MEC 3352

STRENGTH OF MATERIALS II

Rotating Discs and Cylinders – Part 1

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Brainstorming

1. What is a disk?
2. What is a cylinder?
3. What is the difference between the two?
4. Can a disc be considered a cylinder or a cylinder considered a disc?

Rotating Discs and Cylinders

Brake Disc

Flywheel

Concrete Truck Mixer

ROTATING DISCS AND CYLINDERS

Introduction

These notes relate to the stresses and strains existing in rotating discs (like flywheels) and thick walled cylinders.

The primary assumption is that the cylinders are not subject to internal or external pressure.

Symbols/Units

Tensile stresses are considered positive and compressive stresses are negative.

- p_1 = Internal pressure (MPa, N/m^2)
- p_2 = External pressure (MPa, N/m^2)
- σ_r = Radial stress (MPa, N/m^2)
- σ_t = Tangential (Hoop) stress (MPa, N/m^2)
- σ_a = Axial/longitudinal stress (MPa, N/m^2)
- E = Young's modulus (MPa, N/m^2)
- ρ = Density (kg/m^3)

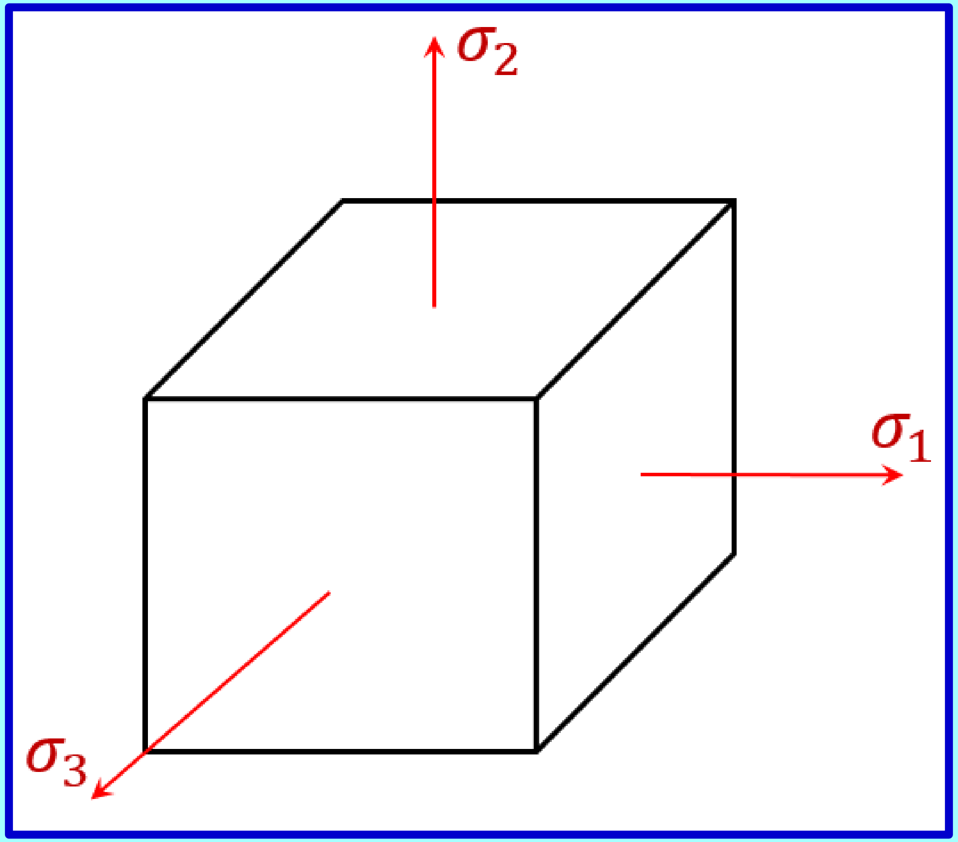
- ν = Poisson's ratio
- r = Radius at point of analysis (m, mm)
- R_1 = Internal radius (m, mm)
- R_2 = External radius (m, mm)
- ε_r = Radial strain
- ε_t = Tangential (Hoop) strain
- ε_a = Axial/longitudinal strain
- u = Radial deflection (m, mm)

Initial Assumptions

For an infinitesimal cube acted upon by the stresses σ_1, σ_2 and σ_3, then ε_1, ε_2 and ε_3 are the strains associated with the stresses.
ν and E are the Poisson's ratio and Young's modulus, respectively.

These strains are given by the relations:

$$\varepsilon_1 = \sigma_1/E - \nu\sigma_2/E - \nu\sigma_3/E$$
$$\varepsilon_2 = \sigma_2/E - \nu\sigma_1/E - \nu\sigma_3/E$$
$$\varepsilon_3 = \sigma_3/E - \nu\sigma_1/E - \nu\sigma_2/E$$



a) Thick Disc Basics

- Consider a “disc”/“thin ring” subject to internal stresses resulting from the internal forces as a result of its rotational velocity, ω.

- Under the action of the internal forces only, the three principal stresses will be the tensile radial stress $\boldsymbol{\sigma_r}$, the tensile tangential stress $\boldsymbol{\sigma_t}$ and an axial stress $\boldsymbol{\sigma_a}$ which is generally also tensile.

