THE UNIVERSITY OF ZAMBIA
SCHOOL OF ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING

MEC 3352- STRENGTH OF MATERIALS II

BENDING OF CURVED BEAMS/BARS

BENDING OF CURVED MEMBERS:

INTRODUCTION

- In this Lecture we will consider the analysis of a *curved beam*, that is, a member that has a curved axis and is subjected to bending.
- Typical examples include **hooks** and **chain links**.
- In all cases, the members are not slender, but rather have a sharp curve, and their cross-sectional dimensions are large compared with their radius of curvature.
- The crane hook represents a typical example of a curved beam.

BENDING OF CURVED MEMBERS:

INTRODUCTION

- Consider an unstressed curved bar (Figure 7.1a) of uniform cross section with a vertical axis of symmetry (Figure 7.1b).
- The outer and inner fibers of the beam are located at radial distances r_o and r_i from the center of curvature, O, respectively.
- The radius of curvature of the centroidal axis is denoted by r_c.
- We focus on a small portion of the bar located between cross sections a-c-e and b-d-f that are separated from each other by a small central angle θ

BENDING OF CURVED MEMBERS:

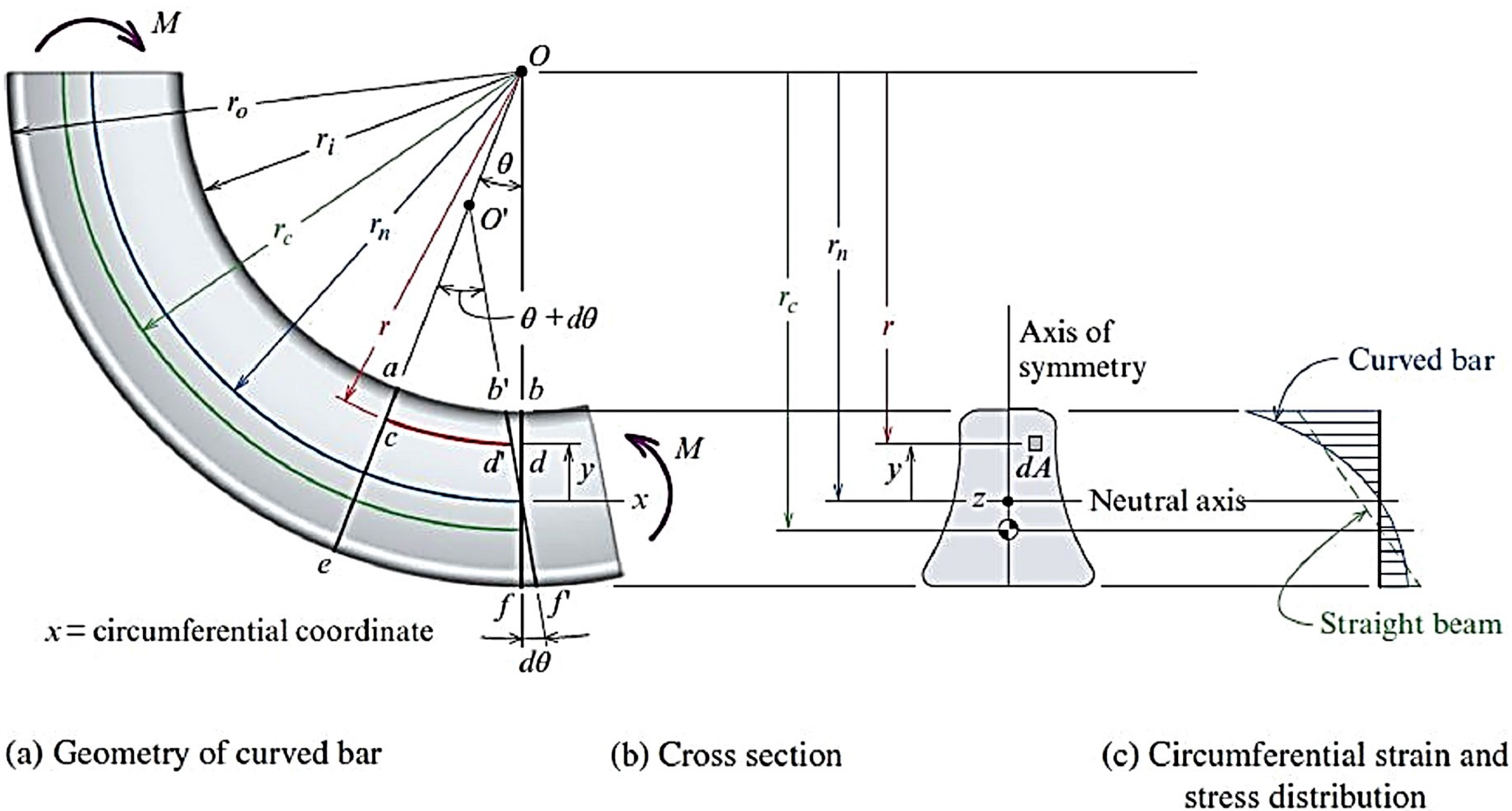


Figure 7.1 Curved bar in pure bending.

BENDING OF CURVED MEMBERS

- The ends of the bar are subjected to bending moments *M* that produce compressive normal stresses on the inner surface of the bar.
- After these moments (M) are applied, the curvature of the bar changes and the center of curvature of the bar moves from its original location *O* to a new location *O*'.
- We assume that plane cross sections in the unstressed bar remain planar after the end moments *M* have been applied.
- On the basis of the above assumption, the deformation of bar fibers must be linearly distributed with respect to a neutral surface that is located at a radius r_n from the center of curvature, *O*.

BENDING OF CURVED MEMBERS:

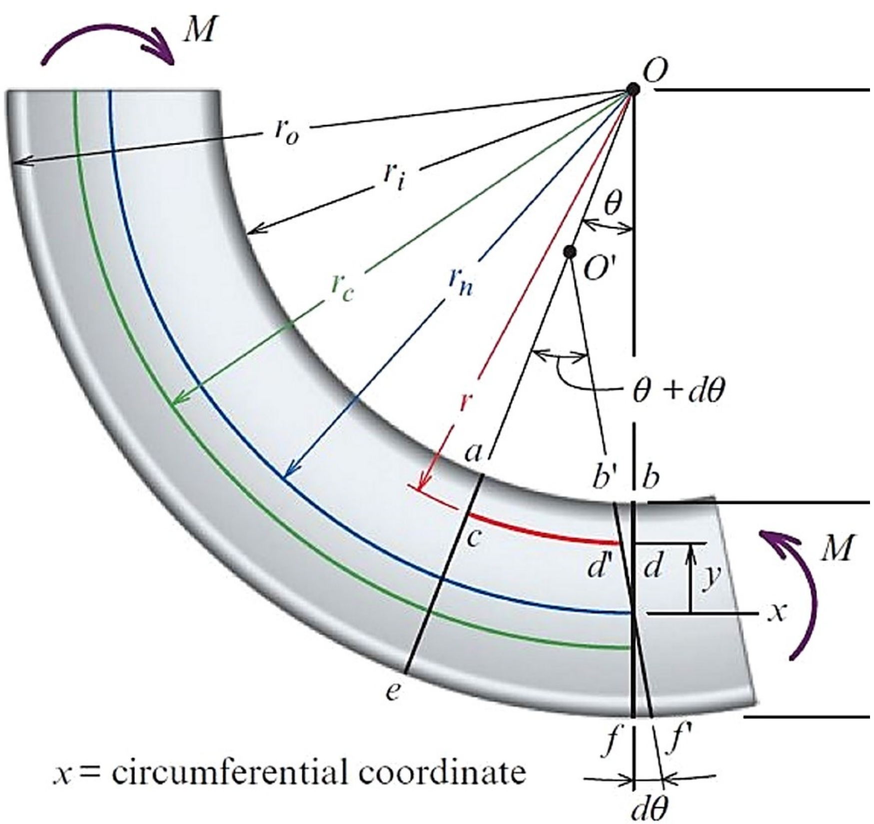


Figure 7.1 (a): Curved bar in pure bending.

