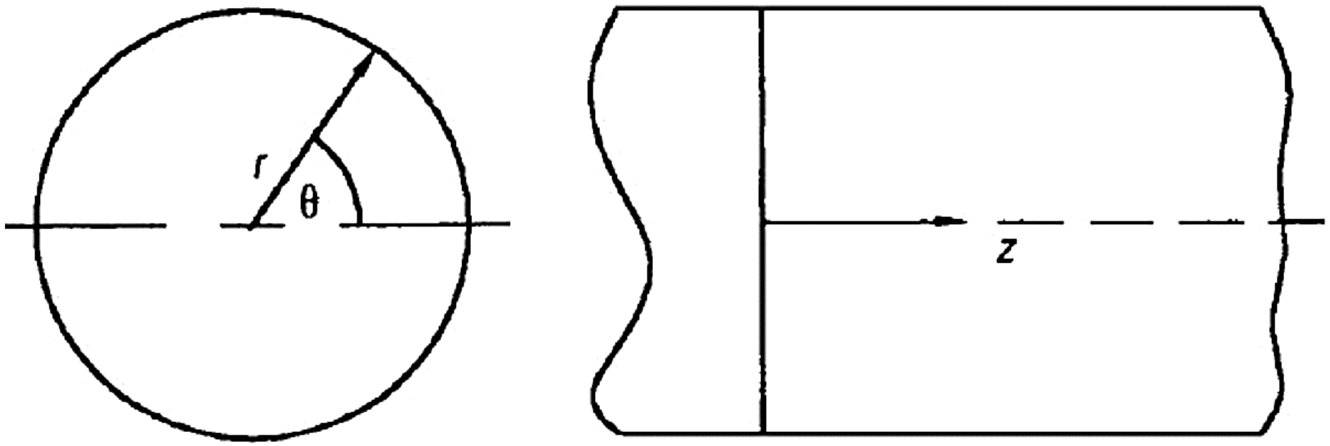


Figure 4.0: Thick Cylinder

Stress and Strain in Thick Circular Cylinders

Longitudinal Stress σ_z

- Consider a cross-section of a thick cylinder with closed ends subjected to an internal pressure P_1 and an external pressure P_2. (Figure 4.1)

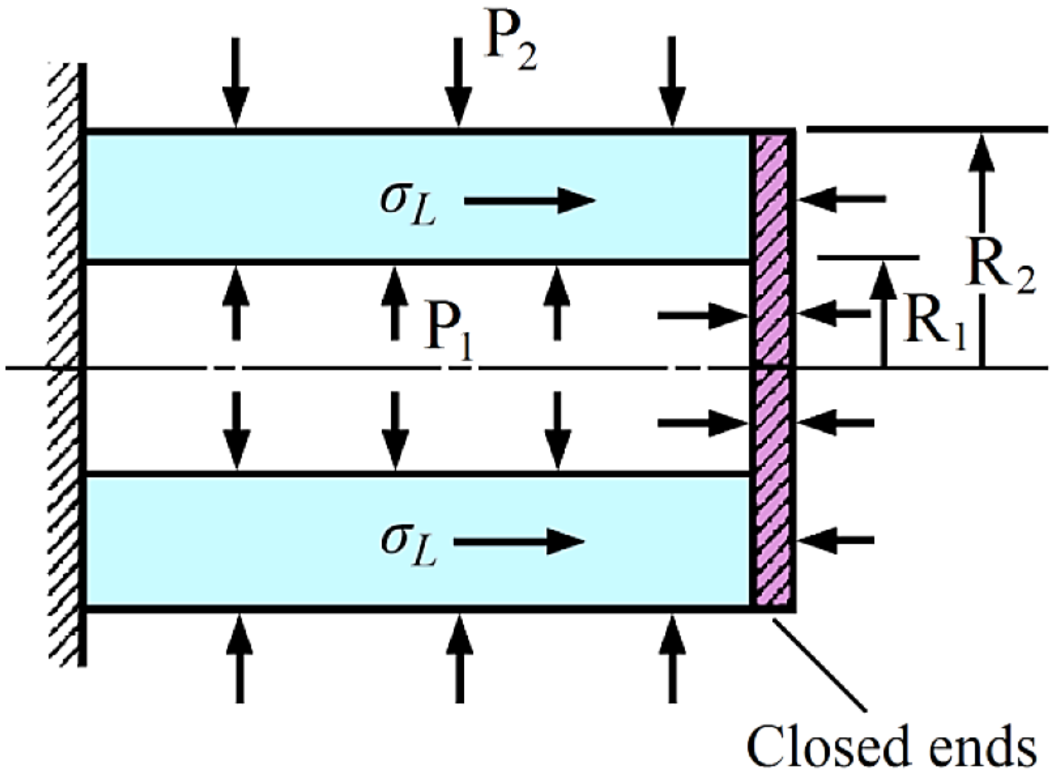


Fig. 4.1: Cylinder longitudinal section

- Remember that, Force = Pressure x Area
- For horizontal equilibrium:

$$P_1 * \pi R_1^2 - P_2 * \pi R_2^2 = \sigma_z * \pi(R_2^2 - R_1^2)$$

$$\sigma_z = \frac{P_1 R_1^2 - P_2 R_2^2}{R_2^2 - R_1^2} \qquad 4.0$$

- We see from equation 4.0 that the longitudinal stress set up in the cylinder walls is constant for the given internal and external pressures

Derivation of the hoop and radial stress equations

From Figure 4.2, it can be seen that at any radius *r*,

$$\varepsilon_\theta = \frac{2\pi(r+w) - 2\pi r}{2\pi r}$$

$$\varepsilon_\theta = \frac{w}{r} \qquad 4.1$$

Similarly,

$$\varepsilon_r = \frac{\delta w}{\delta r} = \frac{dw}{dr} \qquad 4.2$$

From the standard stress-strain relationships

$$E\varepsilon_z = \sigma_z - \nu(\sigma_\theta + \sigma_r) \text{ = a constant}$$

$$E\varepsilon_\theta = E\frac{w}{r} = \sigma_\theta - \nu(\sigma_z + \sigma_r) \qquad 4.3$$

$$E\varepsilon_r = E\frac{dw}{dr} = \sigma_r - \nu(\sigma_\theta + \sigma_z) \qquad 4.4$$

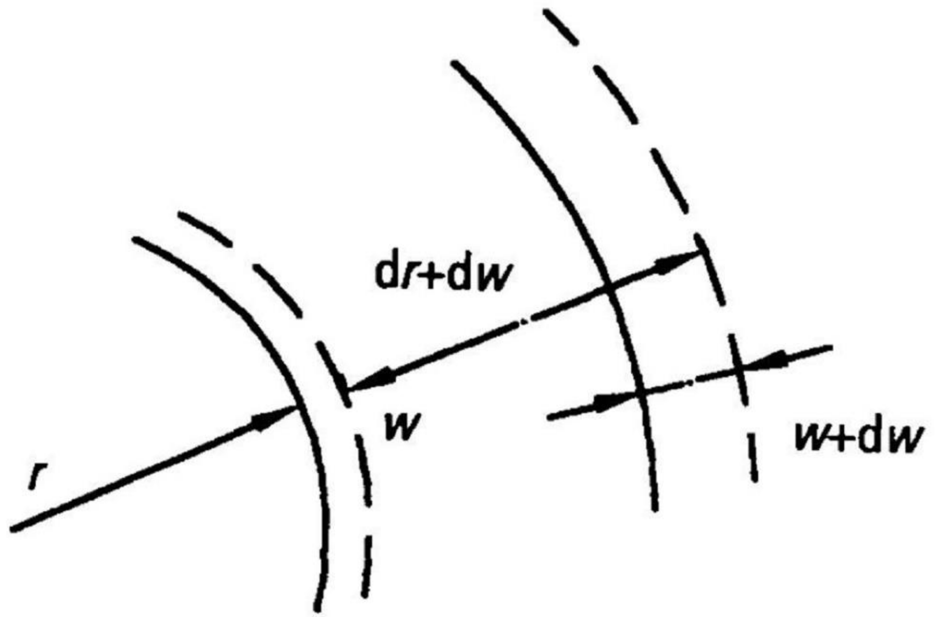


Fig. 4.2: Thick Cylinder Deformation at any radius r

Derivation of the hoop and radial stress equations

$$E\varepsilon_\theta = E\frac{w}{r} = \sigma_\theta - \nu(\sigma_z + \sigma_r) \qquad 4.3$$

Multiplying equation (4.3) by r, we get

$$Ew = [\sigma_\theta - \nu(\sigma_z - \sigma_r)]r \qquad 4.5$$

Differentiating equation (4.5) with respect to r, we get

$$E\frac{dw}{dr} = \sigma_\theta - \nu\sigma_z - \nu\sigma_r + r\left[\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_z}{dr} - \nu\frac{d\sigma_r}{dr}\right] \qquad 4.6$$

$$E\frac{dw}{dr} = \sigma_r - \nu(\sigma_\theta + \sigma_z) \qquad 4.4$$

Subtracting equation (4.4) from equation (4.6),

$$(\sigma_\theta - \sigma_r)(1+\nu) + r\frac{d\sigma_\theta}{dr} - \nu r\frac{d\sigma_z}{dr} - \nu r\frac{d\sigma_r}{dr} = 0 \qquad 4.7$$

As ε_z, is constant

$$\sigma_z - \nu\sigma_\theta - \nu\sigma_r = \text{constant} \qquad 4.8$$

Derivation of the hoop and radial stress equations

$$\sigma_z - \nu\sigma_\theta - \nu\sigma_r = \text{constant} \qquad 4.8$$

Differentiating equation (4.8)with respect to r,

$$\frac{d\sigma_z}{dr} - \nu\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr} = 0$$

Re-arranging the above equation, we get

$$\frac{d\sigma_z}{dr} = \nu\left(\frac{d\sigma_\theta}{dr} + \frac{d\sigma_r}{dr}\right) \qquad 4.9$$

Substituting equation (4.9) into equation (4.7),

$$(\sigma_\theta - \sigma_r)(1+\nu) + r\frac{d\sigma_\theta}{dr} - \nu r\frac{d\sigma_z}{dr} - \nu r\frac{d\sigma_r}{dr} = 0 \qquad 4.7$$

$$(\sigma_\theta - \sigma_r)(1+\nu) + r(1-\nu^2)\frac{d\sigma_\theta}{dr} - \nu r(1+\nu)\frac{d\sigma_r}{dr} = 0 \qquad 4.10$$