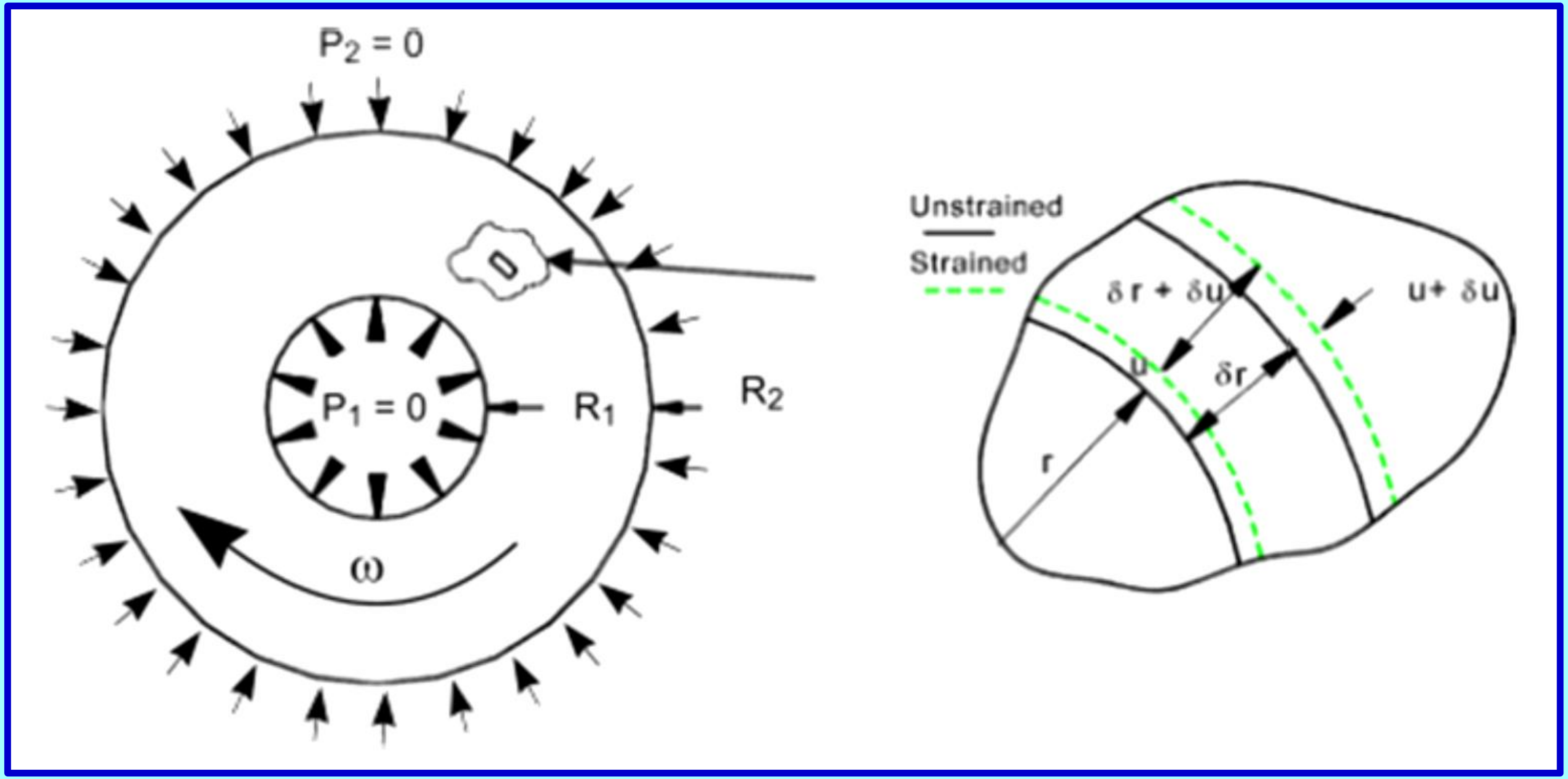
- The stress conditions occur throughout the section and vary primarily relative to the radius r.

- It is assumed that the axial stress σ_a is constant along the length of the section and because the disc is thin compared to its diameter, the axial stress throughout the section is assumed zero.

- It is also assumed that there is no internal or external pressure. So, $P_1 = P_2 = 0$.

Consider a microscopically small element of the cylinder at radius r and δr thick, rotating at an angular velocity of ω, under stresses σ_t, σ_r and σ_a.

Let u be the radial displacement at radius r.



The circumferential (Hoop) strain due to the internal pressure is:

$$\varepsilon_t = \frac{Increase\ in\ circumference}{Original\ circumference} = \frac{2\pi(r+u) - 2\pi r}{2\pi r} = \frac{u}{r}$$

At the outer radius of the small sectional area $(r + \delta r)$, the radius will increase by $(u + \delta u)$. The resulting radial strain as $\delta r \rightarrow 0$ is

$$\varepsilon_r = \frac{Increase\ in\ \delta r}{\delta r} = \frac{u + \delta u - u}{\delta r} = \frac{\delta u}{\delta r}$$

Referring to the stress/strain relationships as stated above. The following equations are derived:

Basis of equations:

We can say:

σ_t is equivalent to σ_1

σ_r is equivalent to σ_2

σ_a is equivalent to σ_3

Derived Equations:

Strictly, the following equations apply:

$$E\varepsilon_a = \sigma_a - \nu\sigma_t - \nu\sigma_r \qquad (1)$$

$$E\varepsilon_t = E\frac{u}{r} = \sigma_t - \nu\sigma_a - \nu\sigma_r \qquad (2)$$

$$E\varepsilon_r = E\frac{du}{dr} = \sigma_r - \nu\sigma_t - \nu\sigma_a \qquad (3)$$

However, because of the assumption that $\sigma_a = 0$ the three equations reduce to:

$$\boldsymbol{E\varepsilon_a = 0 - \nu\sigma_t - \nu\sigma_r = -\nu\sigma_t - \nu\sigma_r} \qquad (1)$$

$$\boldsymbol{E\varepsilon_t = E\frac{u}{r} = \sigma_t - \nu\sigma_r} \qquad (2)$$

$$\boldsymbol{E\varepsilon_r = E\frac{du}{dr} = \sigma_r - \nu\sigma_t} \qquad (3)$$

Multiplying (2) by r

$$Eu = r(\sigma_t - \nu\sigma_r)$$

Differentiating:

$$E\frac{du}{dr} = \sigma_t - \nu\sigma_r + r\left[\frac{d\sigma_t}{dr} - \nu\frac{d\sigma_r}{dr}\right] = \sigma_r - \nu\sigma_t \qquad \text{from (3)}$$

Simplifying by collecting terms:

$$(\sigma_t - \sigma_r)(1 + \nu) + r\left(\frac{d\sigma_t}{dr}\right) - \nu r\left(\frac{d\sigma_r}{dr}\right) = 0 \qquad (4)$$

Now considering the radial equilibrium of the element of the section. Forces based on unit length of cylinder:

Given a small element of unit width,

length $= r\delta\theta$ and thickness $= \delta r$;

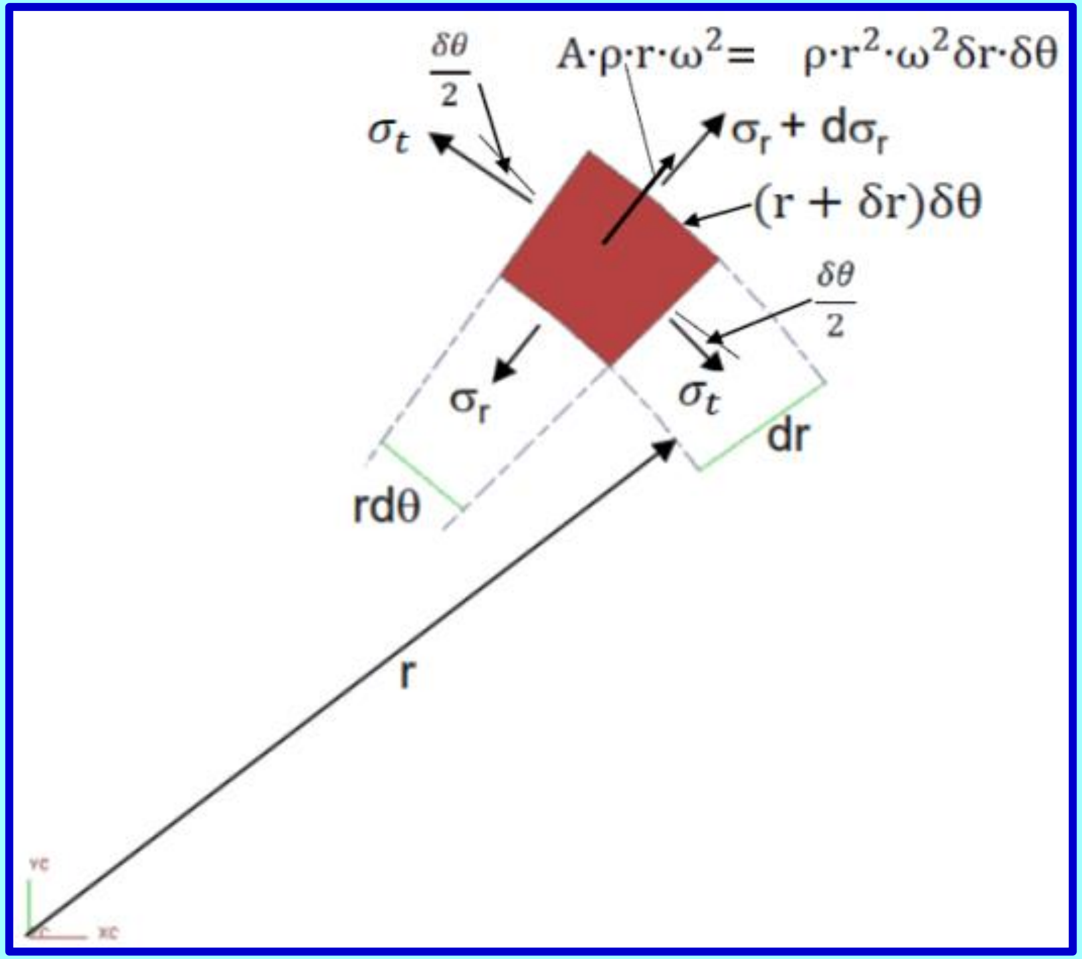
Centrifugal force = $m\omega^2 r = \rho r\delta\theta\delta r\omega^2 r$

$$= \rho r^2\omega^2\delta r\delta\theta$$

And this can be equated to radial forces, i.e.:

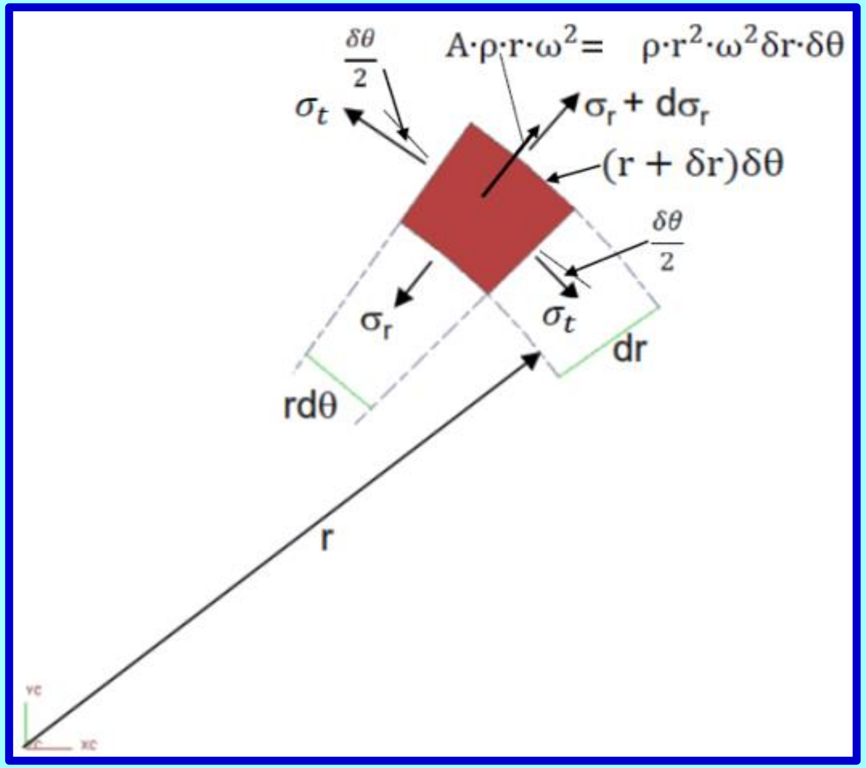
$$2\cdot\sigma_t\cdot\delta r\cdot\sin\left(\frac{\delta\theta}{2}\right)+\sigma_r\delta\theta$$

$$-(\sigma_r+\delta\sigma_r)(r+\delta r)\delta\theta=\rho r^2\omega^2\delta r\delta\theta$$



In the limit, this reduces to

$$\boldsymbol{\sigma_t - \sigma_r - r\left(\frac{d\sigma_r}{dr}\right) = \rho r^2 \omega^2} \qquad (5)$$



Sunstitute for $\sigma_t - \sigma_r$ into Eq. (4) results in

$$\left(r\frac{d\sigma_r}{dr} + \rho r^2\omega^2\right)(1+\nu) + r\frac{d\sigma_t}{dr} - \nu r\frac{d\sigma_r}{dr} = 0$$

Therefore, multiplying out and collecting terms yields:

$$\frac{d\sigma_t}{dr} + \frac{d\sigma_r}{dr} = -\rho r\omega^2(1+\nu)$$

Integrating:

$$\boldsymbol{\sigma_t + \sigma_r = -\frac{\rho r^2 \omega^2 (1+\nu)}{2} + 2A} \qquad (6)$$

Subtract Eq. (5):

$$2\sigma_r + r\frac{d\sigma_r}{dr} = -\frac{\rho r^2 \omega^2 (3+\nu)}{2} + 2A$$

This is the same as:

$$\left(\frac{1}{r}\right)\frac{d(\sigma_r r^2)}{dr} = -\frac{\rho r^2 \omega^2 (3+\upsilon)}{2} + 2A$$

Integrating:

$$\sigma_r r^2 = -\frac{\rho r^4 \omega^2 (3+\nu)}{8} + Ar^2 + B$$

Dividing by r^2:

$$\boldsymbol{\sigma_r = A - \frac{B}{r^2} - \frac{(3+\nu)\rho r^2 \omega^2}{8}} \tag{7}$$

