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Strength of Materials II

Thick Spherical Shells

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Thick Spherical Shells

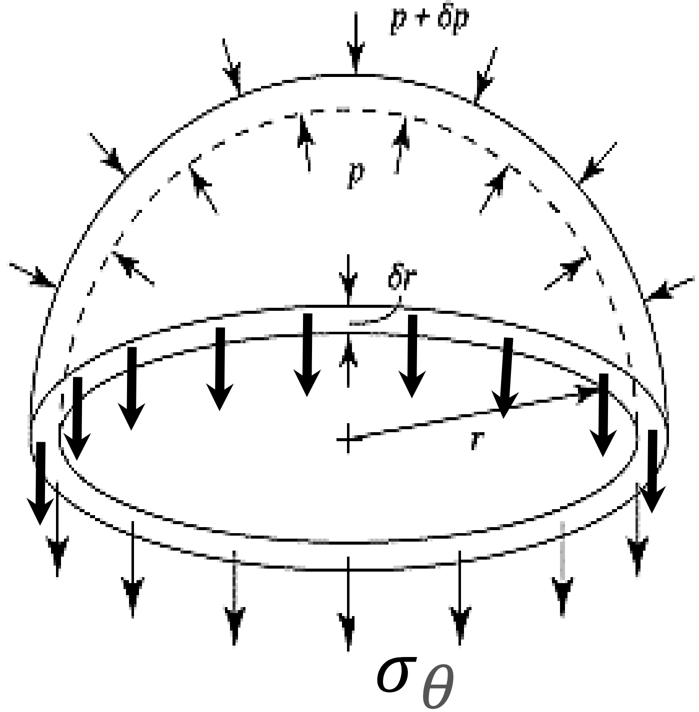

Fig. 4.0: Thick hemispherical shell element

Consider a thick hemispherical shell element of radius *r,* under a compressive radial stress *P,* as shown in Figure 4.0

Let w be the radial deflection at any radius r, so that

$$Hoop\ strain(\sigma_\theta) = {}^{w}/_{r}$$

$$Radial\ \ strain(P) = {}^{dw}/_{dr}$$

From three-dimensional stress-strain relationships,

$$E\,\frac{w}{r} = \sigma - \nu\sigma + \nu P \qquad 4.1$$

$$E\,\frac{dw}{dr} = -P - \nu\sigma - \nu\sigma = -P - 2\nu\sigma \qquad 4.2$$

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Multiplying 4.1 by r

$$Ew = \sigma * r - \nu\sigma * r + \nu P * r$$

And differentiating with respect to r

$$E\frac{dw}{dr} = \sigma + r\frac{d\sigma}{dr} - \nu\sigma - \nu r\frac{d\sigma}{dr} + \nu P + \nu r\frac{dP}{dr}$$

Which gives,

$$E\frac{dw}{dr} = (1-\nu)\left(\sigma - r\frac{d\sigma}{dr}\right) + \nu\left(P + r\frac{dP}{dr}\right) \qquad 4.3$$

Equating 4.2 to 4.3,

$$P - 2\nu\sigma = (1-\nu)\left(\sigma - r\frac{d\sigma}{dr}\right) + \nu\left(P + r\frac{dP}{dr}\right)$$

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Simplifying, we get

$$(1+\nu)(\sigma+P)+r(1-\nu)\frac{d\sigma}{dr}+\nu r\frac{dP}{dr}=0 \qquad 4.4$$

Considering now the equilibrium of the hemispherical shell element,

$$\sigma * 2\pi r * dr = P * \pi r^2 - (P + dP) * \pi * (r + dr)^2 \qquad 4.5$$

Neglecting higher order terms, we get

$$\sigma + P = \left(\frac{-r}{2}\right)\frac{dP}{dr} \qquad 4.6$$

Substituting equation (4.6) into equation (4.4),

$$\left(\frac{-r}{2}\right)\frac{dP}{dr}(1+\nu)+r(1-\nu)\left(\frac{d\sigma}{dr}\right)+\nu r\left(\frac{dP}{dr}\right)=0 \qquad 4.7$$

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Simplifying, we get

$$\frac{d\sigma}{dr} - \frac{1}{2}\frac{dP}{dr} = 0 \qquad 4.7$$

Integrating, we get

$$\sigma - \frac{P}{2} = A \;\Rightarrow \sigma = A + \frac{P}{2} \qquad 4.8$$

Substituting equation (4.8) into equation (4.6)

$$A + \frac{P}{2} + P = \left(\frac{-r}{2}\right)\frac{dP}{dr} \qquad \frac{3P}{2} + A = \left(\frac{-r}{2}\right)\frac{dP}{dr}$$