Dividing equation (4.10) by $(1 + \nu)$, we get

$$\sigma_\theta - \sigma_r + r(1-\nu)\frac{d\sigma_\theta}{dr} - \nu r\frac{d\sigma_r}{dr} = 0 \qquad 4.11$$

Derivation of the hoop and radial stress equations

Consider the radial equilibrium of the shell element of unit thickness shown in Figure 4.3,

$$(\sigma_r + d\sigma_r)(r + dr)d\theta * 1 - \sigma_r * rd\theta * 1 = 2\sigma_\theta * dr * 1 * \sin\left(\frac{d\theta}{2}\right) \qquad 4.12$$

For small angles $sin\left(\frac{d\theta}{2}\right) \cong \frac{\theta}{2}\ rad$

Simplifying and neglecting higher order terms in 4.12, we get

$$\sigma_\theta - \sigma_r - r\frac{d\sigma_r}{dr} = 0 \qquad 4.13$$

$$\sigma_\theta - \sigma_r + r(1+\nu)\frac{d\sigma_\theta}{dr} - \nu r\frac{d\sigma_r}{dr} = 0 \qquad 4.11$$

Subtracting equation (4.11) from equation (4.13) and simplifying,

$$\frac{d\sigma_\theta}{dr} + \frac{d\sigma_r}{dr} = 0 \qquad \therefore \frac{d}{dr}(\sigma_\theta + \sigma_r) = 0 \qquad 4.14$$

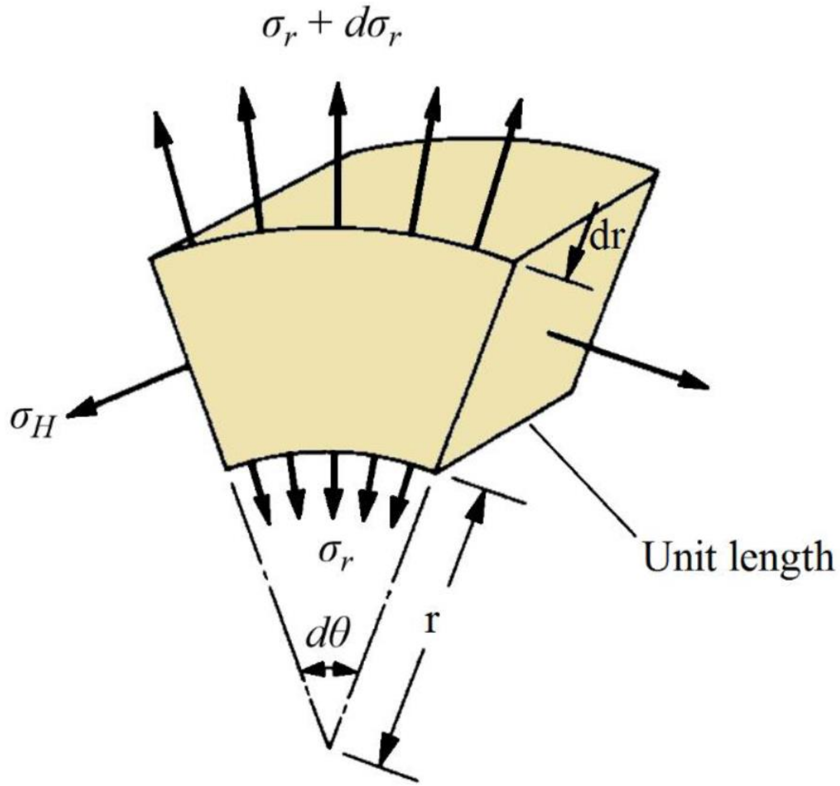


Fig. 4.3 : Shell element

Derivation of the hoop and radial stress equations

$$\frac{d}{dr}(\sigma_\theta + \sigma_r) = 0 \qquad 4.14$$

Integrating equation (4.14), we get

$$\sigma_\theta + \sigma_r = constant = 2A \quad \Rightarrow \sigma_\theta = 2A - \sigma_r \qquad 4.15$$

Note that 2A is simply some convenient constant of integration

Substituting (4.15) into equation (4.13),

$$\sigma_\theta - \sigma_r - r\frac{d\sigma_r}{dr} = 0 \qquad 4.13$$

$$2A - \sigma_r - \sigma_r - r\frac{d\sigma_r}{dr} = 0$$

Multiplying through by ***r*** and rearranging,

$$2r\sigma_r + r^2\frac{d\sigma_r}{dr} - 2Ar = 0$$

$$\therefore \quad \frac{d}{dr}(\sigma_r r^2 - Ar^2) = 0$$

Integrating this, we get:

$$\sigma_r r^2 - Ar^2 = -B$$

$Where\ -B$ is a convenient constant of integration

Derivation of the hoop and radial stress equations

$$\sigma_r r^2 - Ar^2 = -B$$

Simplifying the above equation, we get

$$\sigma_r = A - \frac{B}{r^2} \qquad 4.16$$

$$\sigma_\theta + \sigma_r = constant = 2A \quad \Rightarrow \sigma_\theta = 2A - \sigma_r \qquad 4.15$$

Substituting equation (4.16) into (4.15) and simplifying, we get

$$\sigma_\theta = A + \frac{B}{r^2} \qquad 4.17$$

Compare 4.16 and 4.17 with equations 3.6 and 3.7

$$\boldsymbol{\sigma_r = A - \frac{B}{r^2} - (3 + \nu)\frac{\rho r^2 \omega^2}{8}} \qquad \mathbf{3.6}$$

$$\boldsymbol{\sigma_H = A + \frac{B}{r^2} - (1 + 3\nu)\frac{\rho r^2 \omega^2}{8}} \qquad \mathbf{3.7}$$

Lamé's Equations

- Equations (4.16) and (4.17) are called **Lamé's** Equations
- The **Lamé's** equations when plotted on stress and $1/r^2$ axes produce straight lines, as shown in Figure 4.4 a
- The two lines may be modified to a single a straight line, where (σ_r), lies to the left and (σ_θ), to the right, as shown by Figure 4.4 b.

Graphical Representation of **Lamé's** Equations

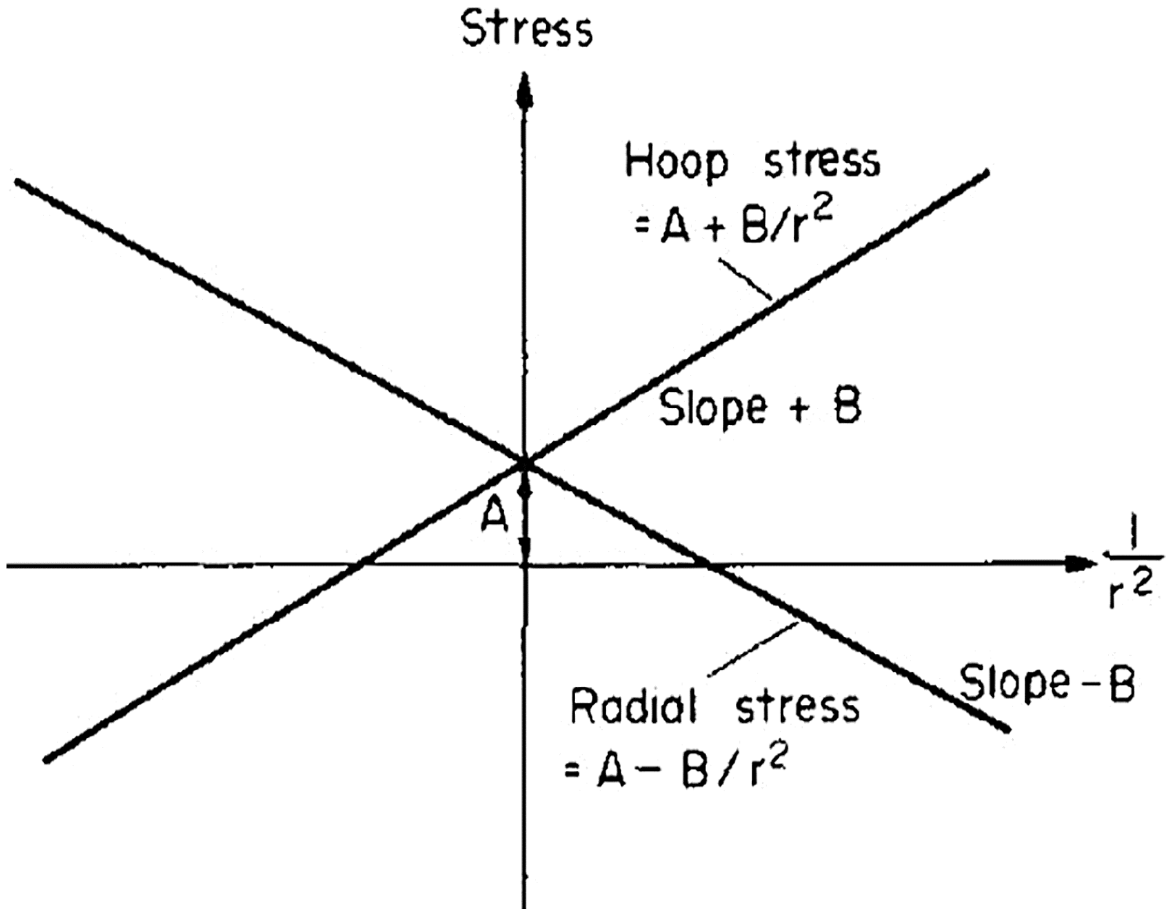


Fig. 4.4 (a): Graphical representation of **Lame** equations - **Lame** line.

- Both lines have exactly the same intercept A and the same magnitude of slope ***B,*** the only difference being the sign of their slopes.
- The two are therefore combined by plotting hoop stress values to the left of the σ axis (again against $1/_{r^2}$) instead of to the right to give the single line shown in Fig. 3.4(b).
- In most questions one value of σ_r, and one value of σ_θ, or alternatively two values of σ_r, are given. In both cases the single line can then be drawn.