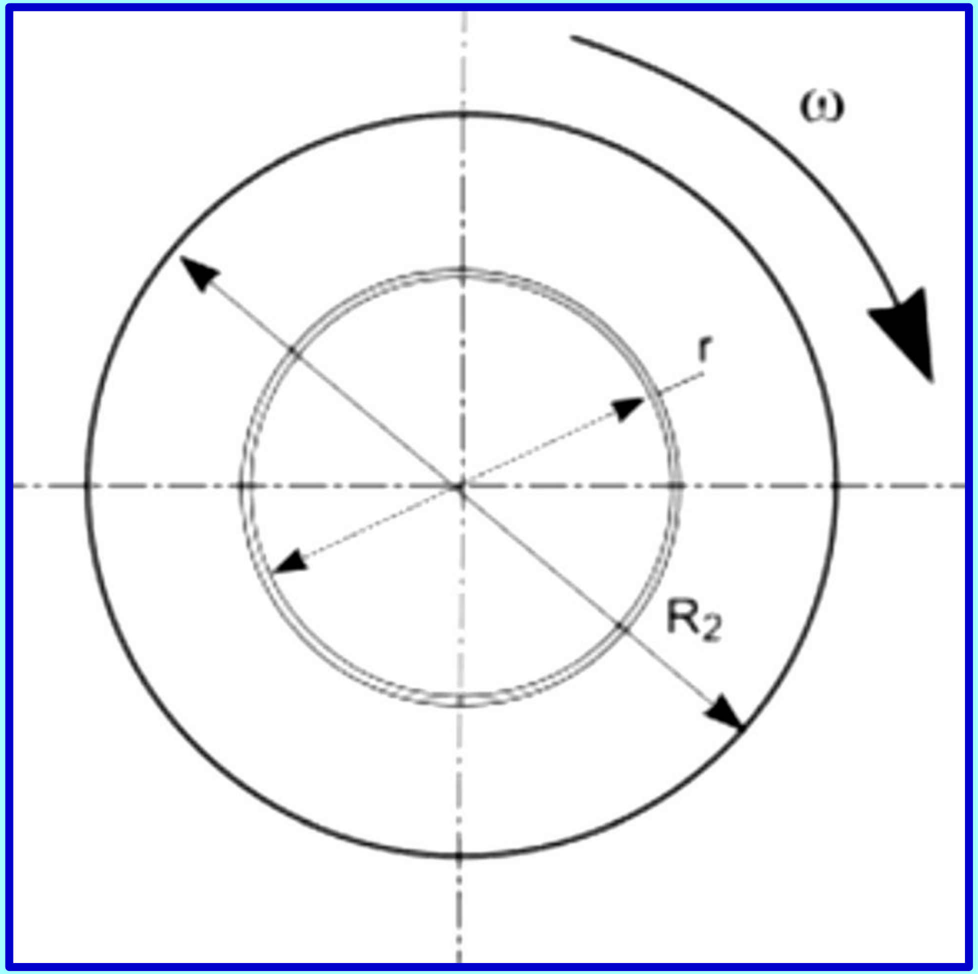
Combining Eq. (7) with Eq. (6) (i.e. substituting σ_r from Eq. (7) into Eq. (6):

$$\boldsymbol{\sigma_t = A + \frac{B}{r^2} - \frac{(1+3\nu)\rho r^2 \omega^2}{8}} \tag{8}$$

The equations, (7) and (8), are the general equations for a rotating disc. What remains now is to apply these equations to specific situations of the disc and apply the relevant boundary conditions to reduce the equations to the specific situation under consideration.

(i) Solid Disk

From (7), it can be seen that at the centre, where $r = R_1 = 0$, the B/r^2 term implies infinite stresses which are clearly not credible and, therefore, B must be 0.



At $r = R_2$ on the outside edge of the disk, the radial stress is equal to the surface stress which is equal to 0.

Therefore, at R_2:

$$\sigma_r = 0 = A - \frac{(3+\nu)\rho R_2^2 \omega^2}{8}$$

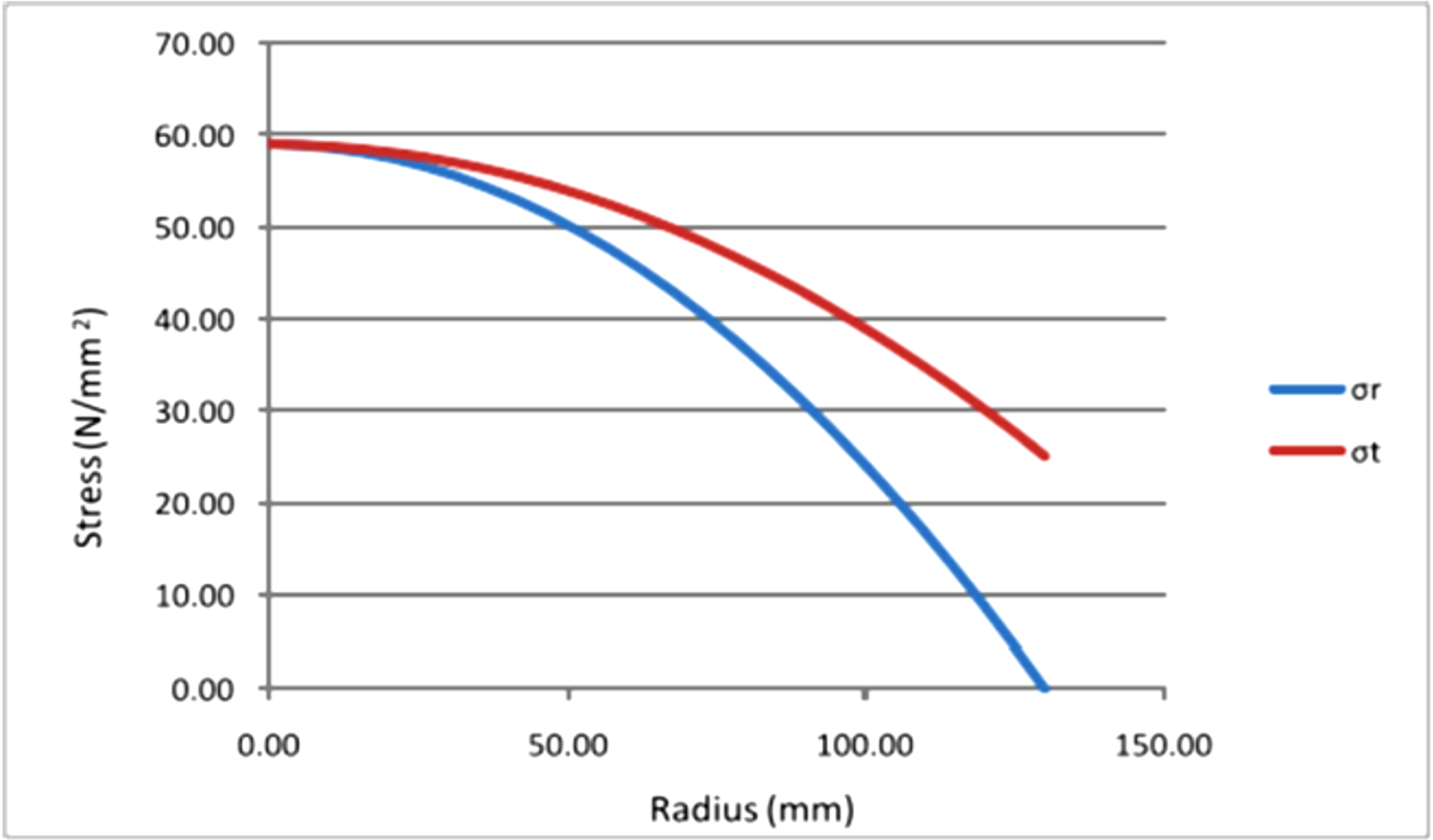
Therefore $A = \frac{(3+\nu)\rho R_2^2\omega^2}{8}$ and with $B = 0$,