- Before the moments *M* are applied, the initial arc length of an arbitrary fiber *cd* of the bar can be expressed in terms of the radial distance *r* and the central angle θ as $cd = r\theta$.
- The initial arc length can also be expressed in terms of the distance from the neutral surface as $cd = (r_n - y)\theta$.

BENDING OF CURVED MEMBERS

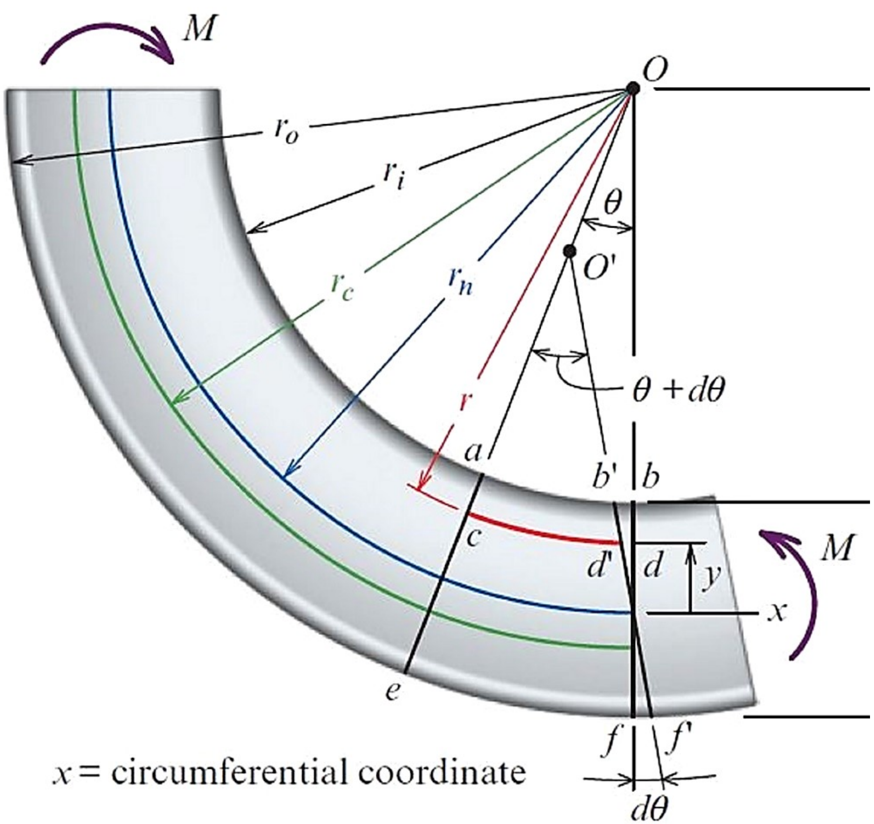


Figure 7.1 (a): Curved bar in pure bending.

- After the moments *M* are applied, the contraction of fiber *cd* can be expressed as

$$dd' = -y\,d\theta\,,$$

- or, in terms of the radial distance to the neutral surface,

$$dd' = -(r_n - r)\,d\theta\,.$$

- The normal strain in the circumferential direction *x* for an arbitrary fiber of the bar is defined as

$$\varepsilon_x = \frac{dd'}{cd} = \frac{-y\,d\theta}{r\theta} = \frac{-(r_n - r)\,d\theta}{r\theta} \quad \text{(i)}$$

BENDING OF CURVED MEMBERS

- Equation (i) shows that the circumferential normal strain ε_x does not vary linearly with the distance *y* from the neutral surface of the bar.
- The distribution of strain is *nonlinear*, as shown in Figure 7.1*c*. The physical reason for this distribution is that the initial lengths of circumferential fibers *cd* vary with *y*, being shorter toward the center of curvature, *O*.
- Thus, while *deformations dd* ' are linear with respect to *y*, these elongations are divided by different initial lengths, so the strain ε_x is not directly proportional to *y*.

BENDING OF CURVED MEMBERS

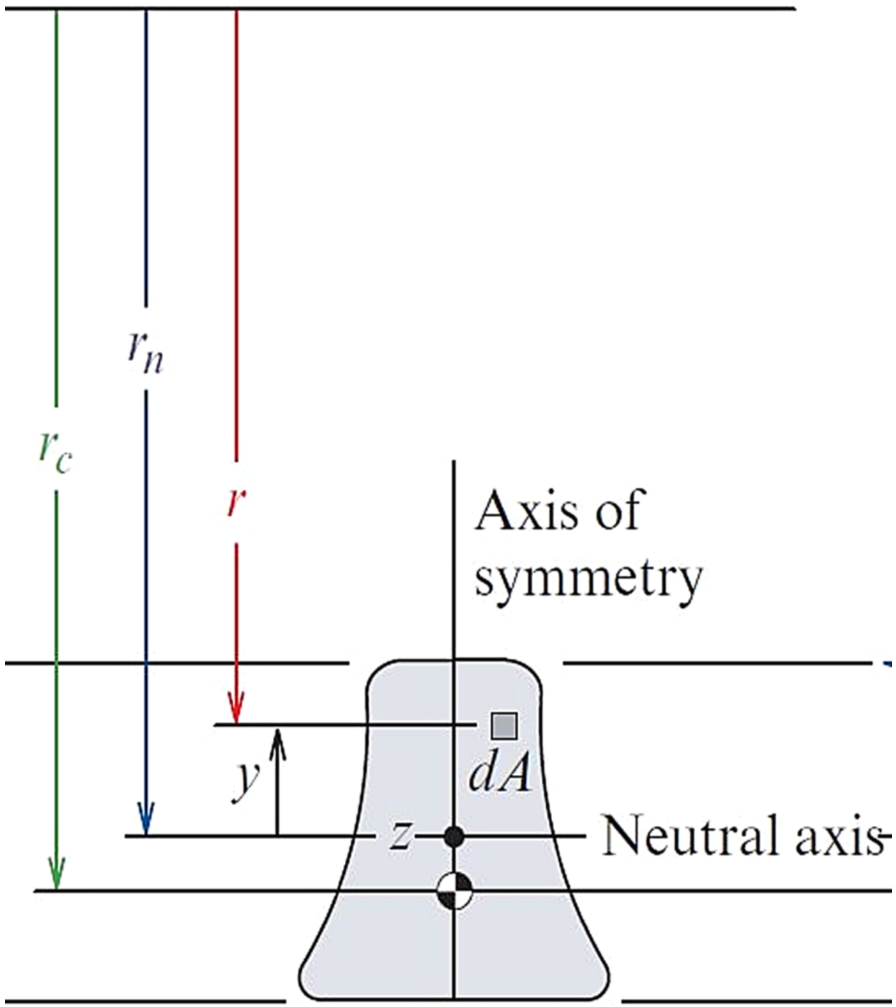


Figure 7.1 (b): Curved bar in pure bending – cross section.

The circumferential normal stress acting on area *dA* can now be obtained from Hooke's law as

$$\sigma_x = -E\frac{(r_n - r)d\theta}{r\theta} \qquad \text{(ii)}$$

The location of the neutral axis (N.A.) follows from the condition that the summation of the forces acting perpendicular to the section must equal zero; that is,

$$\Sigma F_x = 0$$

BENDING OF CURVED MEMBERS

$$\Sigma F_x = 0 \qquad \int_A \sigma_x dA = -\int_A E \frac{(r_n - r)d\theta}{r\theta} dA = 0 \qquad \text{(iii)}$$

However, since r_n, E, θ , and dθ are constant at any one section of a stressed bar, they may be taken outside the integral to obtain

$$\int_A \sigma_x dA = -\frac{E\,d\theta}{\theta} \int_A \frac{r_n - r}{r} dA = -\frac{E\,d\theta}{\theta}\left(r_n \int_A \frac{dA}{r} - \int_A dA \right) = 0 \qquad \text{(iv)}$$

