MEC 3352
STRENGTH OF MATERIALS II

Rotating Discs and Cylinders – Part 2

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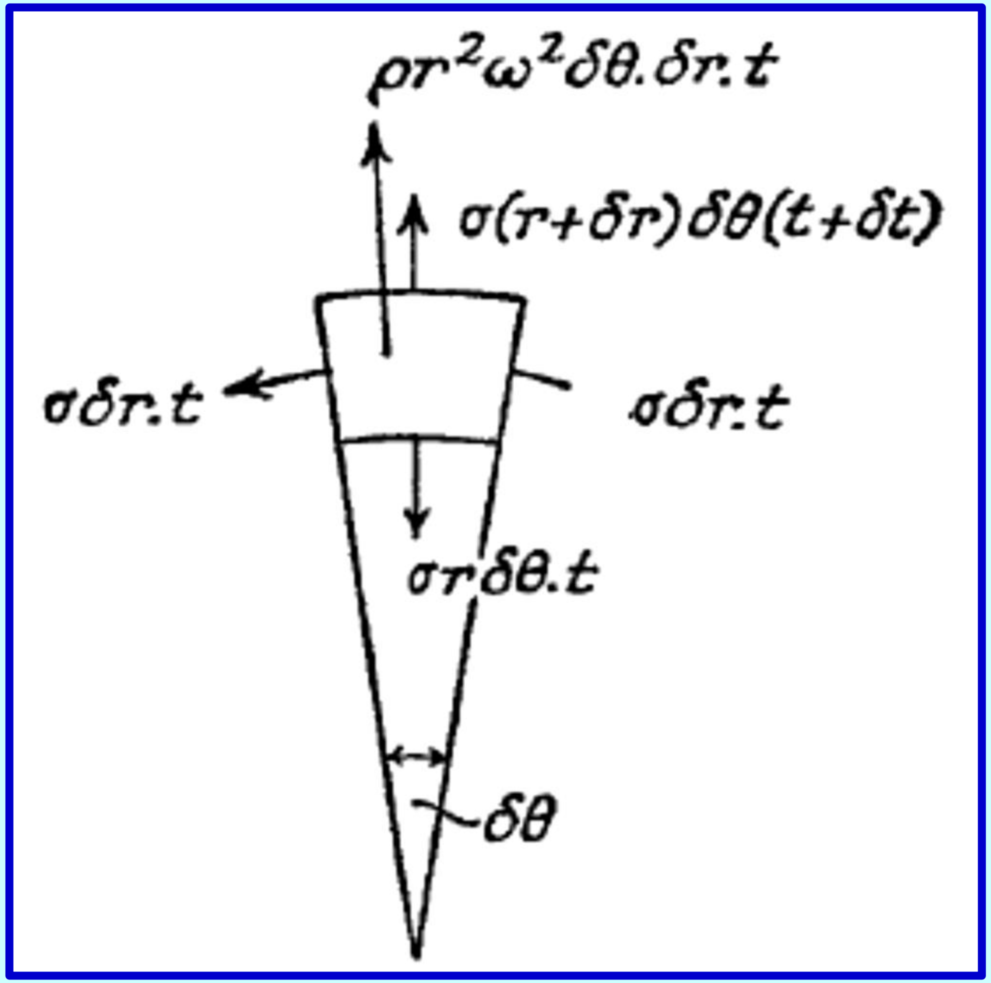
Continued from last time

(iii) Disc of Uniform Strength

Consider the condition of equal stress at all radii, i.e.:

$$\boldsymbol{\sigma_t = \sigma_r = \text{Constant} = \sigma}$$



Let t be the thickness at a radius r,
and $t + \delta t$ at a radius $r + \delta r$

The mass of the element will be:

$$m = \rho r\delta\theta\delta r \cdot t$$

And the centrifugal force $=\rho r^2\omega^2 t\delta\theta\delta r$

Hence the equilibrium equation is:

$$2\sigma\delta r \cdot t \cdot \sin\left(\frac{\delta\theta}{2}\right) + \sigma\delta\theta \cdot t = \sigma(r+\delta r)\delta\theta(t+\delta t) + \rho r^2\omega^2\delta\theta \cdot \sigma r \cdot t$$