$$\sigma_t = \left(\frac{\rho\omega^2}{8}\right)\left[(3+\nu)R_2^2 - (1-3\nu)r^2\right] \qquad (9)$$

$$\sigma_r = \left(\frac{\rho\omega^2}{8}\right)\left[(3+\nu)R_2^2 - r^2\right] \qquad (10)$$

The maximum stress is at the centre as

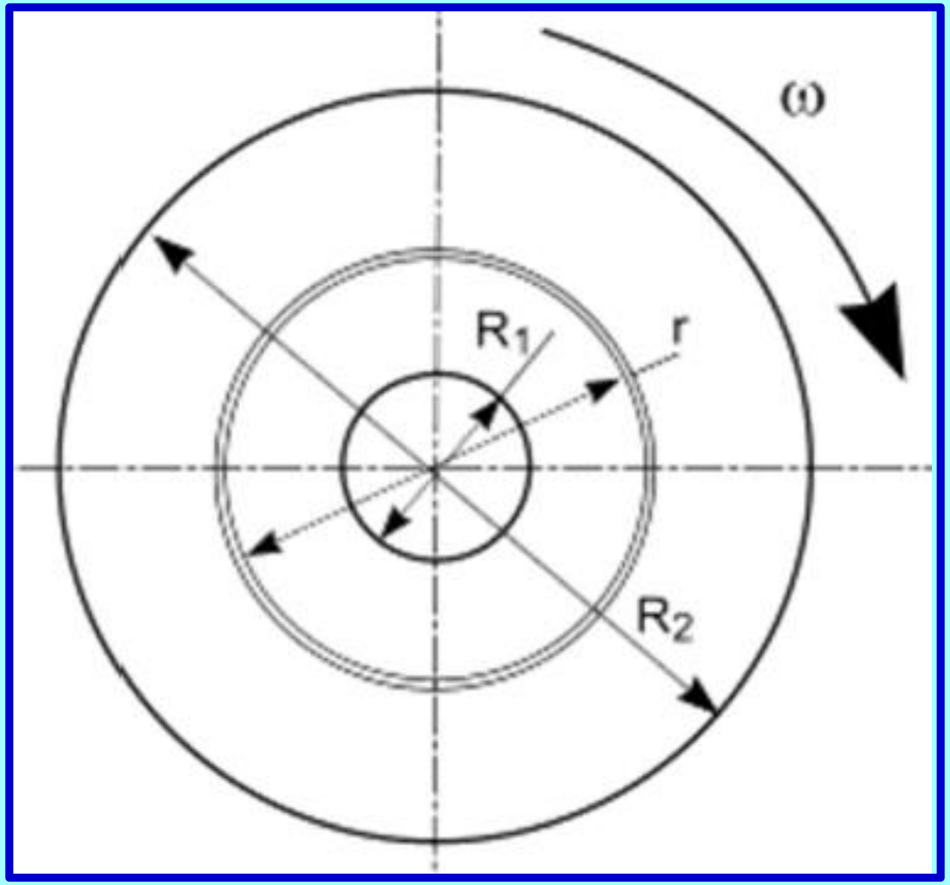
$$\sigma_{t\max} = \sigma_{r\max} = \left(\frac{\rho\omega^2}{8}\right)(3+\nu)R_2^2 \qquad \text{at } r = R_1 = 0 \qquad (11)$$



250mm diameter disc at 10,000 rpm

(ii) Disk with a Central Hole

At the outside edge, $r = R_2$ and at the hole radius, $r = R_1$, the radial stress is assumed to be 0.