To satisfy equilibrium, the value of r_n must be

$$r_n = \frac{A}{\int_A \frac{dA}{r}}$$

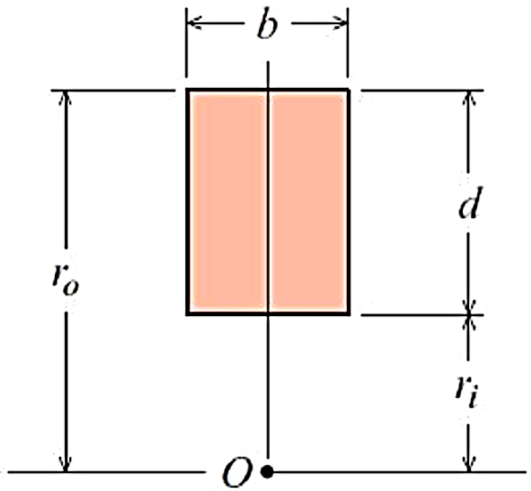
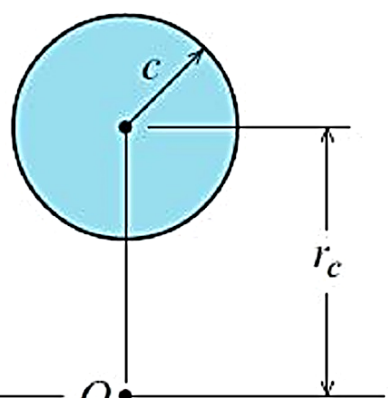
BENDING OF CURVED MEMBERS

$$r_n = \frac{A}{\int_A \frac{dA}{r}} \qquad 7.1$$

- Where *A* is the cross-sectional area of the bar,
- r_n locates the neutral surface of the curved bar relative to the center of curvature.
- Note that the location of the neutral axis does not coincide with that of the centroidal axis.
- Expressions for the areas *A* and the radial distances r_n from the center of curvature, *O*, to the neutral axis are given in Table 7.1 for several typical cross sections.
- These formulas can be combined as necessary for a shape made up of several shapes.

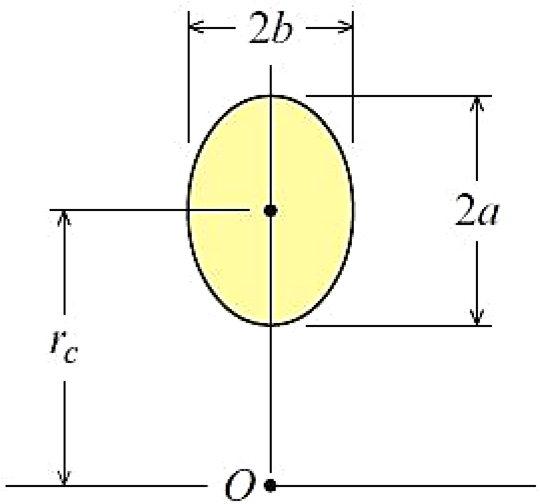
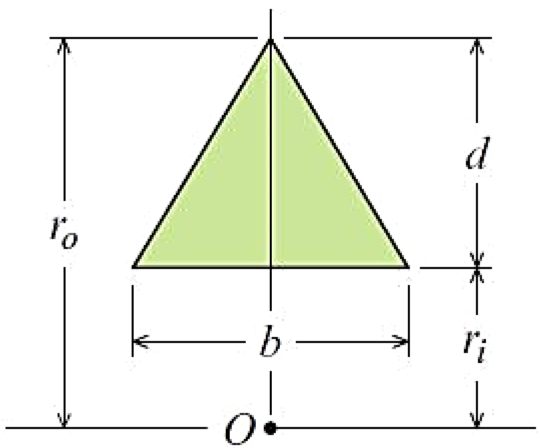
BENDING OF CURVED MEMBERS

Table 7.1 : location of the neutral surface for different shapes

Shape	Cross Section	Radius of Neutral Surface
A. Rectangle		$r_n = \dfrac{A}{b \ln \dfrac{r_o}{r_i}}$ $A = bd$
B. Circle		$r_n = \dfrac{A}{2\pi\left(r_c - \sqrt{r_c^2 - c^2}\right)}$ $A = \pi c^2$

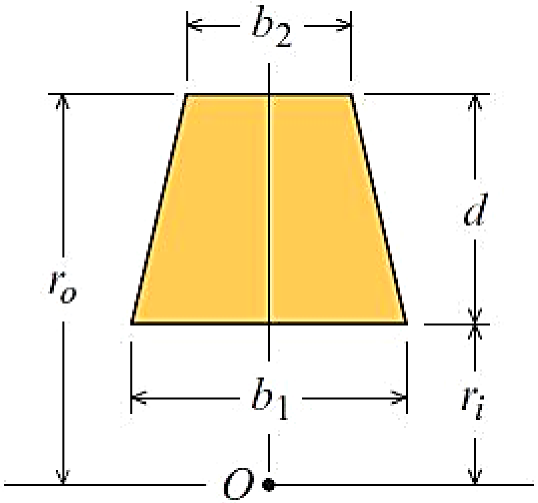
BENDING OF CURVED MEMBERS

Table 7.1 : location of the neutral surface for different shapes

C. Ellipse		$r_n = \dfrac{A}{\dfrac{2\pi b}{a}\left(r_c - \sqrt{r_c^2 - a^2}\right)}$ $A = \pi ab$
D. Triangle		$r_n = \dfrac{A}{\dfrac{br_o}{d}\left(\ln \dfrac{r_o}{r_i}\right) - b}$ $A = \dfrac{1}{2}bd$

BENDING OF CURVED MEMBERS

Table 7.1 : location of the neutral surface for different shapes

E. Trapezoid		$r_n = \dfrac{A}{\dfrac{1}{d}\left[(b_1 r_o - b_2 r_i)\ln\dfrac{r_o}{r_i} - d(b_1 - b_2)\right]}$ $A = \dfrac{1}{2}(b_1 + b_2)d$

BENDING OF CURVED MEMBERS

- After the location of the neutral axis is established, the equation for the circumferential normal stress distribution is obtained by equating the external moment *M* to the internal resisting moment developed by the stresses expressed in Equation (ii).
- The summation of moments is made about the *z* axis, which is placed at the neutral-axis location found in Equation (7.1):

$$\Sigma M_z = 0 \qquad M + \int_A y\sigma_x \, dA = 0 \tag{v}$$

- The distance *y* can be expressed as $y = r_n - r$. The normal stress σ_x was defined in Equation (ii).
- We substitute these expressions into the integral term to obtain:

BENDING OF CURVED MEMBERS

$$M = -\int_A \sigma_x (r_n - r)\, dA = \int_A E \frac{(r_n - r)^2 d\theta}{r\theta} dA \qquad \text{(vi)}$$

- Since the variables *E*, θ , and *d*θ are constant at any one section of a stressed bar, Equation (vi) can be expressed as:

$$M = \frac{E\, d\theta}{\theta} \int_A \frac{(r_n - r)^2}{r} dA \qquad \text{(vii)}$$

- To remove θ from this expression, we once again use Equation (ii), rewriting it, however, as

$$\frac{E\, d\theta}{\theta} = -\frac{\sigma_x r}{r_n - r} \qquad \text{(viii)}$$

Then we substitute Equation (viii) into Equation (vii) to obtain

BENDING OF CURVED MEMBERS

$$M = -\frac{\sigma_x r}{r_n - r}\int_A \frac{(r_n - r)^2}{r}\,dA \qquad \text{(ix)}$$

- We expand Equation (ix) as follows:

$$M = -\frac{\sigma_x r}{r_n - r}\left(\int_A \frac{r_n^2}{r}\,dA - \int_A \frac{r_n r}{r}\,dA - \int_A \frac{r_n r}{r}\,dA + \int_A r\,dA\right)$$

$$= -\frac{\sigma_x r r_n}{r_n - r}\left(r_n \int_A \frac{dA}{r} - \int_A dA\right) - \frac{\sigma_x r}{r_n - r}\left(-r_n \int_A dA + \int_A r\,dA\right) \qquad \text{(x)}$$