In the limit:

$$\sigma t \cdot dr = \sigma r \cdot dt + \sigma t \cdot dr + \rho r^2\omega^2 t \cdot dr$$

Or:

$$\frac{dt}{dr} = -\frac{\rho r\omega^2 t}{\sigma}$$

Re-arranging:

$$\boldsymbol{\frac{dt}{t} = -\frac{\rho r\omega^2}{\sigma}dr}$$

Integrating:

$$\ln t = -\frac{\rho r^2 \omega^2 t}{2\sigma} + C \qquad \text{or} \qquad t = e^{-\frac{\rho r^2 \omega^2 + C}{2\sigma}} = e^C \cdot e^{-\frac{\rho r^2 \omega^2}{2\sigma}}$$

$$t = Ae^{-\frac{\rho r^2 \omega^2}{2\sigma}}$$

At $r = 0,$

$$t = Ae^{-\frac{\rho r^2 \omega^2}{2\sigma}} = Ae^0 \qquad \text{i.e.} \qquad A_{(r=0)} = t_{(r=0)} = t_0$$

Therefore in general:

$$\boxed{\boldsymbol{t = t_0 e^{-\frac{\rho r^2 \omega^2}{2\sigma}}}}$$

Example 2, P. 291: A turbine rotor disc is 0.6 m diameter at the blade ring, and is keyed to a 50 mm diameter shaft. If the minimum thickness is 9.5 mm what should be the thickness at the shaft for a uniform stress of 200 MPa at 10,000 rpm? Density = 7,700 kg/m^3.

In general for a uniform strength rotating disc;

$$t = Ae^{-\rho r^2 \omega^2 / 2\sigma}$$

At r = 0.3 m

$$t = 0.0095 = Ae^{-\rho (0.3)^2 \omega^2 / 2\sigma} = Ae^{-\rho (0.09) \omega^2 / 2\sigma}$$

$$= Ae^{-7700 (0.3)^2 \times 1047.2 / \left(2 \times 200 \times 10^6\right)}$$

$$= Ae^{-1.89}$$

$$0.0095 = 0.1496A$$

$$A = 0.0095/0.1496$$

$$\boldsymbol{A = 0.06352}$$

At r = 0.025 m

$$t_{R_2} = Ae^{-7700(R_2)^2 \times 1047.2^2/(2\times 200\times 10^6)}$$

$$t_{0.025} = Ae^{-7700(0.025)^2 \times 1047.2^2/(2\times 200\times 10^6)}$$

$$= 0.06352e^{-0.0132}$$

$$= 0.06352 \times 0.9869$$

$$= 0.06269 \text{ m} = 62.69 \text{ mm}$$

$$\boldsymbol{t_{0.025} = 63 \text{ mm}}$$

i.e. $\boldsymbol{t_{0.025} = t_{\text{at shaft}} = 63 \text{ mm}}$

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b) Cylinders

The primary difference between a long rotating cylinder and a thin one is that its axial stress σ_a is not equal to zero. The assumption in this case is that the longitudinal strain ε_a is constant, i.e. cross sections remain plain.

Basis of equations … Refer to introductory notes at the top:

σ_t is equivalent to σ_1

σ_r is equivalent to σ_2

σ_a is equivalent to σ_3

Derived equations:

$$\boldsymbol{E\varepsilon_a = \sigma_a - \nu\sigma_t - \nu\sigma_r} \qquad (1)$$

$$\boldsymbol{E\varepsilon_t = E\frac{u}{r} = \sigma_t - \nu\sigma_a - \nu\sigma_r} \qquad (2)$$

$$\boldsymbol{E\varepsilon_r = E\frac{du}{dr} = \sigma_r - \nu\sigma_t - \nu\sigma_a} \qquad (3)$$

Multiplying by (2) r: $\quad Eu = r(\sigma_t - \nu\sigma_a - \nu\sigma_r)$

Differentiating:

$$E\frac{du}{dr} = \sigma_t - \nu\sigma_a - \nu\sigma_r + r\left[\frac{d\sigma_t}{dr} - \nu\frac{d\sigma_a}{dr} - \nu\frac{d\sigma_r}{dr}\right]$$

$$= \sigma_r - \nu\sigma_t - \nu\sigma_a$$ from (3) above.