Graphical Representation of **Lamé's** Equations

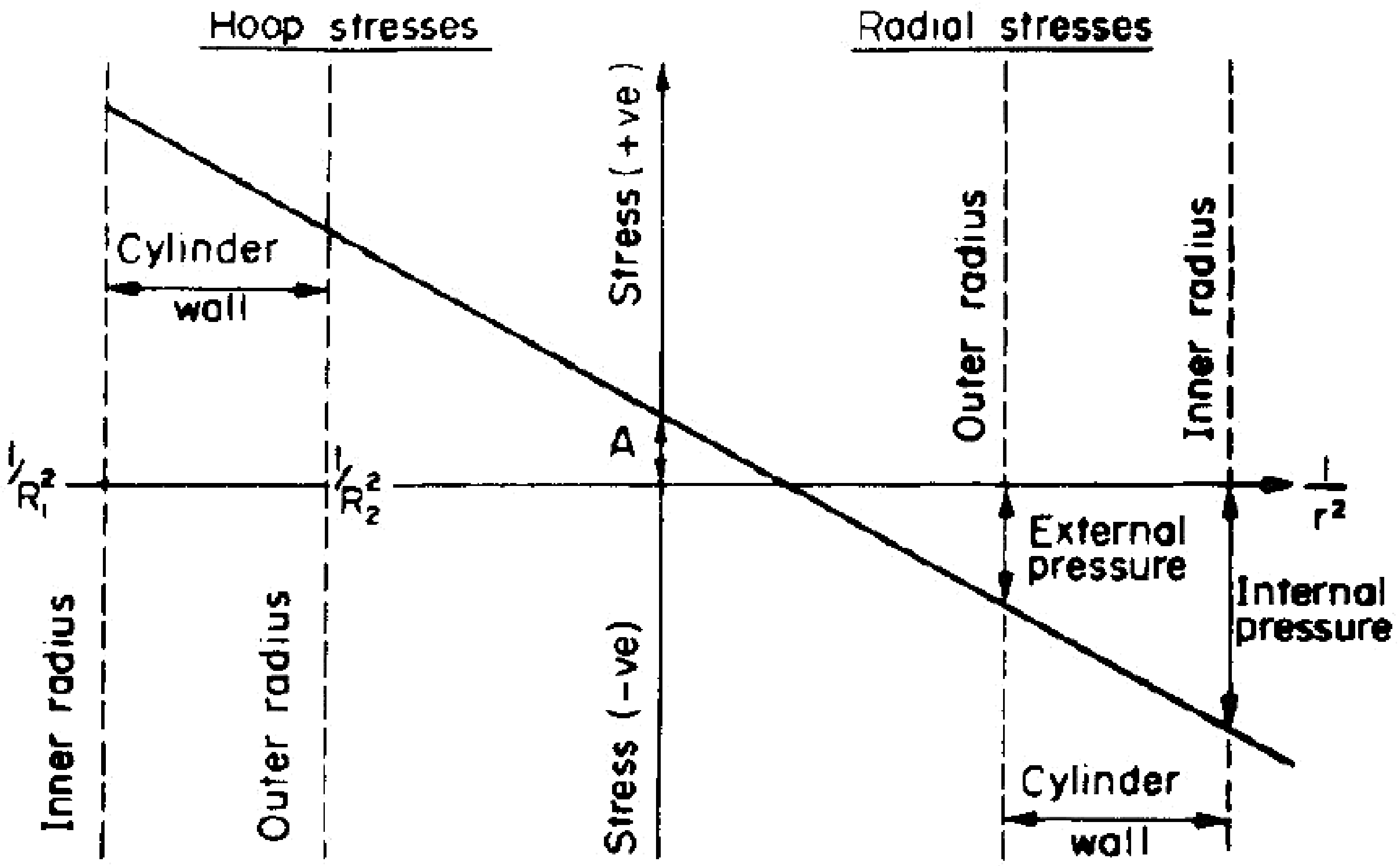


Fig. 4.4 (b) Lame line solution for cylinder with internal and external pressures

Graphical Representation of **Lamé's** Equations

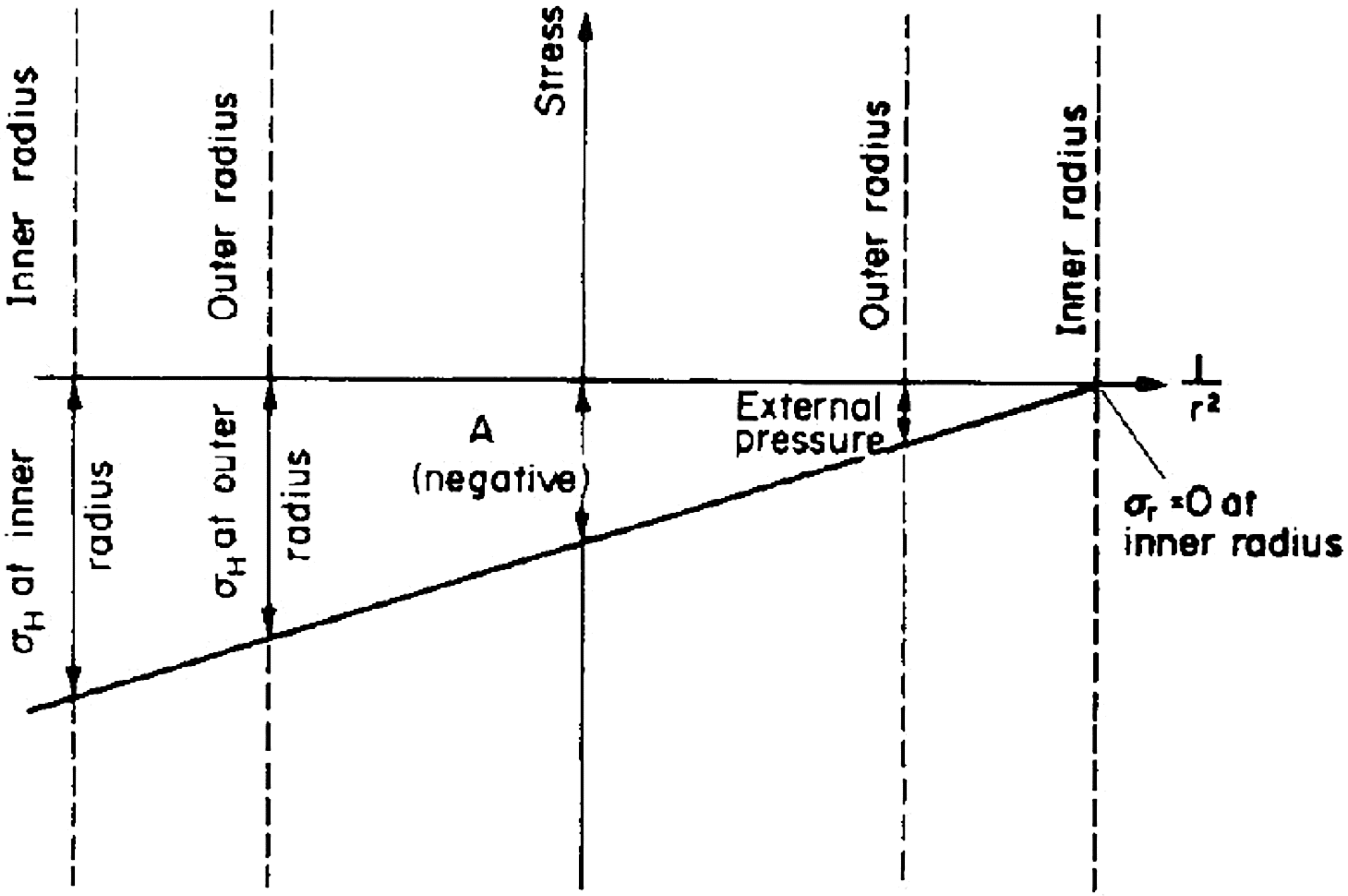


Fig. 4.4 (c) Lame's line solution for cylinder subjected to external pressure only

