MEC 3352
STRENGTH OF MATERIALS II

Torsion of Non-Circular Shafts

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Solid Noncircular Shafts

INTRODUCTION

- It has been demonstrated that when a torque is applied to a shaft having a circular cross section—that is, one that is ***axisymmetric***—the shear strains vary linearly from zero at its center to a maximum at its outer surface.

- Furthermore, due to the uniformity of the shear strain at all points on the same radius, the cross sections do not deform, but rather remain plane after the shaft has twisted.

- Shafts that have a noncircular cross section, however, ***are not axisymmetric***, and so their cross sections will ***bulge*** or ***warp*** when the shaft is twisted. Evidence of this can be seen from the way grid lines deform on a shaft having a square cross section when the shaft is twisted, Fig. 1–1.

- As a consequence of this deformation, the torsional analysis of ***noncircular*** shafts becomes considerably more complicated.

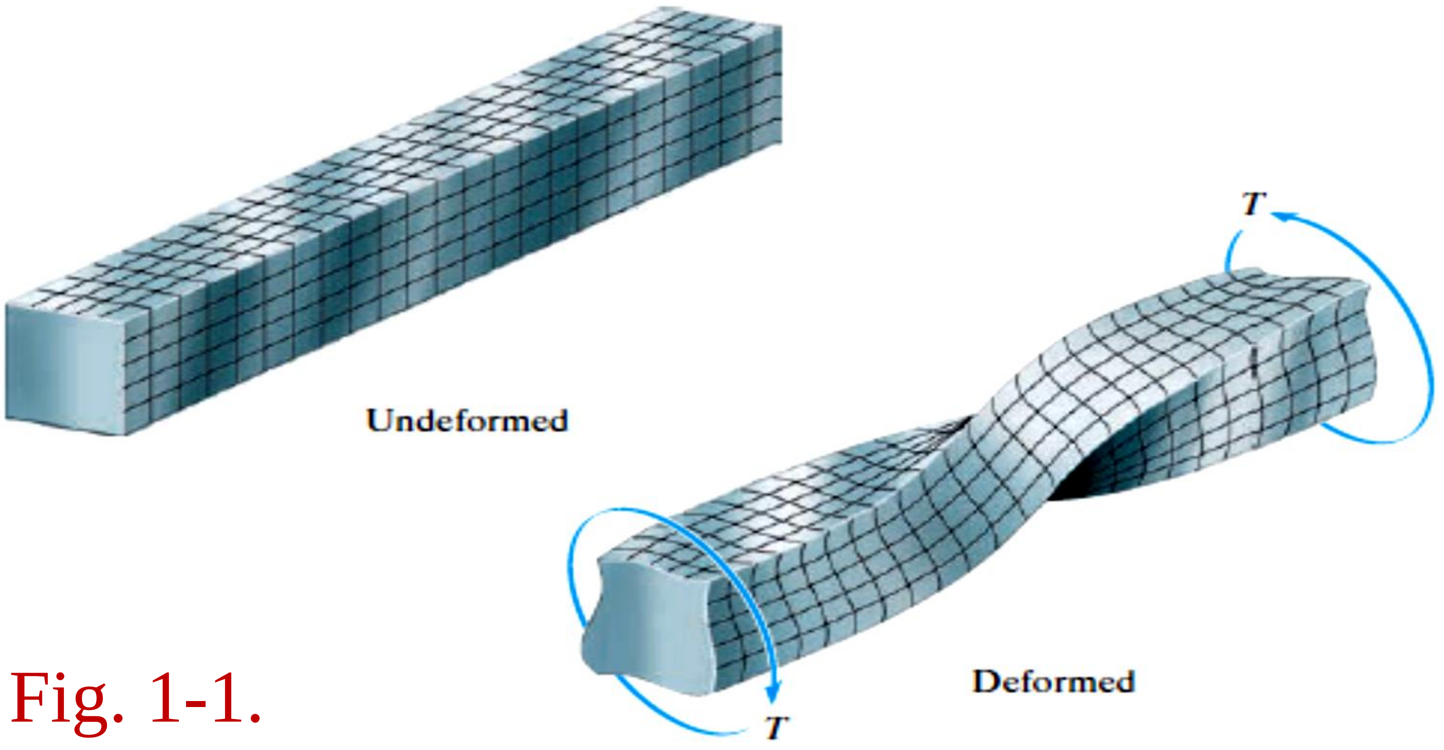


Fig. 1-1.