Simplifying by collecting terms:

$$\boldsymbol{(\sigma_t - \sigma_r)(1+\nu) + r\left(\frac{d\sigma_t}{dr}\right) - \nu r\left(\frac{d\sigma_a}{dr}\right) - \nu r\left(\frac{d\sigma_r}{dr}\right) = 0} \tag{4}$$

Now from (1) above, since ε_a is constant, then

$$\left(\frac{d\sigma_a}{dr}\right) = \nu\left(\frac{d\sigma_t}{dr}\right) + \nu\left(\frac{d\sigma_r}{dr}\right)$$

Substituting for $\left(\frac{d\sigma_a}{dr}\right)$ in Eq. (4):

$$(\sigma_t - \sigma_r)(1+\nu) + r\left(\frac{d\sigma_t}{dr}\right) - \nu r\left[\left(\frac{d\sigma_a}{dr}\right) - \left(\frac{d\sigma_r}{dr}\right)\right] - \nu r\left(\frac{d\sigma_r}{dr}\right) = 0$$

$$(\sigma_t - \sigma_r)(1+\nu) + r(1-\nu^2)\left(\frac{d\sigma_t}{dr}\right) - \nu r(1+\nu)\left(\frac{d\sigma_r}{dr}\right) = 0$$

$$(\sigma_t - \sigma_r) + r(1-\nu)\left(\frac{d\sigma_t}{dr}\right) - \nu r\left(\frac{d\sigma_r}{dr}\right) = 0$$

Now considering the radial equilibrium of the element of the section as shown in the notes above Eq. (5) under rotating disks results

$$\boldsymbol{\sigma_t - \sigma_r - r\left(\frac{d\sigma_r}{dr}\right) = \rho r^2 \omega^2} \tag{5}$$

Substituting for $(\sigma_t - \sigma_r)$

$$r(1-\nu)\left(\frac{d\sigma_t}{dr}\right) + r(1-\nu)\left(\frac{d\sigma_r}{dr}\right) = -\rho r^2 \omega^2$$

Therefore:

$$\left(\frac{d\sigma_t}{dr}\right) + \left(\frac{d\sigma_r}{dr}\right) = -\frac{\rho r^2\omega^2}{(1-\nu)}$$

Integrating:

$$\boldsymbol{\sigma_t + \sigma_r = -\frac{\rho r^2\omega^2}{2(1-\nu)} + 2A} \qquad (6)$$

This is similar to Eq. (6) for Rotating Disk analysis completed above. In fact the rotating disks equation can apply for the long cylinder if $(1+\nu)$ in the disk equations are replaced by $\frac{1}{(1-\nu)}$. Or if ν is replaced by $\frac{\nu}{(1-\nu)}$.

$$2\sigma_r + r\left(\frac{d\sigma_r}{dr}\right) = 2A - \rho r^2\omega^2 - \frac{\rho r^2\omega^2}{2(1-\nu)}$$

Therefore:

$$\frac{1}{r} \cdot \frac{d(\sigma_r r^2)}{dr} = 2A - \frac{\rho r^2 \omega^2 (3 - 2\nu)}{2(1 - \nu)}$$

Therefore:

$$\frac{d(\sigma_r r^2)}{dr} = 2Ar - \frac{\rho r^3 \omega^2 (3 - 2\nu)}{2(1 - \nu)}$$

Integrating:

$$\boldsymbol{\sigma_r = -\frac{\rho r^2 \omega^2 (3 - 2\nu)}{8(1 - \nu)} + A - \frac{B}{r^2}} \tag{7}$$

Now substituting for σ_r in Eq. (7) to obtain: σ_t

$$\boldsymbol{\sigma_t = -\frac{\rho r^2 \omega^2 (1 + 2\nu)}{8(1 - \nu)} + A + \frac{B}{r^2}} \tag{8}$$