Graphical Representation of **Lamé's** Equations

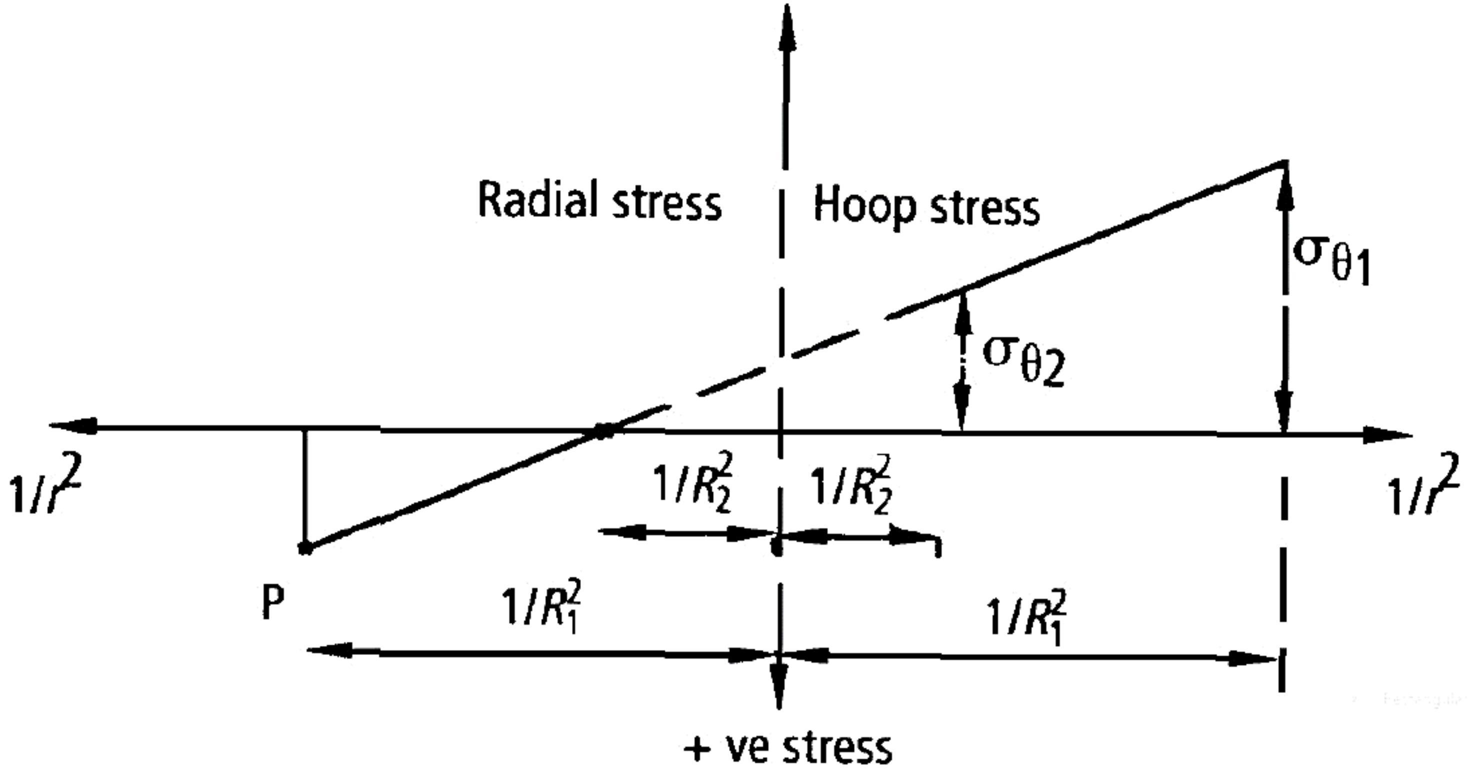


Fig. 4.4 (d): Lame's line for the case of internal pressure

For the case shown in Figure 4.4 (d), (σ_r),is compressive and (σ_θ), tensile, where:

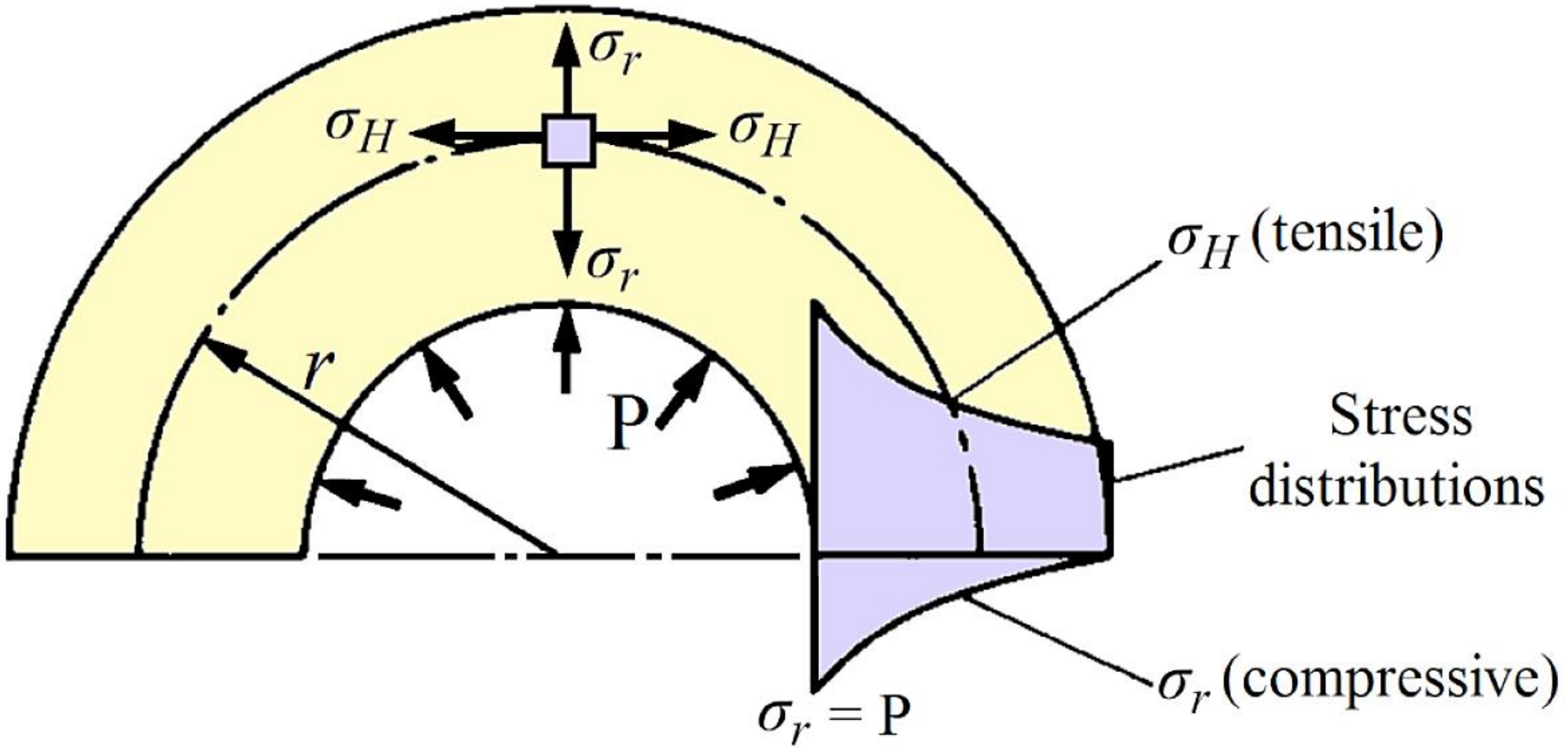
$\sigma_{\theta 1}$ = internal hoop stress, which can be seen to be the maximum stress

$\sigma_{\theta 2}$= external hoop stress

P = internal cylinder pressure

Note that for all graphs in figures 4.4 (a – d) the value of the longitudinal stress σ_z is given by the intercept A on the σ axis.

Thick Cylinders with Internal Pressure only



$\sigma_H = A + B/r^2$ $\sigma_r = A - B/r^2$

Fig. 4.4 (e) Thick cylinder subjected to internal pressure only

Maximum Shear Stress in Thick Cylinders

- The stresses on an element at any point in the cylinder wall are principal stresses.
- Thus, the maximum shear stress at any point will be given by equation 4.18 as follows:

$$\tau_{max} = \frac{\sigma_\theta - \sigma_r}{2}$$

$$\tau_{max} = \frac{1}{2}\left[\left(A + \frac{B}{r^2}\right) - \left(A - \frac{B}{r^2}\right)\right]$$

$$\tau_{max} = \frac{1}{2}\left(2 * \frac{B}{r^2}\right)$$

$$\tau_{max} = \frac{B}{r^2} \qquad 4.18$$

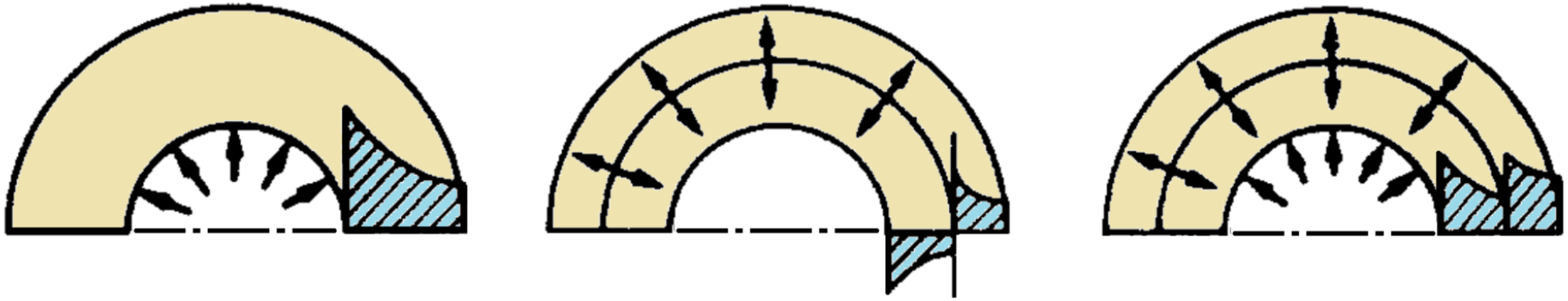
- Greatest value of shear stress normally occurs at the inside radius where r = R_1.

Stresses in Compound Thick Cylinders

- Consider a cylinder shrinked over another cylinder
- The inner cylinder is in initial compression, whereas the outer cylinder is in initial tension
- When the compound cylinder is subjected to internal fluid pressure both the inner and outer cylinders will be subjected to hoop tensile stress.
- The net effect of initial stresses due to shrinkage and those due to internal fluid pressure make the resulting stresses relatively uniform.
- In compound cylinders, a much smaller total fluctuation of hoop stress is obtained.
- A similar effect is obtained if a cylinder is wound with wire or steel tape under tension.

Stresses in Compound Thick Cylinders

- If a compound cylinder is subjected to internal pressure, the resultant hoop stresses will be the algebraic sum of those resulting from internal pressure and those resulting from shrinkage.
- The cylinders are often built up by shrinking one tube on to the outside of another as drawn in Fig. 4.5.



(a) Internal pressure only (b) Shrinkage only (c) Combined shrinkage and internal pressure

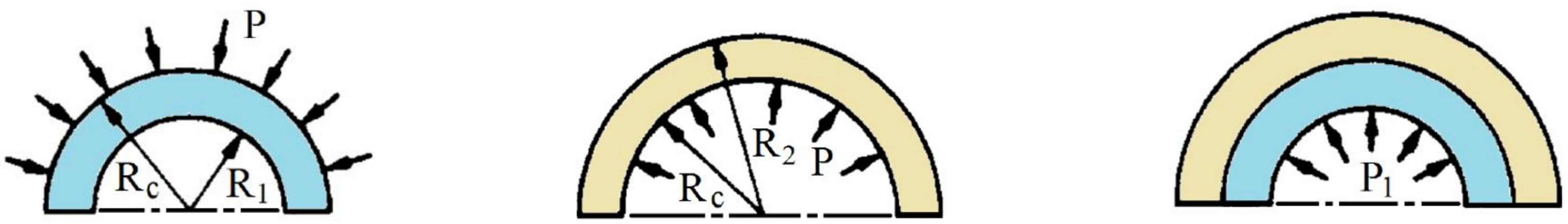
Fig. 4.5: Compound cylinders-combined internal pressure and shrinkage effects

- For each condition the hoop and radial stresses at any radius can be evaluated and the principle of superposition applied.