- Using a ***mathematical analysis*** based on the theory of elasticity, however, it is possible to determine the ***shear-stress distribution*** within a shaft of ***square cross section***.

- Examples of how this shear stress varies along two radial lines of the shaft are shown in Fig. 1–2*a*.

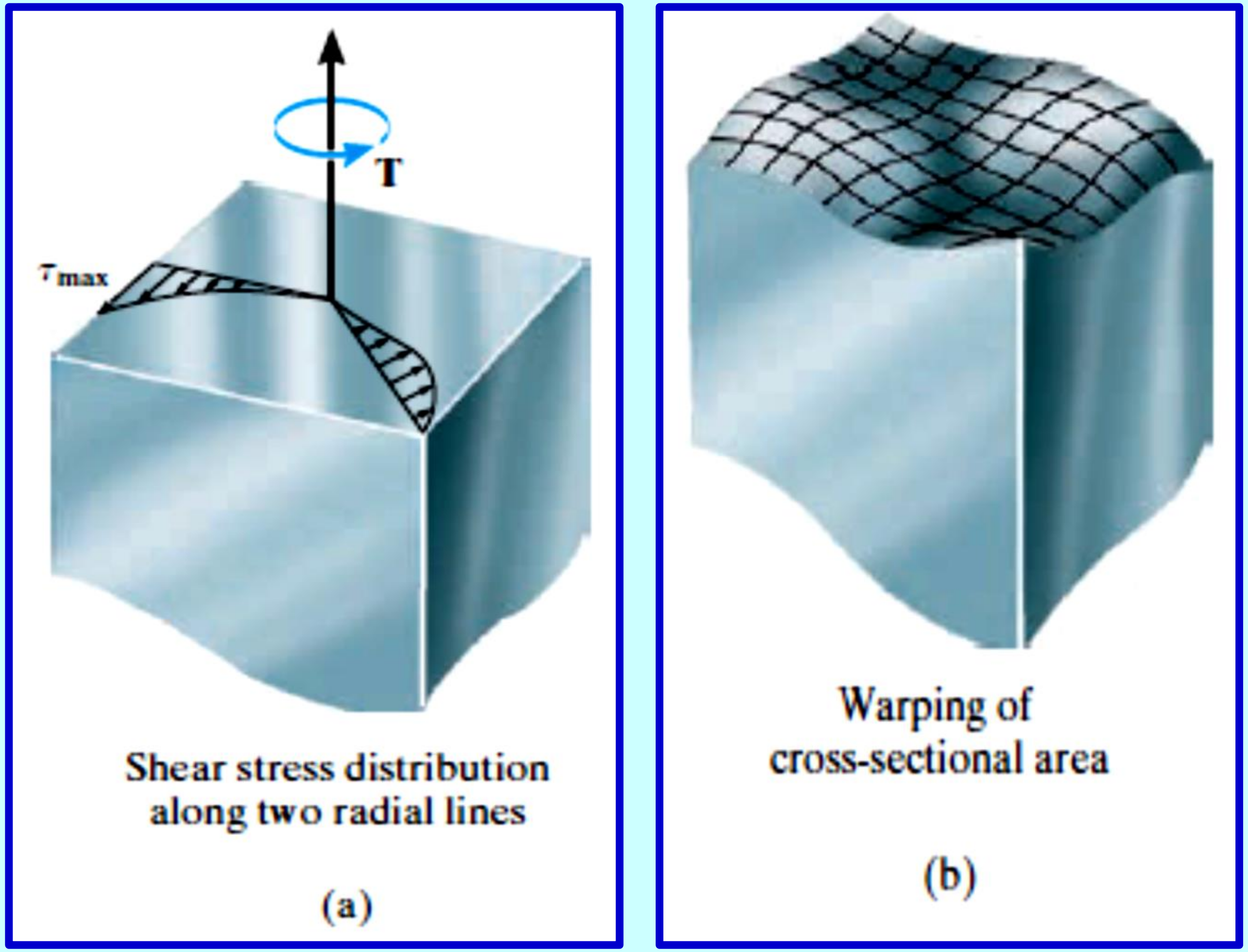


Fig. 1–2

- Because these shear-stress distributions vary in a complex manner, the ***shear strains*** they create will ***warp the cross section*** as shown in Fig. 1–2*b*.