The equations, (7) and (8), are the general equations for a rotating cylinder. What remains now is to apply these equations to specific situations of the disc and apply the relevant boundary conditions to reduce the equations to the specific situation under consideration.

a) Solid Cylinder

Solving for A and B:

At the centre of the cylinder where $R_1 = 0$, the stresses cannot be infinite; so, B is clearly equal to 0.

$$B = 0$$

At the outside diameter of the cylinder, $r = R_2$.

$$\sigma_r = 0 = -\frac{\rho r^2 \omega^2 (3 - 2\nu)}{8(1 - \nu)} + A - 0$$

Therefore:

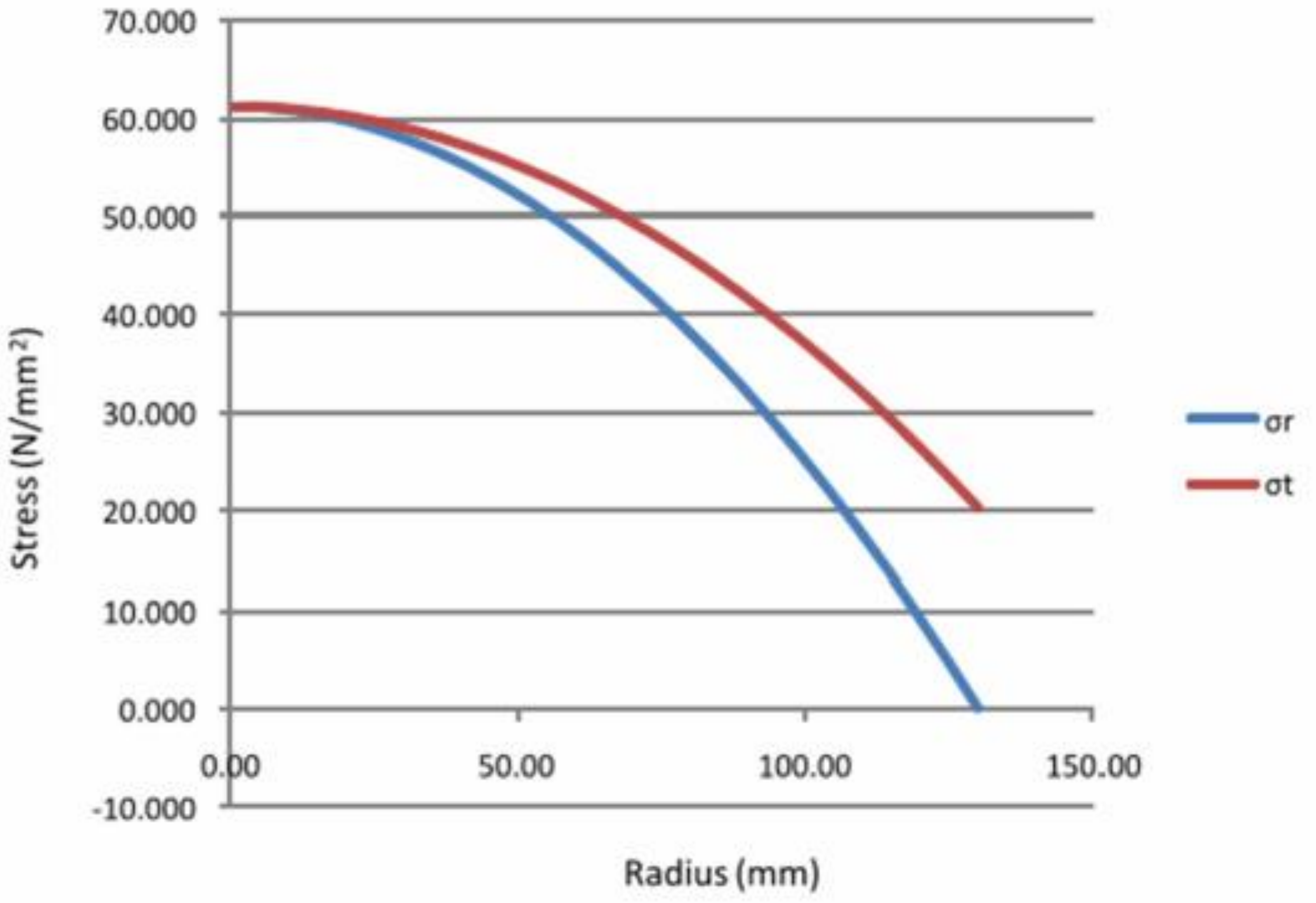
$$A = \frac{\rho r^2 \omega^2 (3 - 2\nu)}{8(1 - \nu)}$$