Stresses in Compound Thick Cylinders

In the analysis of compound cylinders constructed from similar materials, it is easier to break down the problem into three separate effects:

a) Shrinkage pressure only on the inside cylinder.
b) Shrinkage pressure only on the outside cylinder.
c) Internal pressure only on the complete cylinder, as shown in Fig. 4.6.



(a) Shrinkage - internal cylinder

(b) Shrinkage - external cylinder

(c) Internal pressure - compound cylinder

Fig. 4.6: Compound cylinders-combined internal pressure and shrinkage effects

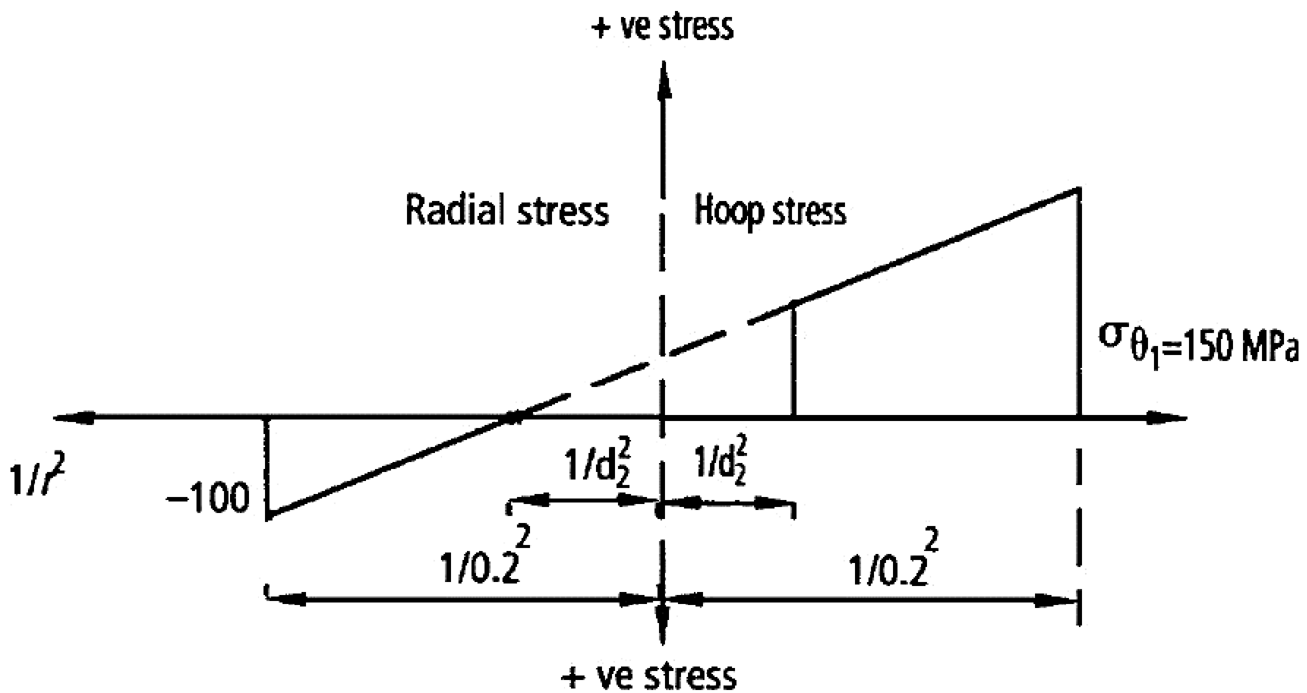
- For each condition the hoop and radial stresses at any radius can be evaluated from the boundary conditions and the principle of superposition applied.

Examples

Question 1

A thick-walled circular cylinder of internal diameter 200 mm is subjected to an internal pressure of 100 MPa. If the maximum permissible stress in the cylinder is limited to 150 MPa, determine the maximum possible external diameter.

Solution



Examples

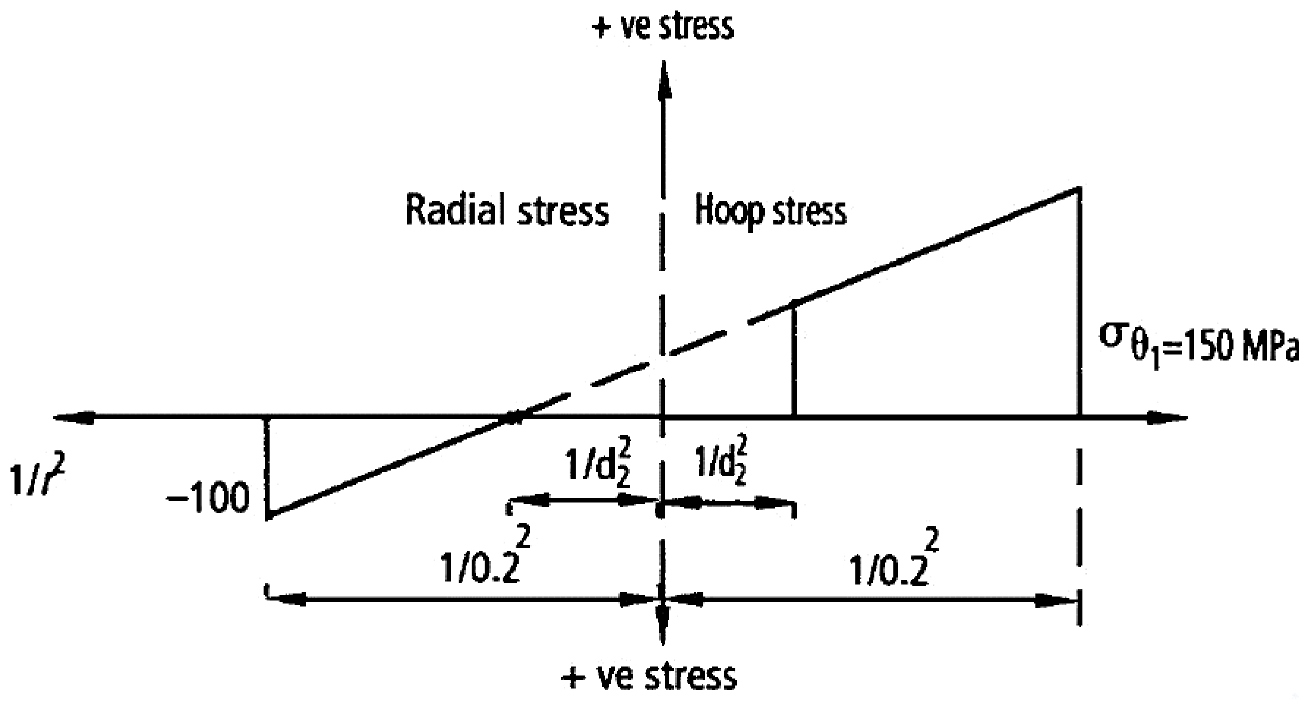
Solution

Using similar triangles

$$\frac{100}{\left(\frac{1}{0.2^2} - \frac{1}{d_2^2}\right)} = \frac{150}{\left(\frac{1}{0.2^2} + \frac{1}{d_2^2}\right)}$$

$$\frac{\left(\frac{1}{0.2^2} + \frac{1}{d_2^2}\right)}{\left(\frac{1}{0.2^2} - \frac{1}{d_2^2}\right)} \times \left(\frac{0.2^2\ d_2^2}{0.2^2\ d_2^2}\right) = 1.5$$

$$\left(\frac{d_2^2 + 0.2^2}{d_2^2 - 0.2^2}\right) = 1.5$$



$$d_2^2 + 0.2^2 = 1.5\left(d_2^2 - 0.2^2\right)$$

$$0.2^2(1+1.5) = d_2^2(1.5-1)$$

$$d_2^2 = 0.2\,\mathrm{m}^2$$

$$d_2 = 0.447\,\mathrm{m}$$

Examples

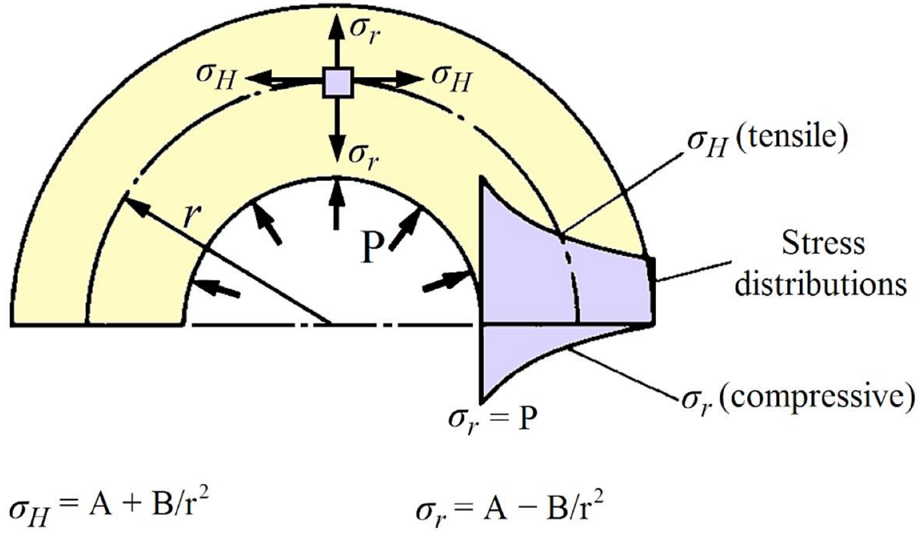
Question 1

A thick-walled circular cylinder of internal diameter 200 mm is subjected to an internal pressure of 100 MPa. If the maximum permissible stress in the cylinder is limited to 150 MPa, determine the maximum possible external diameter, and the hoop stress on the outer surface.