- Notice that the first two integrals in Equation (x) are identical to the terms in parentheses in Equation (iv).
- Accordingly, these two integrals vanish, leaving:

$$M = -\frac{\sigma_x r}{r_n - r}\left(-r_n \int_A dA + \int_A r\,dA\right) \qquad \text{(xi)}$$

BENDING OF CURVED MEMBERS

- The first integral in Equation (xi) is simply the area *A*.
- The second integral is the first moment of area about the center of curvature.
- From the definition of a centroid, this integral can be expressed as $r_c A$, where r_c is the radial distance from the center of curvature, *O*, to the centroid of the cross section.
- We, therefore, get:

$$M = -\frac{\sigma_x r}{r_n - r}(-r_n A + r_c A) = -\sigma_x r\, A\left(\frac{r_c - r_n}{r_n - r}\right) \qquad \text{(xii)}$$

- Solving this equation for the circumferential normal stress created in a curved bar by a bending moment *M* gives

$$\sigma_x = -\frac{M(r_n - r)}{r\, A(r_c - r_n)} \qquad (7.2)$$

BENDING OF CURVED MEMBERS

- In equation 7.2, the bending moment *M* is considered to be a positive value if it creates compressive normal stress on the inner surface (i.e., at radius r_i) of the curved bar.
- A positive bending moment decreases the radius of curvature of the bar.

$$\sigma_x = -\frac{M(r_n - r)}{r A(r_c - r_n)} \qquad (7.2)$$

BENDING OF CURVED MEMBERS

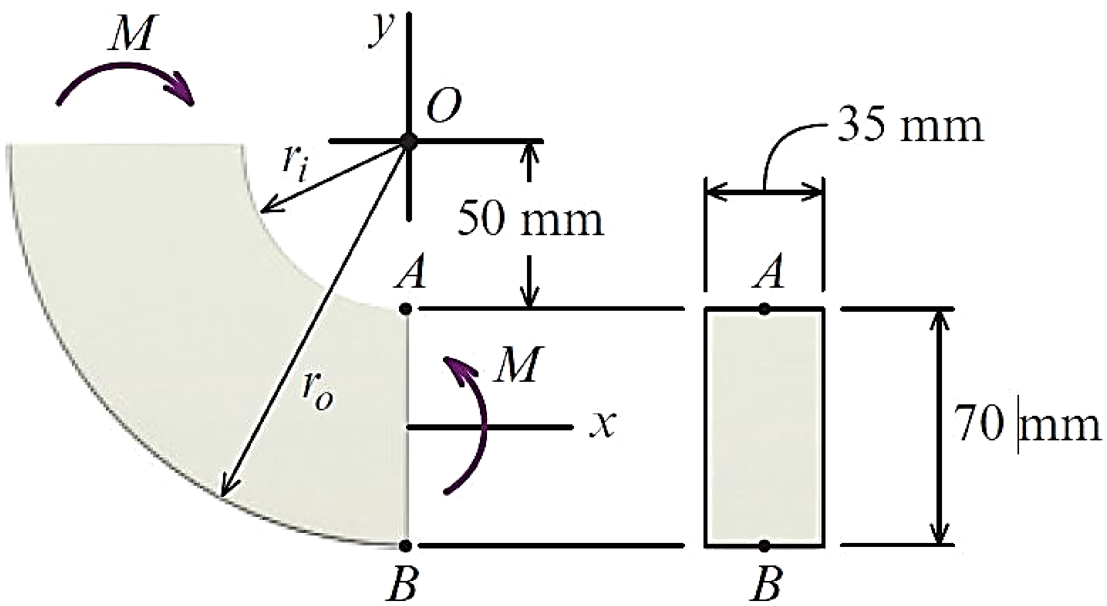
Example 0701

A curved bar with a rectangular cross section is subjected to a bending moment M = 4,500 Nm, acting in the direction shown. The bar has a width of 35 mm and a height of 70 mm. The inside radius of the curved bar is r_i = 50 mm.

a) Determine the bending stresses in the curved bar at points A and B.
b) Sketch the distribution of flexural stresses in the curved bar.
c) Determine the percent error if the flexure formula for straight beams were used for part (a).

BENDING OF CURVED MEMBERS

Solution



A = 35mm x 70mm = 2,450 mm^2.

The radial distance from the center of curvature, O, to the centroid of the cross section is

$$r_c = 50\text{ mm} + \frac{75\text{ mm}}{2} = 85\text{ mm}$$

BENDING OF CURVED MEMBERS

Solution

(a) Bending Stresses

- We determine the location of the neutral axis r_n

$$r_n = \frac{A}{\int_A \frac{dA}{r}}$$

$$\int_A \frac{dA}{r} = \int_{50}^{120} \frac{(35\text{ mm})dr}{r}$$

$$= (35\text{ mm})\ln\left(\frac{120}{50}\right) = 30.641406\text{ mm}$$

- Accordingly, the radial distance from the center of curvature, *O*, to the neutral axis of the curved bar is

$$r_n = \frac{A}{\int_A \frac{dA}{r}} = \frac{2{,}450\text{ mm}^2}{30.641406\text{ mm}} = 79.957167\text{ mm}$$

BENDING OF CURVED MEMBERS

- The curved bar is subjected to a bending moment given as *M* = + 4,500 Nm.
- At point *A*, *r* = 50 mm. Thus, the bending stress at point *A* is

$$\sigma_x = -\frac{M(r_n - r)}{r\,A(r_c - r_n)} = -\frac{(4{,}500\ \text{N}\cdot\text{m})(79.957167\ \text{mm} - 50\ \text{mm})(1{,}000\ \text{mm/m})}{(50\ \text{mm})(2{,}450\ \text{mm}^2)(85\ \text{mm} - 79.957167\ \text{mm})}$$

$$= -218.224\ \text{MPa} = 218\ \text{MPa (C)}$$

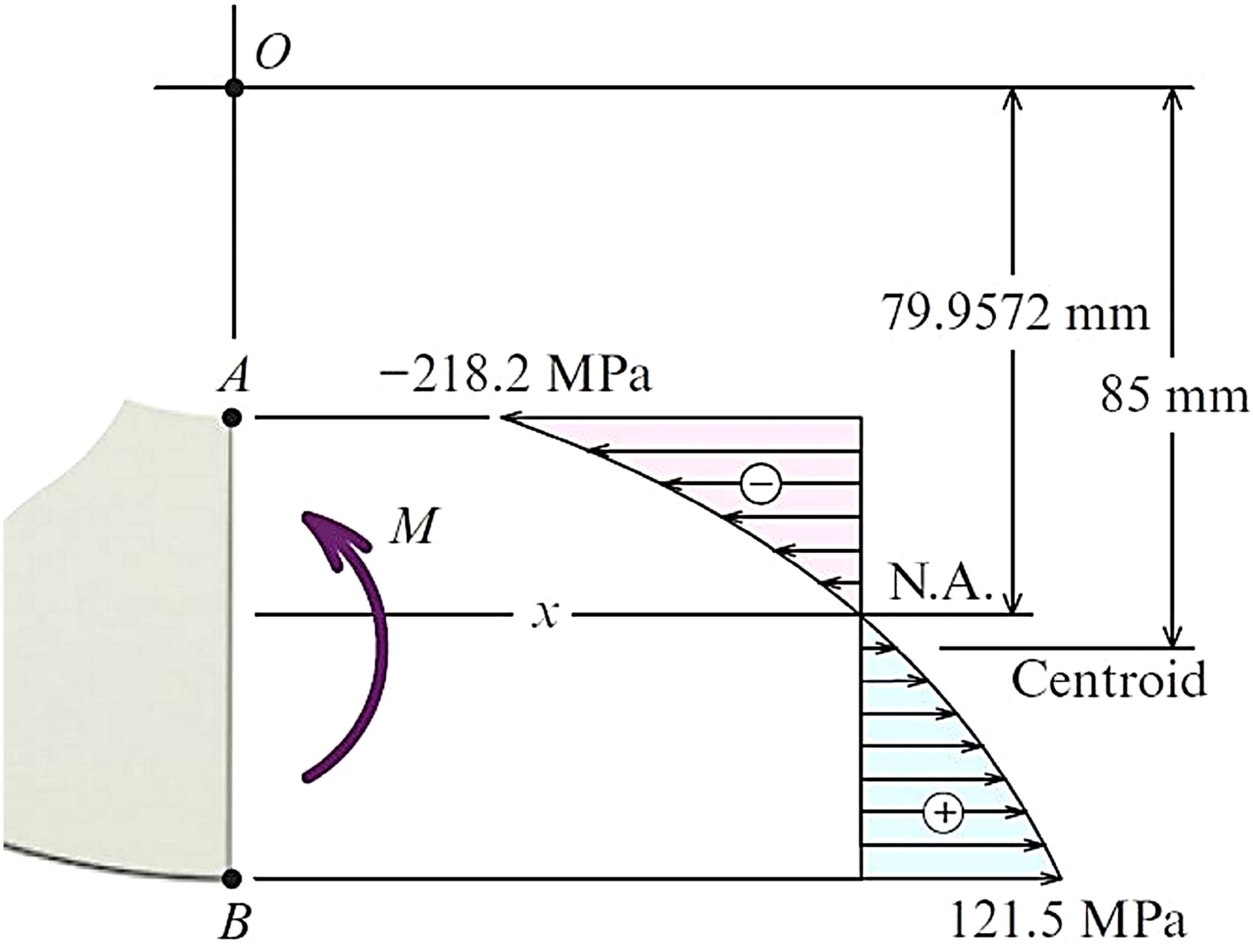
- At point *B*, *r* = 120 mm, making the bending stress at that point