$$\sigma_r = \frac{\rho\omega^2(3-2\nu)}{8(1-\nu)}\left(R_2^2 - r^2\right) \tag{9}$$

$$\sigma_t = \frac{\rho\omega^2}{8(1-\nu)}\left[(3-2\nu)R_2^2 - (1-2\nu)r^2\right] \tag{10}$$

Therefore the maximum radial and tangential stresses are equal at $r = 0$

$$\sigma_{r\max} = \sigma_{t\max} = \frac{\rho\omega^2(3-2\nu)}{8(1-\nu)}R_2^2 \tag{11}$$



250mm dia solid cylinder at 10000rpm

b) Hollow Cylinder

R_1 and at R_2 are the inner and outer radii.

Solving for A and B:

At $r = R_1$ and at $r = R_2$, the radial stress $\sigma_r = 0$:

Therefore:

$$0 = \sigma_r = -\rho R_1^2 \omega^2 \frac{(3-2\upsilon)}{8(1-\upsilon)} + A - \frac{B}{R_1^2}$$

Therefore:

$$\rho R_1^2 \omega^2 \frac{(3-2\upsilon)}{8(1-\upsilon)} + \frac{B}{R_1^2} = \rho R_2^2 \omega^2 \frac{(3-2\upsilon)}{8(1-\upsilon)} + \frac{B}{R_2^2}$$

$$\rho (R_1^2 - R_2^2) \omega^2 \frac{(3-2\upsilon)}{8(1-\upsilon)} = \rho \frac{B}{R_2^2} - \frac{B}{R_1^2}$$

Therefore:

$$B = \frac{\rho\omega^2(3-2\nu)}{8(1-\nu)} R_1^2 R_2^2$$

Solving for A:

$$0 = \sigma_r = -\rho R_1^2 \omega^2 \frac{(3-2\nu)}{8(1-\nu)} + A - \rho R_2^2 \omega^2 \frac{(3-2\upsilon)}{8(1-\upsilon)}$$

Therefore:

$$A = \frac{\rho\omega^2(3-2\nu)}{8(1-\nu)} \left[-r^2 + (R_2^2 + R_1^2) - \frac{(R_1^2 \cdot R_2^2)}{r^2} \right]$$

Resulting in:

$$\boldsymbol{\sigma_r = \frac{\rho\omega^2(3-2\nu)}{8(1-\nu)}\left[-r^2 + \left(R_2^2 + R_1^2\right) - \frac{\left(R_1^2 \cdot R_2^2\right)}{r^2}\right]} \tag{12}$$

And

$$\boldsymbol{\sigma_{r\,\mathrm{max}} = \frac{\rho\omega^2(3-2\nu)}{8(1-\nu)}\left(R_2^2 - R_1^2\right)} \tag{13}$$