Solution (alternative approach)

$$\sigma_r = A - \frac{B}{r^2} \qquad 4.16$$

$$\sigma_\theta = A + \frac{B}{r^2} \qquad 4.17$$



At the inner surface

$r_1 = 100 \qquad \sigma_r = -100\ MPa$

$-100 = A - \frac{B}{100^2} \qquad -100 \times 100^2 = 100^2 A - B$

Examples

Solution (alternative approach)

$-100 \times 100^2 = 100^2 A - B$ (1)

At the inner surface

$r_1 = 100 \qquad \sigma_\theta = +150\ MPa$

$\sigma_\theta = A + \frac{B}{r^2}$

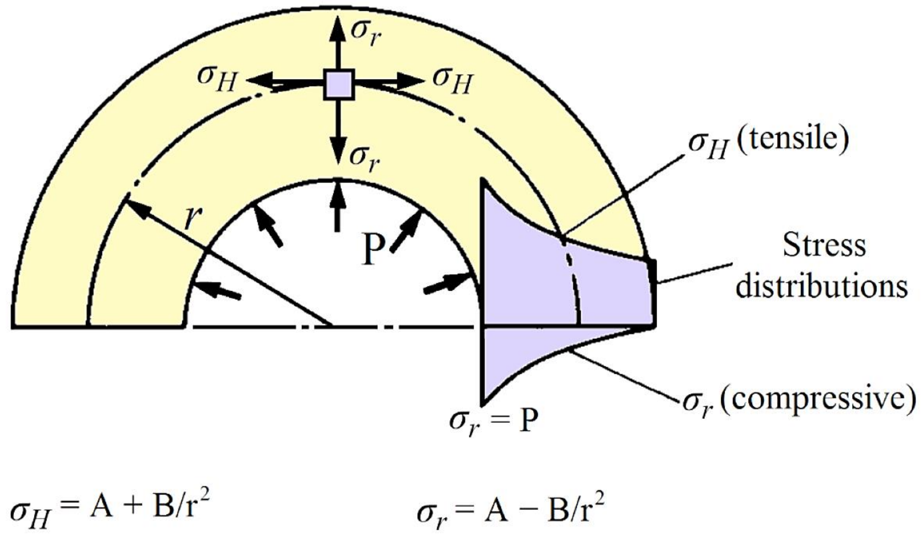
$150 = A + \frac{B}{100^2}$

$150 \times 100^2 = 100^2 A + B$ (2)

(2) - (1) $\quad (150 + 100) \times 100^2 = 2B$

$B = \frac{250 \,.\, 100^2}{2}$

$B = 125 \,.\, 100^2$



Substitute *B* into (1)

$-100 \times 100^2 = 100^2 A - 125 \,.\, 100^2$

$-100 = A - 125$

$A = 25$

At the outer surface

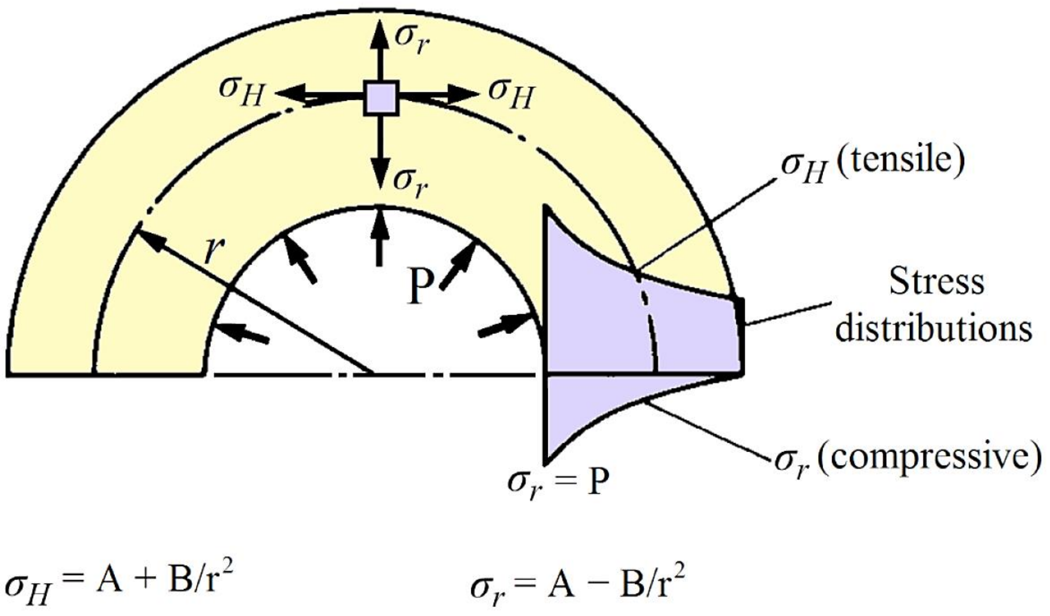
$r = r_2 \qquad \sigma_r = 0\ MPa \qquad \sigma_r = A - \frac{B}{r^2}$

Examples

Solution (alternative approach)

At the outer surface

$$r = r_2 \qquad \sigma_r = 0\ MPa \qquad \sigma_r = A - \frac{B}{r^2}$$



$$0 = 25 - \frac{125.100^2}{r_2^2}$$

$$25 = \frac{125.100^2}{r_2^2}$$

$$25r_2^2 = 125.100^2$$

$$r_2^2 = 5.100^2$$

$$r_2 = 2.2361(100)$$

$$r_2 = 223.61\ mm$$

$$d_2 = 447.22\ mm$$

With the value of d_2 known, we can now determine the hoop stress on the outer surface

$$\sigma_\theta = A + \frac{B}{r^2}$$

$$\sigma_\theta = 25 + \frac{(125)100^2}{223.61^2}$$

$$\sigma_\theta = 25 + 24.99 \qquad \sigma_\theta = 49.99\ MPa$$

Check the value of σ_θ you get using Lame Line

Examples

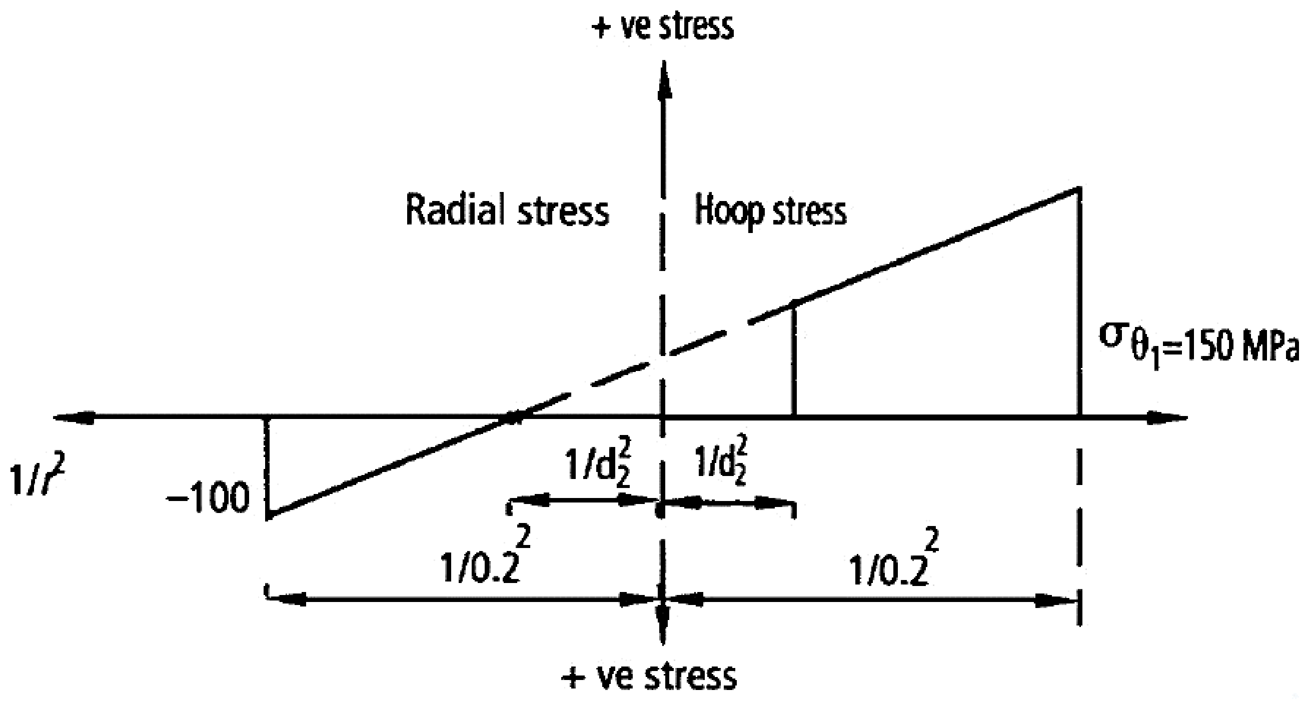
Solution

Using similar triangles

$$\frac{100}{\left(\frac{1}{0.2^2} - \frac{1}{d_2^2}\right)} = \frac{150}{\left(\frac{1}{0.2^2} + \frac{1}{d_2^2}\right)}$$

$$\frac{\left(\frac{1}{0.2^2} + \frac{1}{d_2^2}\right)}{\left(\frac{1}{0.2^2} - \frac{1}{d_2^2}\right)} \times \left(\frac{0.2^2\ d_2^2}{0.2^2\ d_2^2}\right) = 1.5$$