$$\sigma_x = -\frac{M(r_n - r)}{r\,A(r_c - r_n)} = -\frac{(4{,}500\ \text{N}\cdot\text{m})(79.957167\ \text{mm} - 120\ \text{mm})(1{,}000\ \text{mm/m})}{(120\ \text{mm})(2450\ \text{mm}^2)(85\ \text{mm} - 79.957167\ \text{mm})}$$

$$= 121.539\ \text{MPa} = 121.5\ \text{MPa (T)}$$

BENDING OF CURVED MEMBERS

(b)Flexural Stress Distribution



BENDING OF CURVED MEMBERS

(c) Comparison with Stresses from Straight-Beam Flexure Formula

- The section modulus of the rectangular shape is

$$S = \frac{bd^2}{6} = \frac{(35\ \text{mm})(70\ \text{mm})^2}{6} = 28{,}583.333\ \text{mm}^3$$

- If the beam were initially straight, the stresses at *A* and *B* for M = 4,500 Nm would be

$$\sigma_x = \pm\frac{M}{S} = \pm\frac{(4500\ \text{N}\cdot\text{m})(1000\ \text{mm/m})}{28{,}583.333\ \text{mm}^3} = \pm 157.434\ \text{MPa}$$

- Therefore, the errors between the actual stress (determined from the curved-bar formula) and the stress obtained by using the flexure formula for straight beams is

BENDING OF CURVED MEMBERS

$$\left|\frac{-157.434 \text{ MPa} - (-218.224 \text{ MPa})}{-218.224 \text{ MPa}}\right|(100\%) = 27.9\% \text{ low at } A$$

$$\left|\frac{157.434 \text{ MPa} - (121.539 \text{ MPa})}{121.539 \text{ MPa}}\right|(100\%) = 29.5\% \text{ high at } B$$

Neutral axis of a transverse section

- We recall from statics that a couple **M** actually consists of two equal and opposite forces.
- The sum of the components of these forces in any direction is therefore equal to zero.
- Moreover, the moment of the couple is the same about *any* axis perpendicular to its plane, and is zero about any axis contained in that plane.
- we express the equivalence of the elementary internal forces and of the couple **M** by writing that,

IMPORTANT POINTS

- The **curved-beam formula** should be used to determine the circumferential stress in a beam when the radius of curvature is less than five times the depth of the beam ($r < 5h$).
- Due to the curvature of the beam, the **normal strain** in the beam **does not vary linearly with depth as** in the case of a straight beam. As a result, the neutral axis does not pass through the centroid of the cross section.
- The radial stress component caused by bending can generally be neglected, especially if the cross section is a solid section and not made from thin plates.

CONCLUSION

- The **curved-beam formula** is normally used when the curvature of the member is very pronounced, as in the case of hooks or rings.

- However, if the radius of curvature is greater than five times the depth of the member, the **flexure formula** can normally be used to determine the stress.

EXAMPLE 0702

The curved bar has a cross-sectional area shown in Fig. Q1. If it is subjected to bending moments of 4kN.m, determine the maximum normal stress developed in the bar.

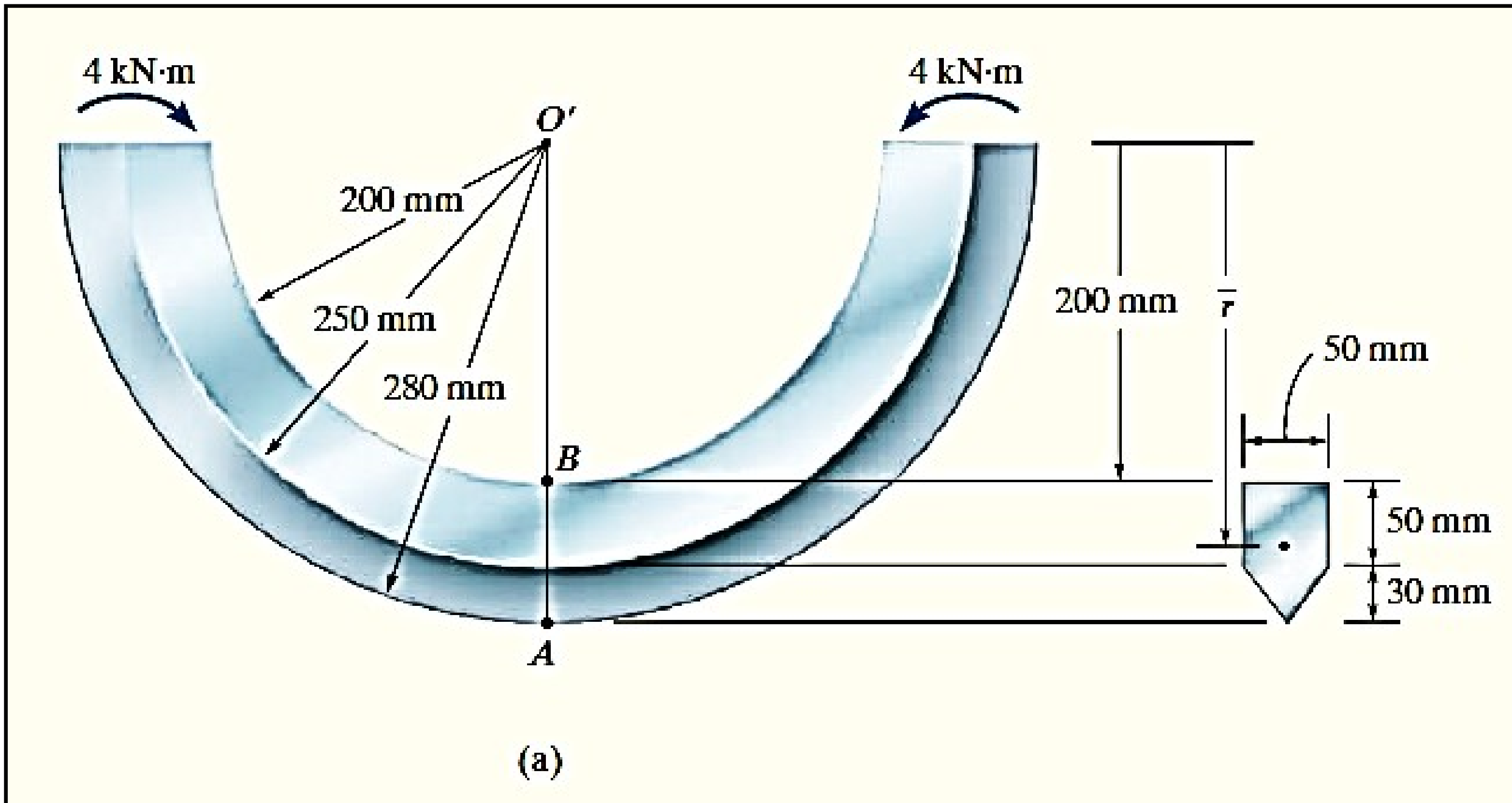


FIGURE. EQ 1

EXAMPLE 0702

SOLUTION

Internal Moment. Each section of the bar is subjected to the same resultant internal moment of 4 kN · m. Since this moment tends to decrease the bar's radius of curvature, it is negative. Thus, $M = -4\ \text{kN} \cdot \text{m}$.