and it is located at $r = \sqrt{(R_1 \cdot R_2)}$

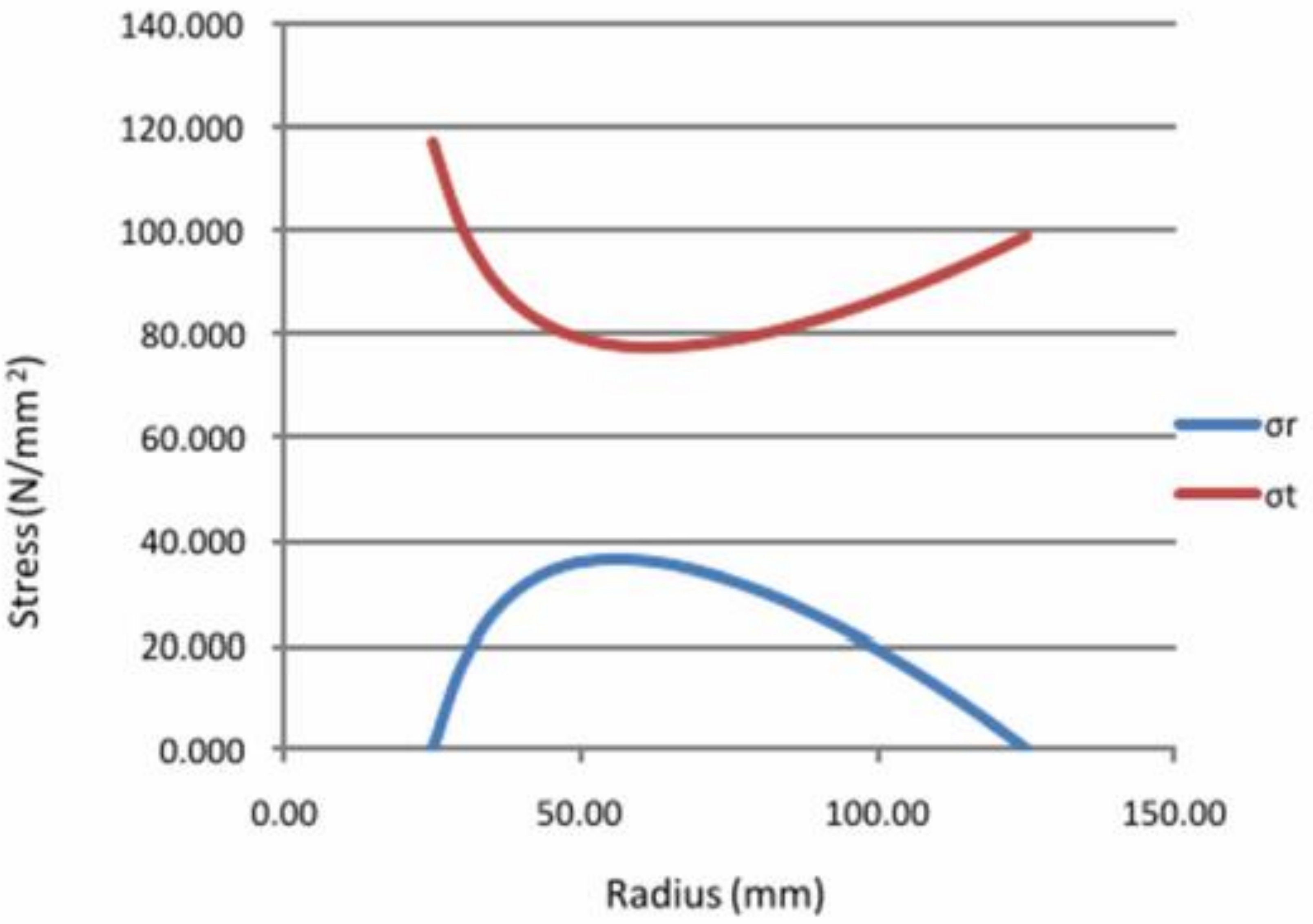
Similarly:

$$\boldsymbol{\sigma_t = \frac{\rho\omega^2}{8(1-\nu)}\left[-r^2(1+2\nu)+(3-2\nu)\left\{\left(R_2^2+R_1^2\right)+\frac{\left(R_1^2\cdot R_2^2\right)}{r^2}\right\}\right]} \quad (14)$$

And

$$\boldsymbol{\sigma_{t\,\mathrm{max}} = \frac{\rho\omega^2}{4(1-\nu)}\left[(1-2\nu)R_1^2+(3-2\nu)R_2^2\right]} \quad (15)$$

and it is located at $r = R_1$



250mm OD x 50mm ID cylinder at 10000rpm

c) Temperature Stresses in Uniform Disc

Let T be the temperature rise above that of the unstressed state in a disc. Then, following the earlier procedures, the *stress-strain* equations are:

$$\boldsymbol{E\frac{du}{dr} = \sigma_r - \nu\sigma_t + E\alpha T} \qquad (1)$$

$$\boldsymbol{E\frac{u}{r} = \sigma_t - \nu\sigma_r + E\alpha T} \qquad (2)$$

where α is the coefficient of linear expansion.