Therefore, from (7):

$$\sigma_{r_{R_2}} = 0 = A - \frac{B}{R_2^2} - (3+\nu)\left(\frac{\rho R_2^2 \omega^2}{8}\right)$$

$$\sigma_{r_{R_1}} = 0 = A - \frac{B}{R_1^2} - (3+\nu)\left(\frac{\rho R_1^2 \omega^2}{8}\right)$$

Solving:

$$B = -(3+\nu)\frac{\rho\omega^2}{8} \cdot (R_1^2 \cdot R_2^2)$$

$$A = (3+\nu)\frac{\rho\omega^2}{8} \cdot (R_1^2 + R_2^2)$$

Therefore:

$$\boldsymbol{\sigma_r = (3+\nu)\frac{\rho\omega^2}{8} \cdot \left(R_1^2 + R_2^2 - \frac{R_1^2 \cdot R_2^2}{r^2} - r^2\right)} \qquad (12)$$

and

$$\boldsymbol{\sigma_t = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_1^2 + R_2^2 + \frac{R_1^2 \cdot R_2^2}{r^2}\right) - (1+3\nu)r^2\right]} \qquad (13)$$

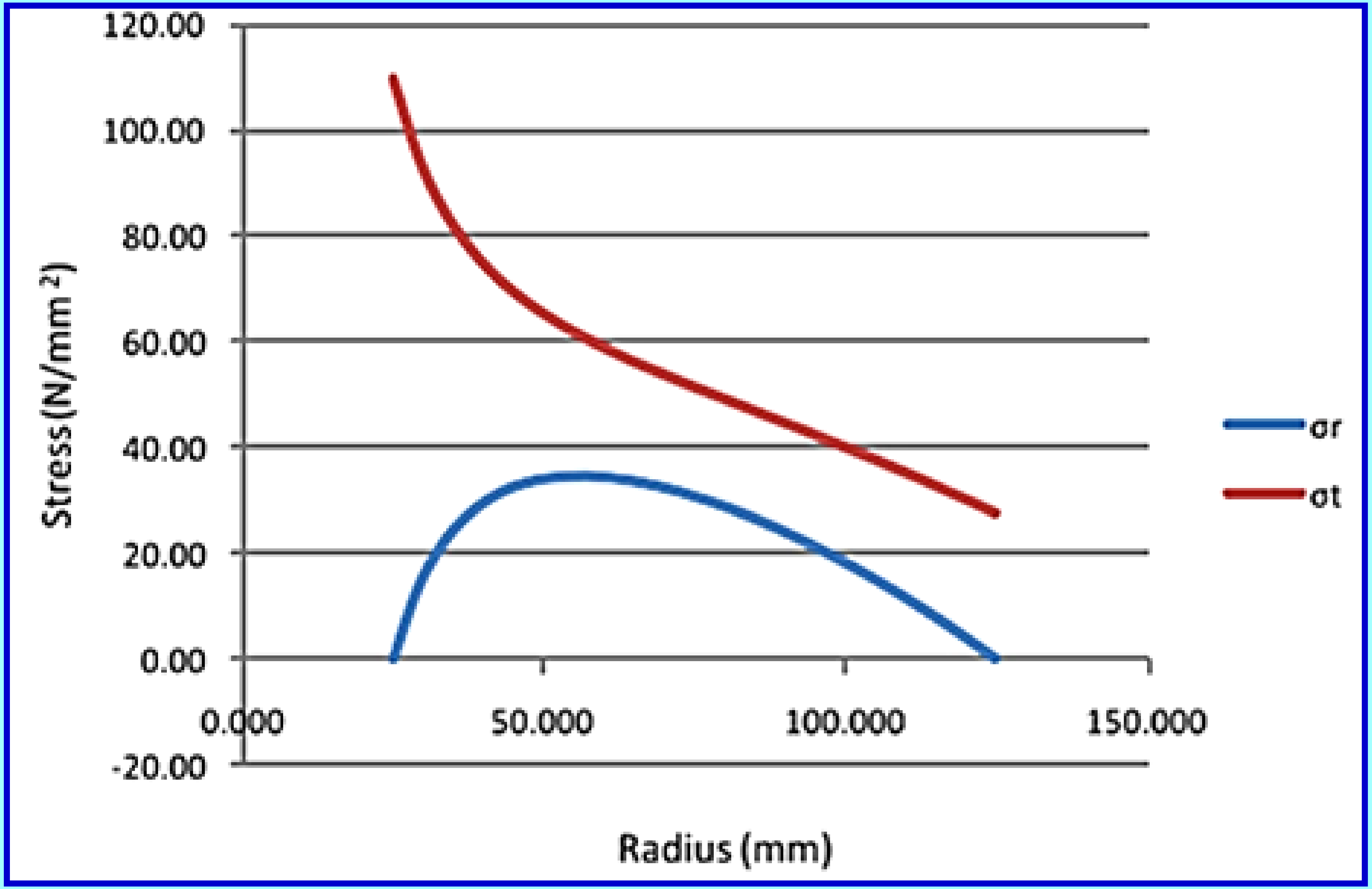
The maximum tangential stress σ_t is at the inside hole surface, where $r = R_1$ and equals:

$$\boldsymbol{\sigma_{t\max} = \left(\frac{\rho\omega^2}{2}\right)\left[(1-\nu)R_1^2 + (3+\nu)R_2^2\right]} \qquad \text{at } r = R_1 \qquad (14)$$

The maximum radial stress σ_r equals:

$$\boldsymbol{\sigma_{r\max} = (3+\nu)\left(\frac{\rho\omega^2}{8}\right)(R_2 - R_1)^2} \qquad \text{at } r = \sqrt{R_1 R_2} \qquad (15)$$

Eq. (15) is obtained by differentiating (12) and equating to 0, $\boldsymbol{\frac{d\sigma_r}{dr} = 0}$, to get a local maximum or a minimum. The second derivative must be negative, i.e. $\boldsymbol{\frac{d^2\sigma_r}{dr^2} < 0}$, for a maximum point. Solving yields $r = \sqrt{R_1 R_2}$ in (15).