Section Properties. Here we will consider the cross section to be composed of a rectangle and triangle. The total cross-sectional area is

$$\Sigma A = (0.05\ \text{m})^2 + \frac{1}{2}(0.05\ \text{m})(0.03\ \text{m}) = 3.250(10^{-3})\ \text{m}^2$$

The location of the centroid is determined with reference to the center of curvature, point O', Fig. 6–41a.

$$\bar{r} = \frac{\Sigma \tilde{r} A}{\Sigma A}$$

$$= \frac{[0.225\ \text{m}](0.05\ \text{m})(0.05\ \text{m}) + [0.260\ \text{m}]\frac{1}{2}(0.050\ \text{m})(0.030\ \text{m})}{3.250(10^{-3})\ \text{m}^2}$$

$$= 0.23308\ \text{m}$$

The location of the centroid is determined with reference to the center of curvature, point O', Fig. 6–41*a*.

$$\bar{r} = \frac{\Sigma \tilde{r} A}{\Sigma A}$$

$$= \frac{[0.225 \text{ m}](0.05 \text{ m})(0.05 \text{ m}) + [0.260 \text{ m}]\frac{1}{2}(0.050 \text{ m})(0.030 \text{ m})}{3.250(10^{-3}) \text{ m}^2}$$

$$= 0.23308 \text{ m}$$

We can find $\int_A dA/r$ for each part using Table 6–1. For the rectangle,

$$\int_A \frac{dA}{r} = 0.05 \text{ m}\left(\ln\frac{0.250 \text{ m}}{0.200 \text{ m}}\right) = 0.011157 \text{ m}$$

And for the triangle,

$$\int_A \frac{dA}{r} = \frac{(0.05 \text{ m})(0.280 \text{ m})}{(0.280 \text{ m} - 0.250 \text{ m})}\left(\ln\frac{0.280 \text{ m}}{0.250 \text{ m}}\right) - 0.05 \text{ m} = 0.0028867 \text{ m}$$

Thus the location of the neutral axis is determined from

$$R = \frac{\Sigma A}{\Sigma \int_A dA/r} = \frac{3.250(10^{-3})\ \text{m}^2}{0.011157\ \text{m} + 0.0028867\ \text{m}} = 0.23142\ \text{m}$$

Note that $R < \bar{r}$ as expected. Also, the calculations were performed with sufficient accuracy so that $(\bar{r} - R) = 0.23308\ \text{m} - 0.23142\ \text{m} = 0.00166\ \text{m}$ is now accurate to three significant figures.

Normal Stress. The maximum normal stress occurs either at A or B. Applying the curved-beam formula to calculate the normal stress at B, $r_B = 0.200$ m, we have

$$\sigma_B = \frac{M(R - r_B)}{A r_B(\bar{r} - R)} = \frac{(-4\ \text{kN}\cdot\text{m})(0.23142\ \text{m} - 0.200\ \text{m})}{3.250(10^{-3})\ \text{m}^2(0.200\ \text{m})(0.00166\ \text{m})}$$

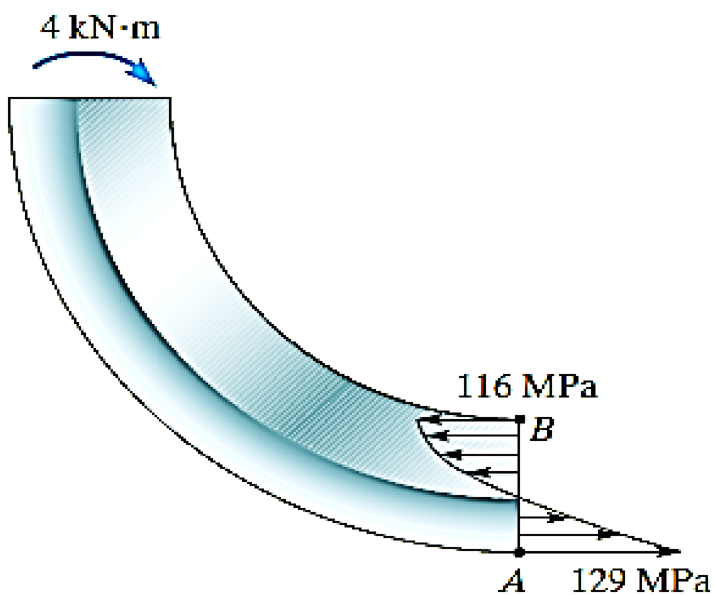
$$= -116\ \text{MPa}$$

At point A, $r_A = 0.280$ m and the normal stress is

$$\sigma_A = \frac{M(R - r_A)}{Ar_A(\bar{r} - R)} = \frac{(-4\text{ kN}\cdot\text{m})(0.23142\text{ m} - 0.280\text{ m})}{3.250(10^{-3})\text{ m}^2(0.280\text{ m})(0.00166\text{ m})}$$

$$= 129\text{ MPa} \qquad \textit{Ans.}$$

By comparison, the maximum normal stress is at A. A two-dimensional representation of the stress distribution is shown in Fig. 6–41b.

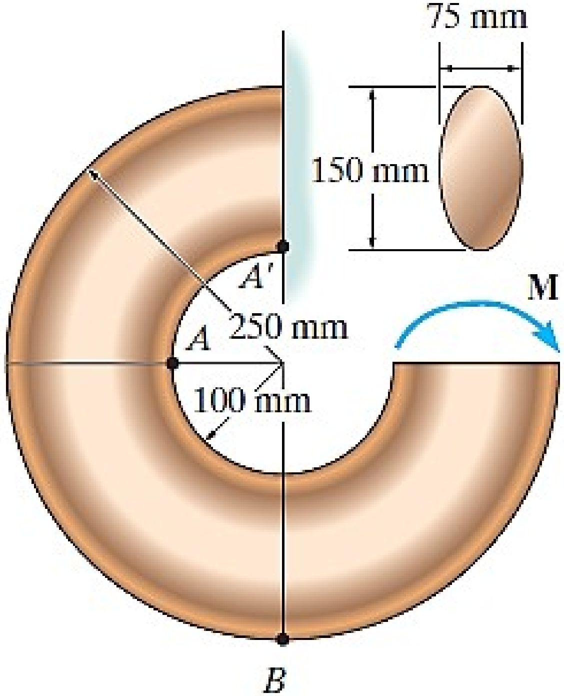


(b)

Fig. 6–41 (cont.)

EXAMPLE 0702

The member has an elliptical cross section. If it is subjected to a moment of M = 50 N.m , determine the stress at points *A* and *B*. Is the stress at point A' , which is located on the member near the wall, the same as that at *A*? Explain.