Eliminating u between (1) and (2) gives:

$$(\sigma_t - \sigma_r)(1+\nu) + r\frac{d\sigma_t}{dr} - \nu r\frac{d\sigma_r}{dr} + E\alpha r\frac{dT}{dr} = 0 \qquad (3)$$

The *equilibrium* equation is unchanged:

$$\sigma_t - \sigma_r + r\frac{d\sigma_r}{dr} = \rho r^2 \omega^2 \tag{4}$$

Substituting for $\sigma_t - \sigma_r$ from (4) into (3) and re-arranging:

$$\frac{d\sigma_t}{dr} + \frac{d\sigma_r}{dr} = -(1+\nu)\rho r \omega^2 - E\alpha\frac{dT}{dr}$$

Integrating yields:

$$\sigma_t + \sigma_r = -\frac{(1+\nu)\rho r^2 \omega^2}{2} - E\alpha T + 2A \tag{5}$$

Subtracting (4) from (5), re-grouping and integrating as before, yields:

$$\boldsymbol{\sigma_r = A - \frac{B}{r^2} - \frac{\rho r^2 \omega^2 (3+\nu)}{8} - \frac{E\alpha}{r^2}\int Tr\,dr} \tag{6}$$

Then from (5) to obtain: σ_t

$$\boldsymbol{\sigma_t = A + \frac{B}{r^2} - \frac{\rho r^2 \omega^2 (1+\nu)}{8} - E\alpha T + \frac{E\alpha}{r^2}\int Tr\,dr} \tag{7}$$

Example 3, p.292: Suppose a disc in Example 1 has a linear variation of temperature of 45 ^{o}C between the inner and outer (hotter edges). Calculate the new value of the maximum stress. $E = 205{,}000$ MPa, $\alpha = 11 \times 10^{-6}$ per ^{o}C.

(**Example 1, P. 289**: A thin uniform steel disc of 25 cm diameter, with a central hole of 5 cm diameter, runs at 10,000 rpm. Calculate the maximum principal stress and the maximum shearing stress in the disc if $\nu = 0.3$ and density = 7.7 Mg/m^3.)

Solution:

The variation of temperature with radius may be written as:

$$T = 450(r - 0.025)$$

Assuming no external radial pressure, the radial stress may be equated to zero at $r = 0.025$ and $r = 0.125$, i.e. from (6)

$$A - \frac{B}{0.025^2} - 7700\,\frac{(3 + 0.3) \times 0.025^2}{8}\left(\frac{10{,}000 \times 2\pi}{60}\right)^2 = 0$$

$$\boldsymbol{A - 1600B = 2.18 \times 10^6} \qquad \text{(i)}$$

and

$$A - \frac{B}{0.125^2} - 7700\,\frac{0.125^2(3+0.3)}{8}\left(\frac{10{,}000\times 2\pi}{60}\right)^2 - \left(205\times 10^9 \times \frac{11\times 10^{-6}}{0.125^2}\right)\left[\frac{450r^3}{3} - 450\,\frac{0.025r^2}{2}\right]_{0.025}^{0.125} = 0$$

$$\boldsymbol{A - 64B = 83.4\times 10^6} \qquad \text{(ii)}$$

From (i) and (ii)

$$\boldsymbol{A = 86.8\times 10^6}$$

$$\boldsymbol{B = 53.0\times 10^6}$$

The maximum stress again occurs at $r = 0.025$ m

$$\sigma_1 = 86.8 \times 10^6 + 84.8 \times 10^6 - 0.086 \times 10^6 \text{ N/m}^2$$

$$\boldsymbol{\sigma_1 = 171} \textbf{ N/mm}^2$$

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d) Plastic Collapse of Rotating Discs

- It has been seen that the centrifugal forces in a rotating disc set up a two-dimensional tensile stress system, and in all the cases considered, the hoop stress is greater than or equal to the radial stress at a given radius.