We get,

$$3P + (r)\frac{dP}{dr} = -2A \qquad \Longrightarrow \frac{d(Pr^3)}{dr} = -2Ar^2$$

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Integrating, $$\frac{d(Pr^3)}{dr} = -2Ar^2$$

$$Pr^3 = \frac{-2Ar^3}{3} + B$$

$$P = \frac{B}{r^3} - \frac{2A}{3} \qquad 4.9$$

Substituting equation (4.9) into equation (4.8)

$$\sigma = A + \frac{1}{2}\left(\frac{B}{r^3} - \frac{2A}{3}\right)$$

$$\sigma = A + \left(\frac{1}{2} * \frac{B}{r^3} - \frac{A}{3}\right)$$

$$\therefore \qquad \sigma_\theta = \frac{2A}{3} + \frac{B}{2r^3} \qquad 4.10$$

By putting: $a = \frac{2A}{3}$ and $b = 8B$

We then have the general equations:

$$P = -a + \frac{b}{d^3} \tag{4.11}$$

and

$$\sigma_\theta = a + \frac{b}{2d^3} \tag{4.12}$$

If the inside and outside diameters are d_1 and d_2 and the pressures on these surfaces are P_1 and P_2 respectively, we can write Equation (4.11):

$$P_1 = -a + \frac{b}{d_1^3} \quad \text{and} \quad P_2 = -a + \frac{b}{d_2^3}$$

Solving these equations for a and b gives:

$$b = \frac{(P_1 - P_2) d_1^3 d_2^3}{d_2^3 - d_1^3}$$

and

$$a = \frac{b}{d_1^3} - P_1 = \frac{P_1 d_1^3 - P_2 d_2^3}{d_2^3 - d_1^3}$$

Hence, from Equations (4.11) and (4.12);

$$P = \frac{P_2 d_2^3 - P_1 d_1^3}{d_2^3 - d_1^3} + \frac{(P_1 - P_2) d_1^3 d_2^3}{(d_2^3 - d_1^3) d^3} \tag{4.13}$$

and

$$\sigma_\theta = \frac{P_1 d_1^3 - P_2 d_2^3}{(d_2^3 - d_1^3)} + \frac{(P_1 - P_2) d_1^3 d_2^3}{(d_2^3 - d_1^3) 2 d^3} \tag{4.14}$$

Equations (4.13) and (4.14) are the general equations for thick spherical shells whose inside and outside diameters are d_1 and d_2 and whose pressures on these surfaces are P_1 and P_2, respectively.

Specific cases are then determined according to specific scenarios.

1. **If there is internal pressure only (P_2 = 0):**

The Equations (4.13) and (4.14) become:

$$P = \frac{P_1 d_1^3}{d_2^3 - d_1^3}\left(\frac{d_2^3}{d^3} - 1\right) \tag{4.15}$$

and

$$\sigma_\theta = \frac{P_1 d_1^3}{d_2^3 - d_1^3}\left(\frac{d_2^3}{2d^3} + 1\right) \tag{4.16}$$

The Maximum Stress is the value of σ_θ at the inside radius i.e.

$$\sigma_{\theta_{\max}} = \frac{P_1\left(d_2^3+2d_1^3\right)}{2\left(d_2^3-d_1^3\right)} \tag{4.17}$$

And the maximum Shear Stress, at the inner radius:

$$\tau\mathrm{max} = \frac{1}{2}(\sigma_\theta + P_1) = \frac{3P_1 d_2^3}{4\left(d_2^3-d_1^3\right)} \tag{4.18}$$

2. If there is external pressure only (P_1 = 0):

This is an unusual situation for thick spherical shells, but the student may easily determine the equations for radial and hoop stresses at any diameter, *d*, using equations (4.13) and (4.14).

Example

A thick spherical shell of 200mm internal diameter is subjected to an internal fluid pressure of 7N/mm^2. If the permissible stress in the shell material is 8N/mm^2, compute the;

(a) thickness of the shell
(b) minimum value of the hoop stress
(c) maximum shear stress

Assignment 2

For a thick spherical shell of internal and external radii r_1 and r_2, respectively, under external pressure p_o, find the expressions for the following:

(a) The radial stress at any radius r;

(b) The circumferential stress at any radius r;

(c) The maximum radial stress and where it occurs;

(d) The maximum circumferential stress and where it occurs; and

(e) The maximum shear stress and where it occurs.

Grazie Fratelli