Example 0702: SOLUTION

$$\int_A \frac{dA}{r} = \frac{2\pi\, b}{a}\left(\bar{r} - \sqrt{\bar{r}^2 - a^2}\,\right)$$

$$= \frac{2\pi(0.0375)}{0.075}\left(0.175 - \sqrt{0.175^2 - 0.075^2}\,\right) = 0.053049301 \text{ m}$$

$$A = \pi\, ab = \pi(0.075)(0.0375) = 2.8125(10^{-3})\pi$$

$$R = \frac{A}{\int_A \frac{dA}{r}} = \frac{2.8125(10^{-3})\pi}{0.053049301} = 0.166556941$$

$$\bar{r} - R = 0.175 - 0.166556941 = 0.0084430586$$

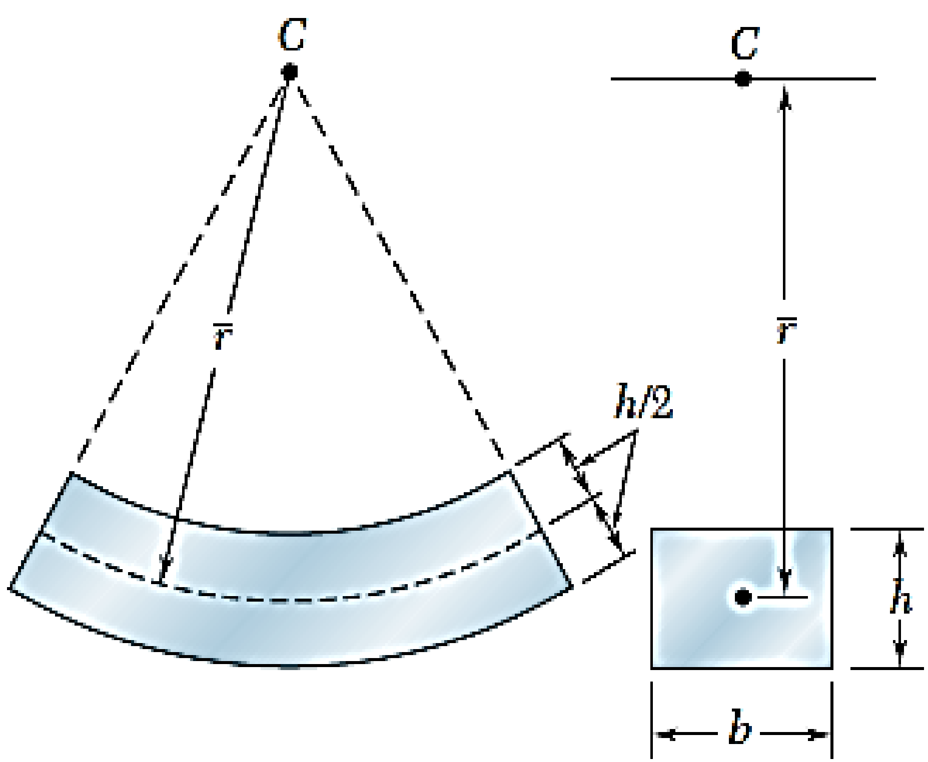
$$\sigma_A = \frac{M(R - r_A)}{Ar_A\,(\bar{r} - R)} = \frac{50(0.166556941 - 0.1)}{2.8125(10^{-3})\pi\,(0.1)(0.0084430586)} = 446\text{k Pa (T)} \qquad \textbf{Ans.}$$

$$\sigma_B = \frac{M(R - r_B)}{Ar_B\,(\bar{r} - R)} = \frac{50(0.166556941 - 0.25)}{2.8125(10^{-3})\pi\,(0.25)(0.0084430586)} = 224 \text{ kPa (C)} \qquad \textbf{Ans.}$$

No, because of localized stress concentration at the wall. **Ans.**

EXAMPLE 0703

A curved rectangular bar has a mean radius $\bar{r} = 152$mm, and a cross section of width b =63.5mm and a depth h = 38mm (Fig.). Determine the distance e between the centroid and the neutral axis of the cross section.



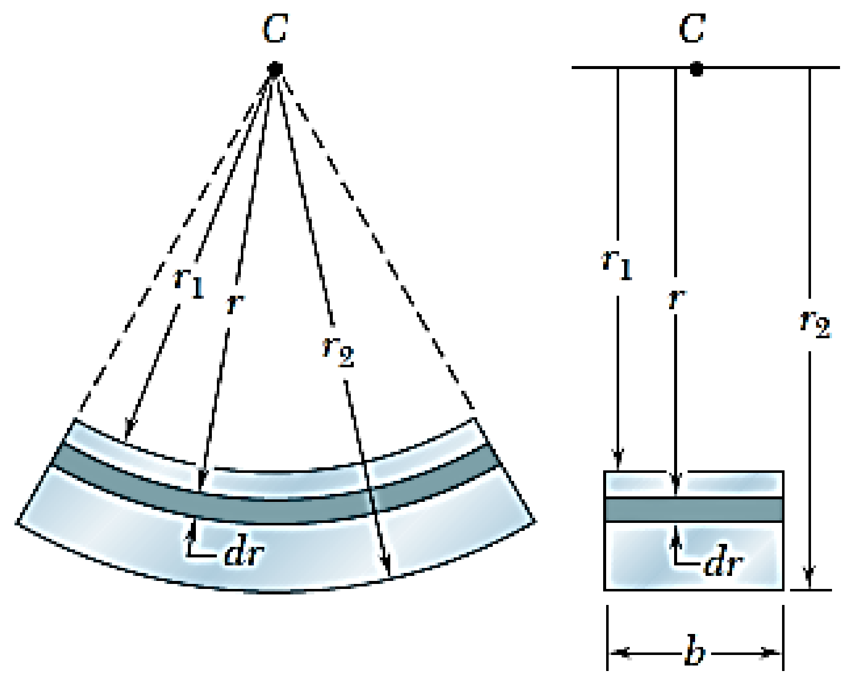
EXAMPLE 0703: SOLUTION

Solution

- We first derive the expression for the radius *R* of the neutral surface Denoting by r_1 andr_2, respectively, the inner and outer radius of the bar (Fig. 4.75), we use $r = \frac{A}{\int \frac{dA}{r}}$ and write:

$$R = \frac{A}{\int_{r_1}^{r_2} \frac{dA}{r}} = \frac{bh}{\int_{r_1}^{r_2} \frac{b\,dr}{r}} = \frac{h}{\int_{r_1}^{r_2} \frac{dr}{r}}$$

$$R = \frac{h}{\ln \frac{r_2}{r_1}}$$



For the given data, we have:

$$r_1 = \bar{r} - \frac{1}{2}h = 152 - 19 = 133\text{mm}$$

$$r_2 = \bar{r} + \frac{1}{2}h = 152 + 19 = 171\text{mm}$$

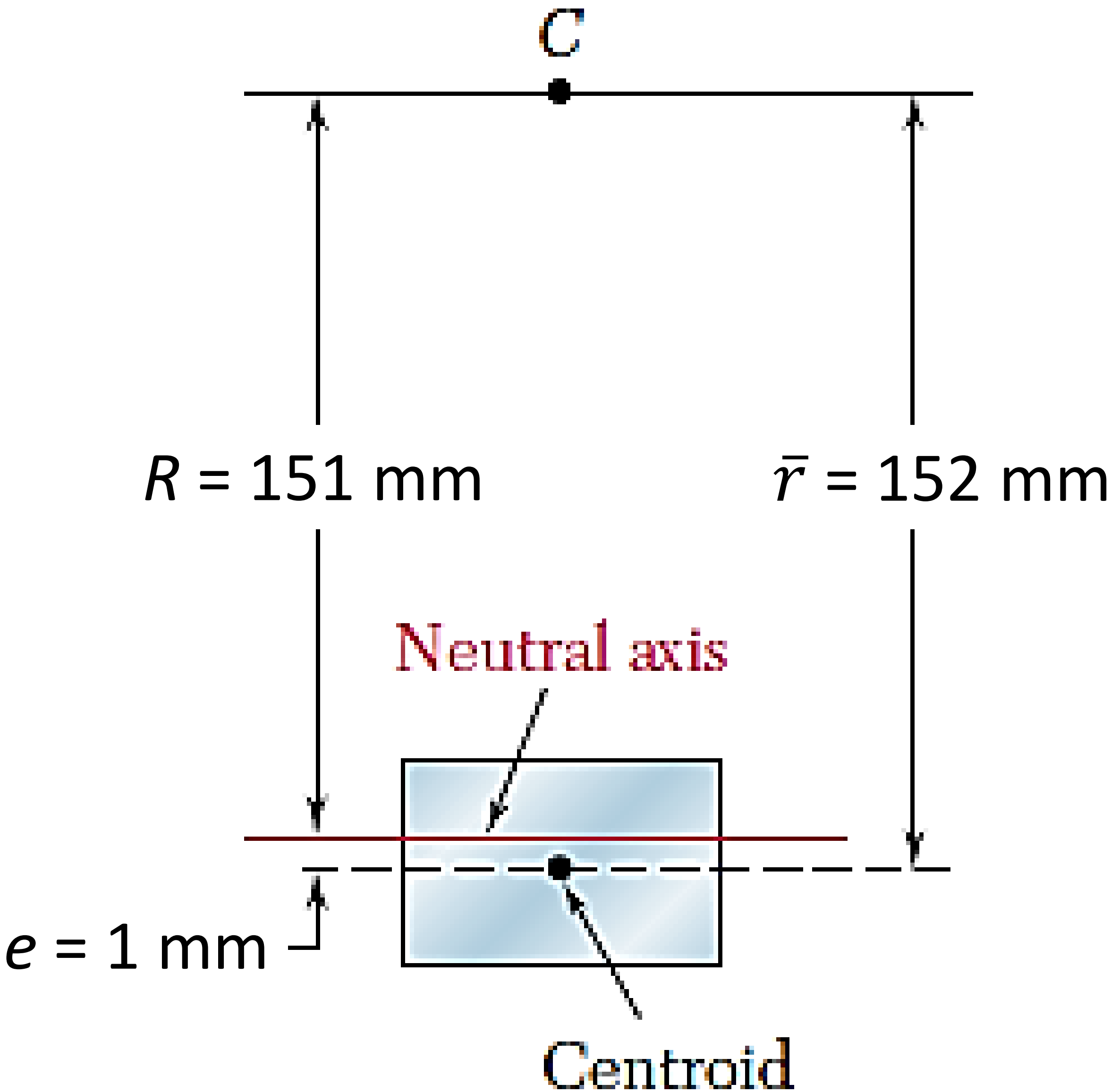
Substituting for h, r_1 and r_2, we have:

$$R = \frac{h}{\ln\frac{r_2}{r_1}} = \frac{38}{\ln\frac{171}{133}} = 151\text{mm}$$

The distance between the centroid and the neutral axis of the cross section (Fig. 4.76) is thus:

$$r_n = \bar{r} - R = 152 - 151 = 1\text{mm}$$

Please note that it is sometimes necessary to calculate R with a high degree of accuracy (up to two or three decimal places when working in millimetres in order to obtain e with a good degree of accuracy.



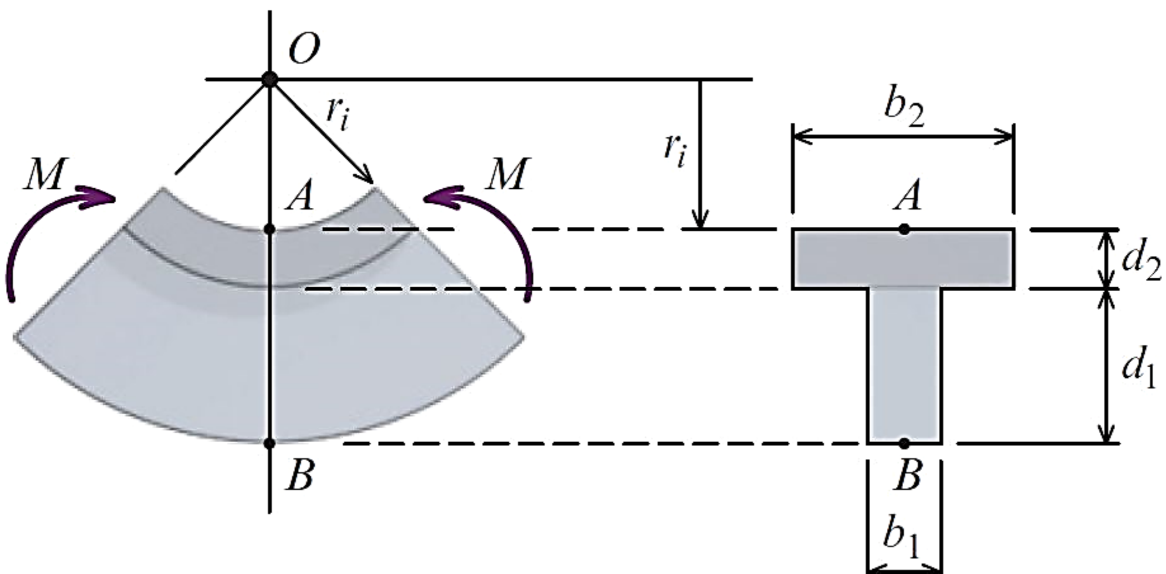
C
R = 151 mm
r̄ = 152 mm
Neutral axis
e = 1 mm
Centroid

Questions 1

The curved tee shape shown in Figure P8.83 is subjected to a bending moment M = 2,700 N · m. The dimensions of the cross section are b_1 = 15 mm, d_1 = 70 mm, b_2 = 50 mm, and d_2 = 20 mm. The radial distance from the center of curvature, O, to A is r_i = 85 mm. Determine

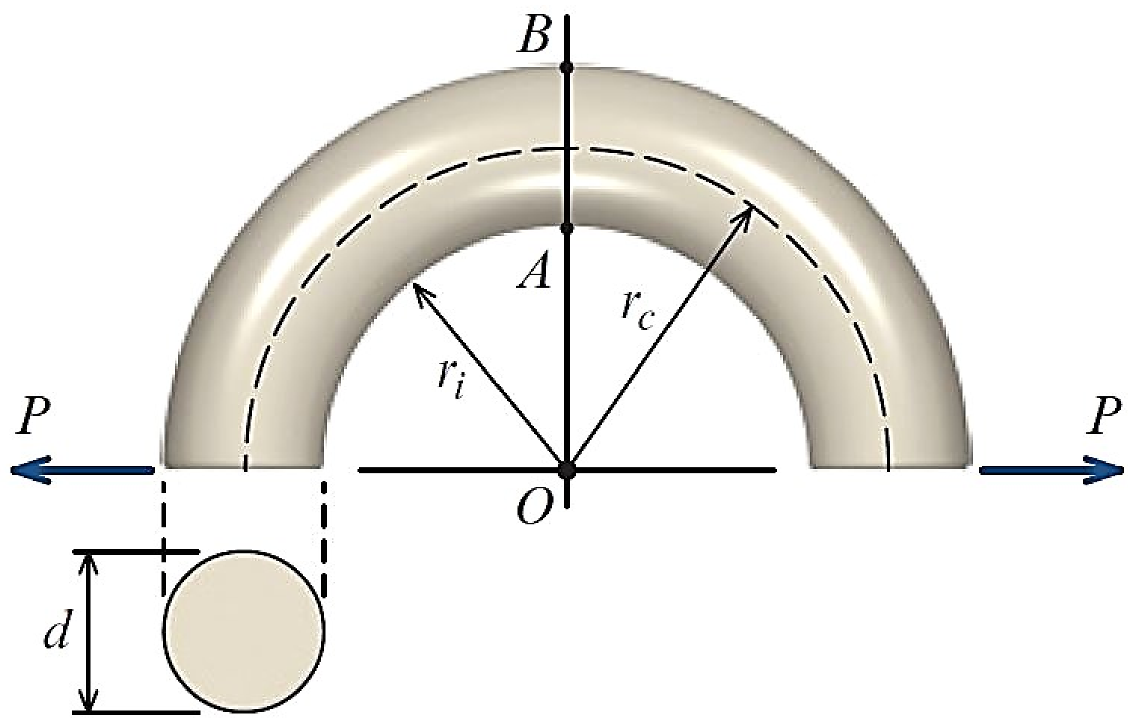
(a) the radial distance from O to the neutral axis.

(b) the stresses at points A and B.



Question 2

A solid circular rod of diameter *d* is bent into a semicircle as shown in the Figure. The radial distance to the centroid of the rod is to be $r_c = 3d$, and the curved rod is to support a load P = 5,000 N. If the allowable stress must be limited to 135 MPa, what is the smallest diameter *d* that may be used for the rod?



The End