$$\left(\frac{d_2^2 + 0.2^2}{d_2^2 - 0.2^2}\right) = 1.5$$



$$d_2^2 + 0.2^2 = 1.5\left(d_2^2 - 0.2^2\right)$$

$$0.2^2(1+1.5) = d_2^2(1.5-1)$$

$$d_2^2 = 0.2\,\text{m}^2$$

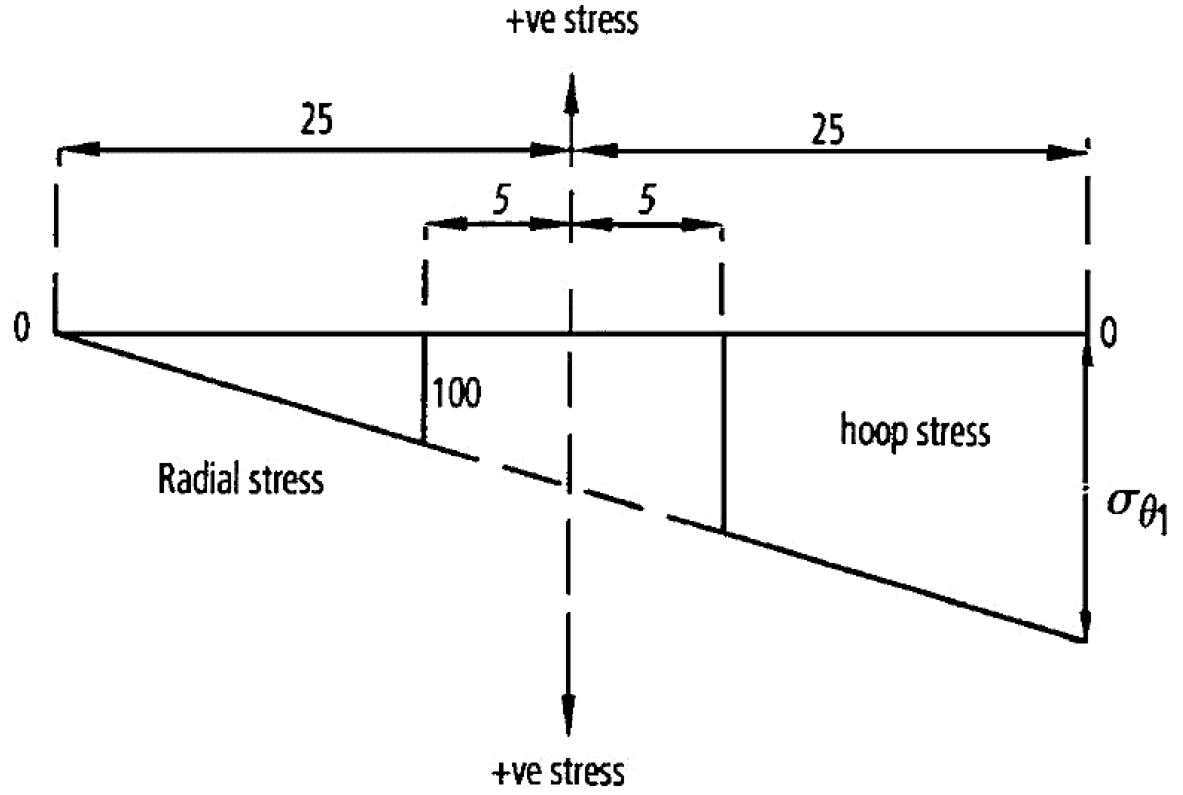
$$d_2 = 0.447\,\text{m}$$

Examples

Question 2

If the cylinder in the previous problem were subjected to an external pressure **of** 100 MPa and an internal pressure of zero, what would be the maximum magnitude of stress.

Solution



Examples

Solution

Now $\frac{1}{d_1^2} = 25$ and $\frac{1}{d_2^2} = 5,$

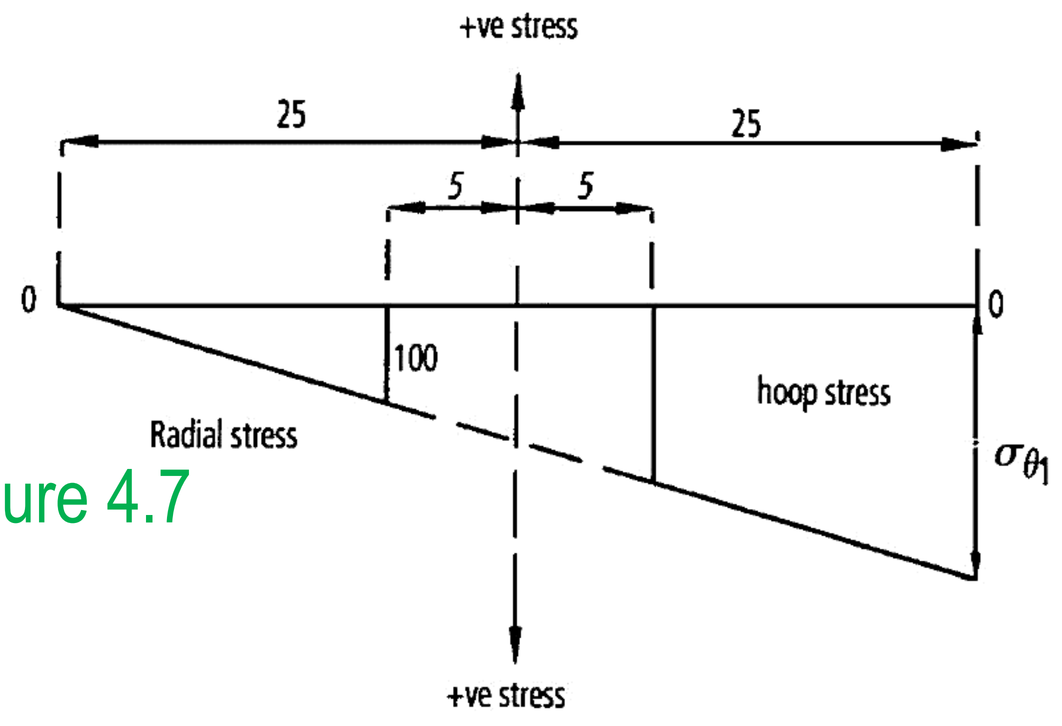


Fig. 4.7 Lame line for external pressure only.

Thus, Lame line would take the form shown in Figure 4.7

Using similar triangles,

$$\frac{-100}{(25 - 5)} = \frac{\sigma_{\theta i}}{25 + 25}$$

Note that $\sigma_{\theta 1}$ is the maximum hoop stress at the inner cylinder surface

$$\therefore \sigma_{\theta i} = \frac{-50 \times 100}{20}$$

$$= -250 \text{ MPa}$$

Use boundary conditions to solve this problem and compare results.

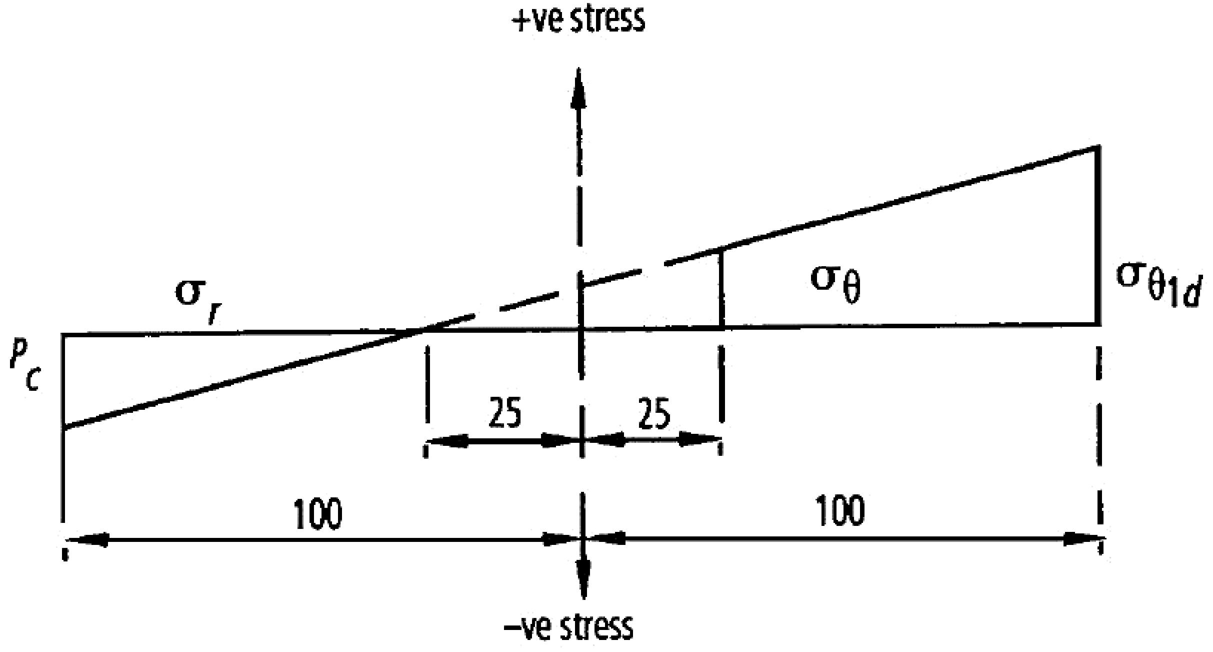
Examples

Question 3

A steel disc of external diameter 200 mm and internal diameter 100 mm is shrunk onto a solid steel shaft of external diameter 100 mm, where all the dimensions are nominal. If the interference fit, based on diameters, between the shaft and the disc at the common surface is 0.2 mm, determine the maximum stress in the assembly and comment on the final dimensions of the shaft and the disc.

For steel, $\boldsymbol{E} = 2 \times 10^{11}$N/m^2, $\nu = 0.3$

Solution



Examples

Solution

- Consider the steel disc
- Let the radial stress on the internal surfaces be the unknown Pc.
- The Lame line will take the form shown in Figure 4.8.