- The maximum values occur at the minimum radius.

- It follows that, as the speed is increased, yield will first occur in the circumferential direction when $\boldsymbol{\sigma_t = \sigma_y}$ (the yield stress in tension).

- A state of collapse will be reached when this stress condition extends to the outer surface of the disc (assuming an ideal elastic-plastic material).

Equilibrium equation becomes:

$$\sigma_y - \sigma_r - r\frac{d\sigma_r}{dr} = \rho r^2 \omega^2$$

Integrating:

$$\sigma_r r = \sigma_y r - \frac{\rho r^3 \omega^2}{3} + A$$

i) Solid Disc

Since the stresses are infinite at $r = 0$, then $A = 0$.

At $r = R$,

$$\sigma_2 = 0 = \sigma_y - \frac{\rho r^3 \omega^2}{3}$$

Giving a collapse speed of:

$$\boxed{\boldsymbol{\omega = \frac{1}{R}\sqrt{\frac{3\sigma_y}{\rho}}}}$$

ii) Disc with Central Hole

At $r = R_1$, $\sigma_r = 0$, giving:

$$A = \left(\frac{\rho R_1^2 \omega^2}{3} - \sigma_y\right) R_1$$

At $r = R_2$,

$$\sigma_r = 0 = \sigma_y - \frac{\rho R_2^2 \omega^2}{3} + \left(\frac{\rho R_1^2 \omega^2}{3} - \sigma_y\right) \frac{R_1}{R_2}$$

Giving a collapse speed of:

$$\boldsymbol{\omega = \sqrt{\left(\frac{3\sigma_y}{\rho} \cdot \frac{R_1 + R_2}{R_2^3 - R_1^3}\right)}}$$

Example:

Substituting the values of Example 1 and assuming a yield stress of 280 MPa gives a collapse speed of:

$$\omega = \frac{1}{R}\sqrt{\left(\frac{3\sigma_y}{\rho}\cdot\frac{R_2 + R_1}{R_2^3 - R_1^3}\right)}$$

$$= \frac{1}{0.025}\sqrt{\left(\frac{3 \times 280 \times 10^6}{7700}\cdot\frac{(0.125 + 0.025)}{(0.125^3 - 0.025^3)}\right)}$$

$$\boldsymbol{\omega = 2,906\ \mathrm{rad/s} \quad \mathrm{or} \quad 27,752\ \mathrm{rpm}}$$

SUMMARY

Uniform Disc:

Solid: $$\sigma_{t\max} = \sigma_{r\max} = \frac{\rho\omega^2(3+\nu)}{8}R_2^2$$

Hollow: $$\sigma_{t\max} = \left(\frac{\rho\omega^2}{2}\right)\left[(1-\nu)R_1^2 + (3+\nu)R_2^2\right]$$

Long Cylinder:

Solid: $$\sigma_{t\max} = \frac{\rho\omega^2(3-2\nu)}{8(1-\nu)}R^2$$

Hollow: $$\sigma_{t\max} = \frac{\rho\omega^2}{4(1-\nu)}\left[(1-2\nu)R_1^2 + (3-2\nu)R_2^2\right]$$

Disc of Uniform Strength: $$t = t_0 e^{-\frac{\rho r^2 \omega^2}{2\sigma}}$$

Collapse Speed: Solid: $$\omega = \frac{1}{R}\sqrt{\frac{3\sigma_y}{\rho}}$$

Hollow: $$\omega = \sqrt{\left(\frac{3\sigma_y}{\rho} \cdot \frac{R_1 + R_2}{R_2^3 - R_1^3}\right)}$$

The End

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ME 3352: Strength of Materials II

For it is, this far, the best there can be among courses!!!