250mm OD × 50mm ID ring running at 10,000 rpm

Example 1, P. 289: A thin uniform steel disc of 25 cm diameter, with a central hole of 5 cm diameter, runs at 10,000 rpm. Calculate the maximum principal stress and the maximum shearing stress in the disc if $\nu = 0.3$ and density = 7.7 Mg/m^3.

The maximum principal stress is

$$\hat{\sigma}_1 = (\rho\omega^2/4)[(1-\nu)R_1{}^2 + (3+\nu)R_2{}^2]$$

$$= \frac{7700}{4}\left(\frac{10{,}000 \times 2\pi}{60}\right)^2 (0{\cdot}7 \times 0{\cdot}025^2 + 3{\cdot}3 \times 0{\cdot}125^2)\ \mathrm{N/m^2}$$

$$= 110\ \mathrm{N/mm^2}$$

The maximum shearing stress at any radius

$$= \tfrac{1}{2}(\sigma_1 - \sigma_2)$$
$$= (\rho\omega^2/8)[(3+\nu)R_1{}^2R_2{}^2/r^2 + (1-\nu)r^2]$$

It is clear from Fig. 16.3 that the greatest stress difference occurs at $r = R_1$.

Then maximum shearing stress

$$= \frac{7700}{8}\left(\frac{10{,}000 \times 2\pi}{60}\right)^2\left(3{\cdot}3 \times \frac{0{\cdot}025 \times 0{\cdot}125^2}{0{\cdot}025^2} + 0{\cdot}7 \times 0{\cdot}025^2\right)\ \mathrm{N/m^2}$$

$$= 55\ \mathrm{N/mm^2}$$

Note that if R_1 is very small, $\hat{\sigma}_1 \to (3+\nu)(\rho\omega^2 R_2^2/4)$, which is *twice the value for a solid disc* (Para. 16.2).

At the outside

$$\sigma_1 = (\rho\omega^2/4)[(3+\nu)R_1^2 + (1-\nu)R_2^2]$$

If $R_1 \to R_2 = R$, then

$$\hat{\sigma}_1 \to \rho\omega^2 R^2$$

as in the case of a thin rotating cylinder (Para. 15.7).

The variation of stresses is shown in Fig. 16.3.

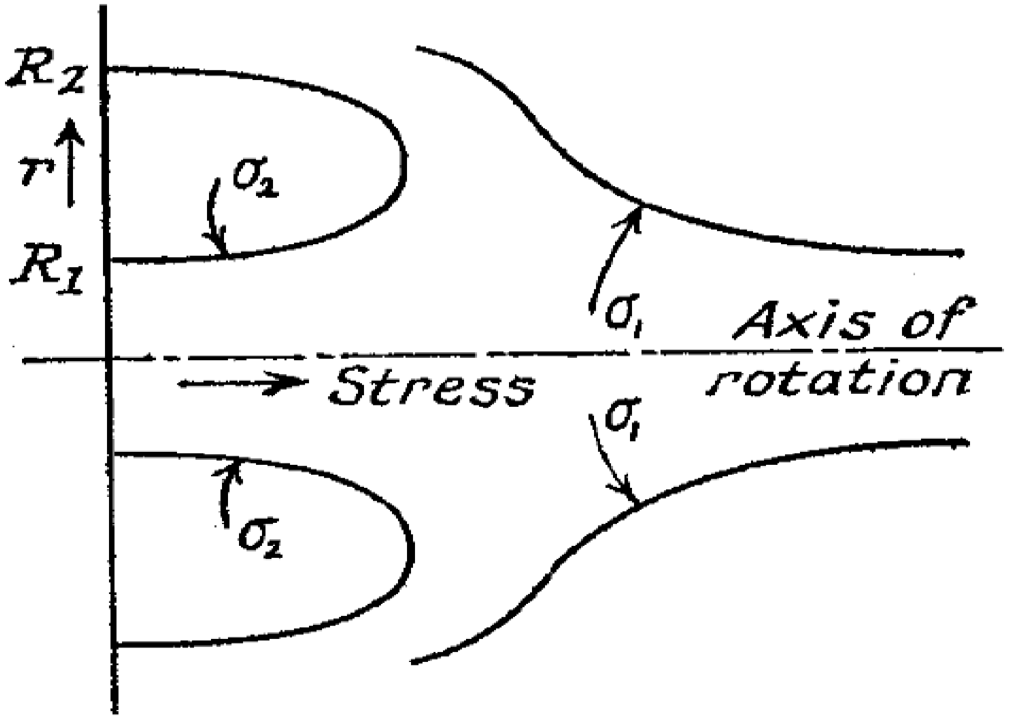


Fig. 16.3

To be continued.

Enjoy ME 3352.

ME 3352: Strength of Materials II

For it is, this far, the best there can be among courses!!!