$\sigma_{\theta 1d}$ = hoop stress (maximum stress) on the internal surface of the disc

σ_{r1d} = radial stress on the internal surface of the disc

Using similar triangles,

$$\frac{P_c}{(100 - 25)} = \frac{\sigma_{\theta 1d}}{100 + 25}$$

$$\therefore \sigma_{\theta 1d} = \frac{125 P_c}{75}$$

$$= 1.667 P_c$$

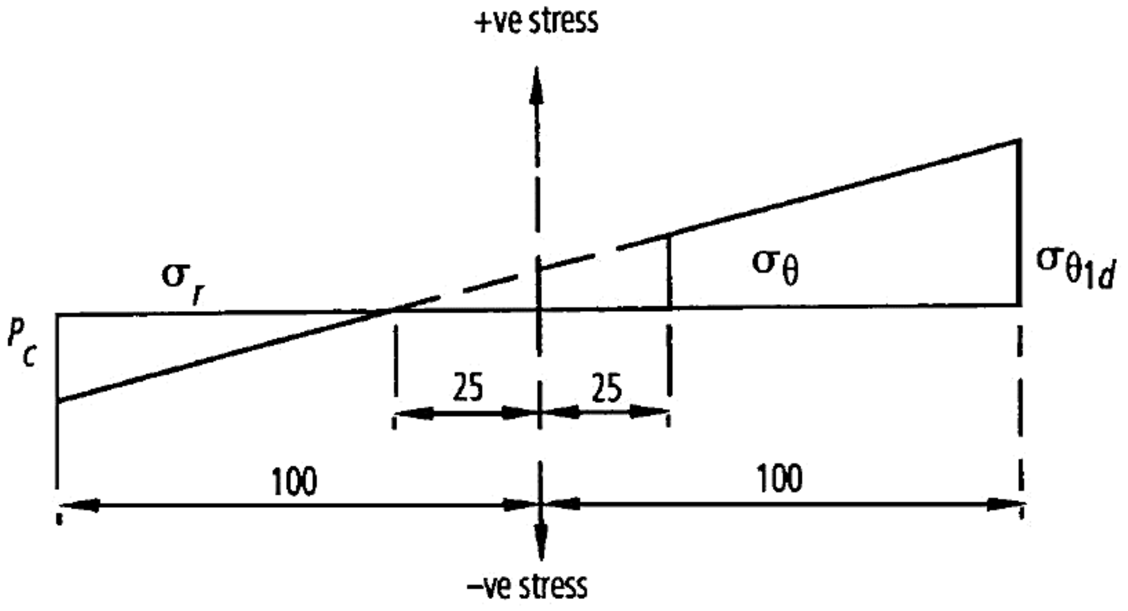


Figure 4.8 Lame line for the **steel ring.**

Examples

Solution

- We now consider the solid shaft.
- In this case, the internal diameter of the shaft is zero and **$1/0^2$ is ∞,**
- **Thus,** the Lame line must be horizontal or the shaft's hoop stress will be **infinity,** which is impossible;

Let

$$P_c = \text{external pressure on the shaft}$$

$$\therefore \sigma_r = \sigma_\theta = -P_c \text{ (everywhere)}$$

Let also,

$$w_d = \text{increase in the radius of the disc at its inner surface}$$

$$w_s = \text{increase in the radius of the shaft at its outer surface}$$

We now, apply the expression

$$E\varepsilon_\theta = \frac{w}{r} = \sigma_\theta - \nu\sigma_r - \nu\sigma_x$$

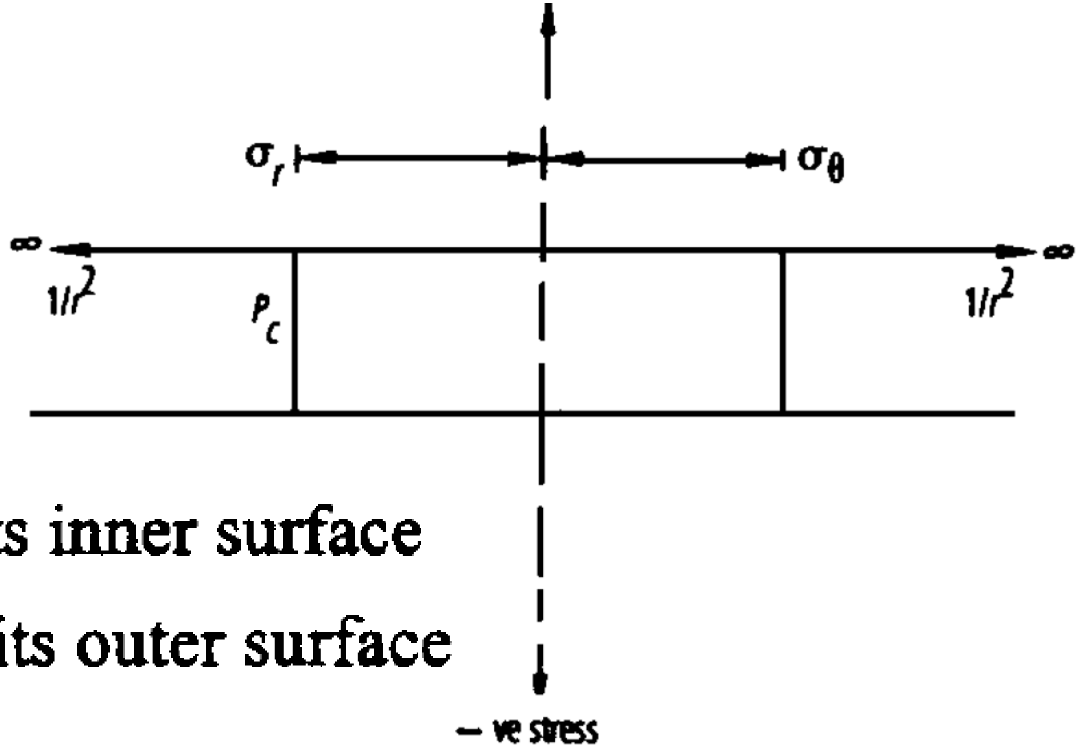


Figure 4.9 Lame line for the **solid shaft.**

Examples

Solution

We now, apply the expression

$$E\varepsilon_\theta = \frac{w}{r} = \sigma_\theta - \nu\sigma_r - \nu\sigma_x$$

Applying the above equation to the inner surface ***of*** the disc

$$\frac{EW_d}{5 \times 10^{-2}} = \sigma_{\theta 1d} - \nu\sigma_{r1d}$$

But

$$\sigma_{r1d} = -P_c$$

$$\sigma_{\theta 1d} = 1.667\ P_c$$

Therefore,

$$\frac{2 \times 10^{11} \times w_d}{5 \times 10^{-2}} = 1.667\ P_c + 0.3\ P_c$$

$$w_d = 4.918 \times 10^{-13}\ P_c$$

Examples

Solution

We now, apply the expression

$$E\varepsilon_\theta = \frac{w}{r} = \sigma_\theta - \nu\sigma_r - \nu\sigma_x$$

Similarly, for the **shaft**

$$\frac{Ew_s}{5 \times 10^{-2}} = \sigma_{\theta s} - \nu\sigma_{rs}$$

But $\sigma_{\theta s} = \sigma_{rs} = P_c$

$$\therefore \frac{2 \times 10^{11}\, w_s}{5 \times 10^{-2}} = -P_c\,(1 - \nu)$$

$$w_s = -1.75 \times 10^{-13}\, P_c$$

But $w_d - w_s = 2 \times 10^{-3}/2$

$$(4.918 \times 10^{-13} + 1.75 \times 10^{-13})\, P_c = 1 \times 10^{-4} \qquad \therefore P_c = 150 \text{ MPa}$$

Examples

Solution

$$\therefore P_c = 150 \text{ MPa}$$

We now compute the maximum stress

$$\sigma_{\theta 1d} = 1.667 P_c$$

$$\approx 250 \text{ MPa}$$

We now compute the dimensions:

Shaft:

$$w_s = -1.75 \times 10^{-13} P_c$$

Disc:

$$w_d = 4.918 \times 10^{-13} P_c$$

Evaluate and conclude

MEC 3352 QUIZ 3

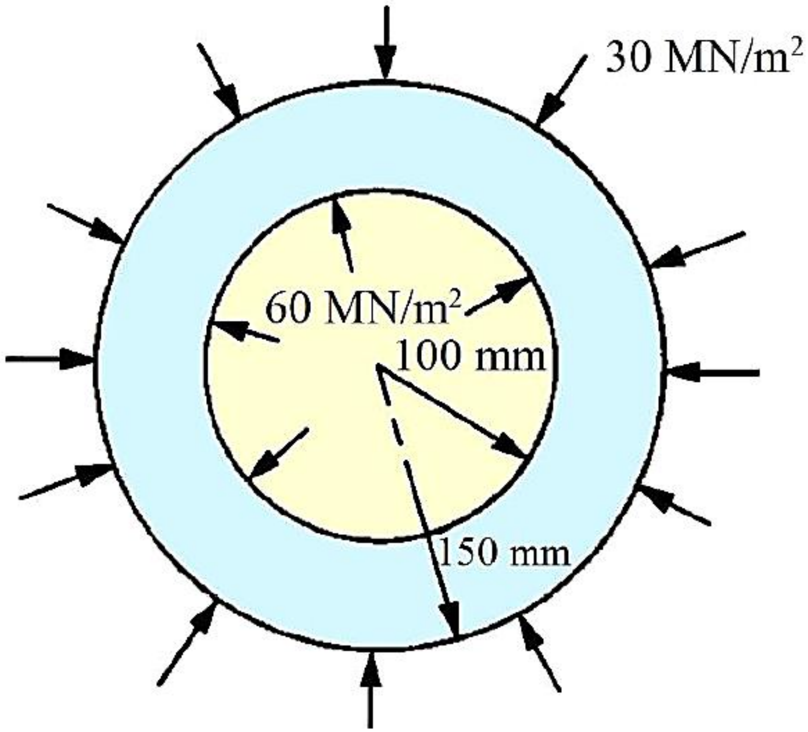
A thick cylinder of 200 mm internal diameter and 150 mm external radius is subjected to an internal pressure of 60MPa and an external pressure of 30MPa. Determine the hoop and radial stresses at the inside and outside of the cylinder together with the longitudinal stress if the cylinder is assumed to have closed ends. Comment of the value of the longitudinal stress.

MEC 3352 QUIZ 3

Solution

at $r = 0.1$ m, $\sigma_r = -60$ MN/m^2

at $r = 0.15$ m, $\sigma_r = -30$ MN/m^2



$$\sigma_\theta = A + \frac{B}{r^2} \qquad \sigma_r = A - \frac{B}{r^2}$$

$$-30 = A - 44.5B \qquad (1)$$

$$-60 = A - 100B \qquad (2)$$

Subtracting (2) from (1), $-30 = -55.5B$

$$B = 0.54$$

From (1) $A = -60 + (100 \times 0.54) \qquad A = -6$

Therefore, at $r = 0.1$ m,

$$\sigma_H = A + \frac{B}{r^2} = -6 + 0.54 \times 100$$

$$= \mathbf{48\ MN/m^2}$$

Questions

Question 1

Determine the maximum and minimum hoop stress across the section of pipe of 400mm internal diameter and 100mm thick, the pipe contains a fluid at a pressure of $8 N/mm^2$. Also sketch the radial pressure distribution and hoop stress distribution across the section.

Question 2

Find the thickness of metal necessary for a cylindrical shell of internal diameter 160mm to withstand an internal pressure of $8 N/mm^2$. The maximum hoop stress in the section is not to exceed $35 N/mm^2$.

Ti Amo Signore