- In particular notice that the ***corner points*** of the shaft must be subjected to zero shear stress and therefore ***zero shear strain***.

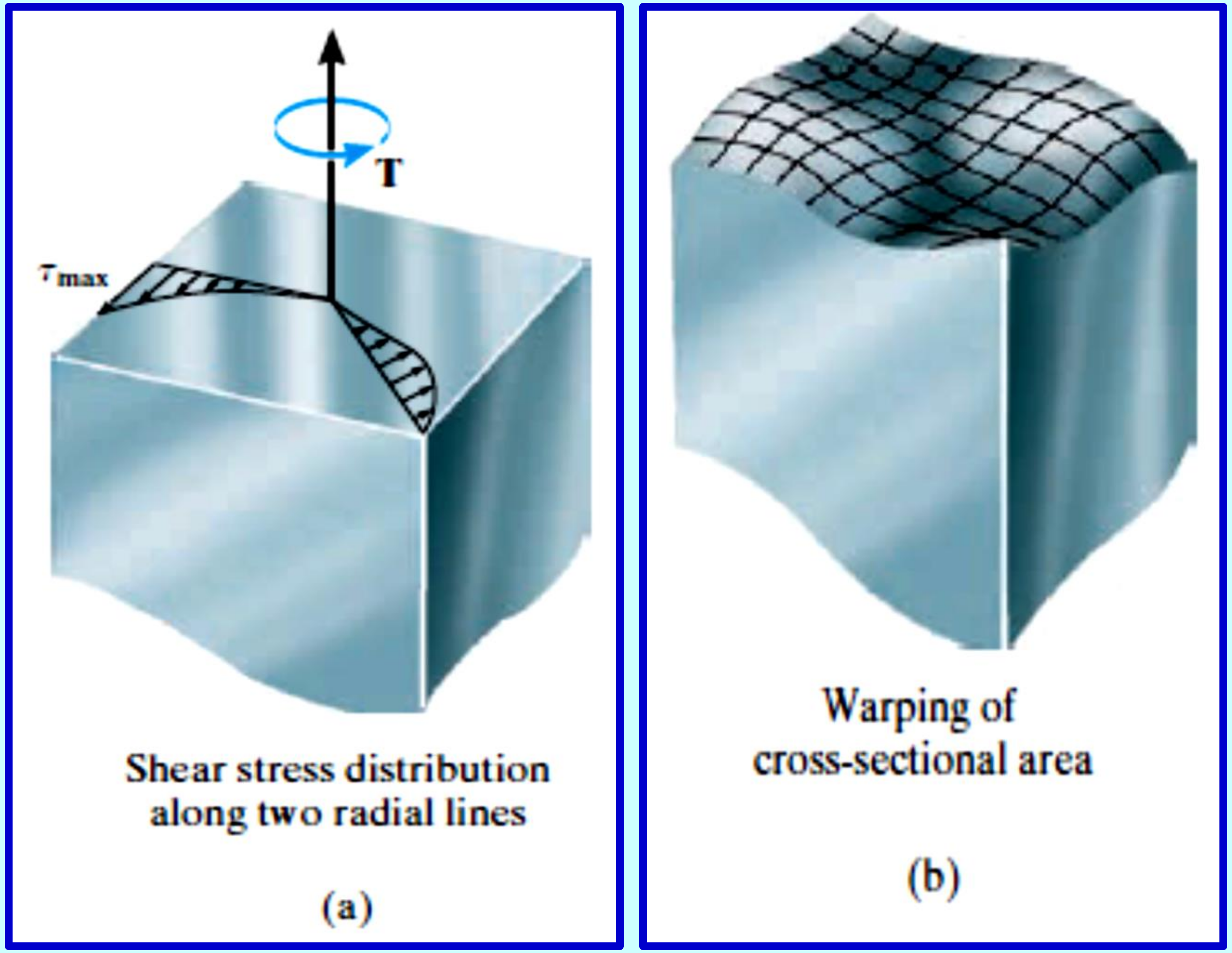


Fig. 1–2

- The reason for this can be shown by considering an element of material located at one of these points, Fig.1 –2*c*.
- One would expect the top face of this element to be subjected to a shear stress in order to aid in resisting the applied torque ***T***.
- This, however, cannot occur since the **complementary shear stresses τ and τ', acting on the *outer surface* of the shaft, must be zero**.

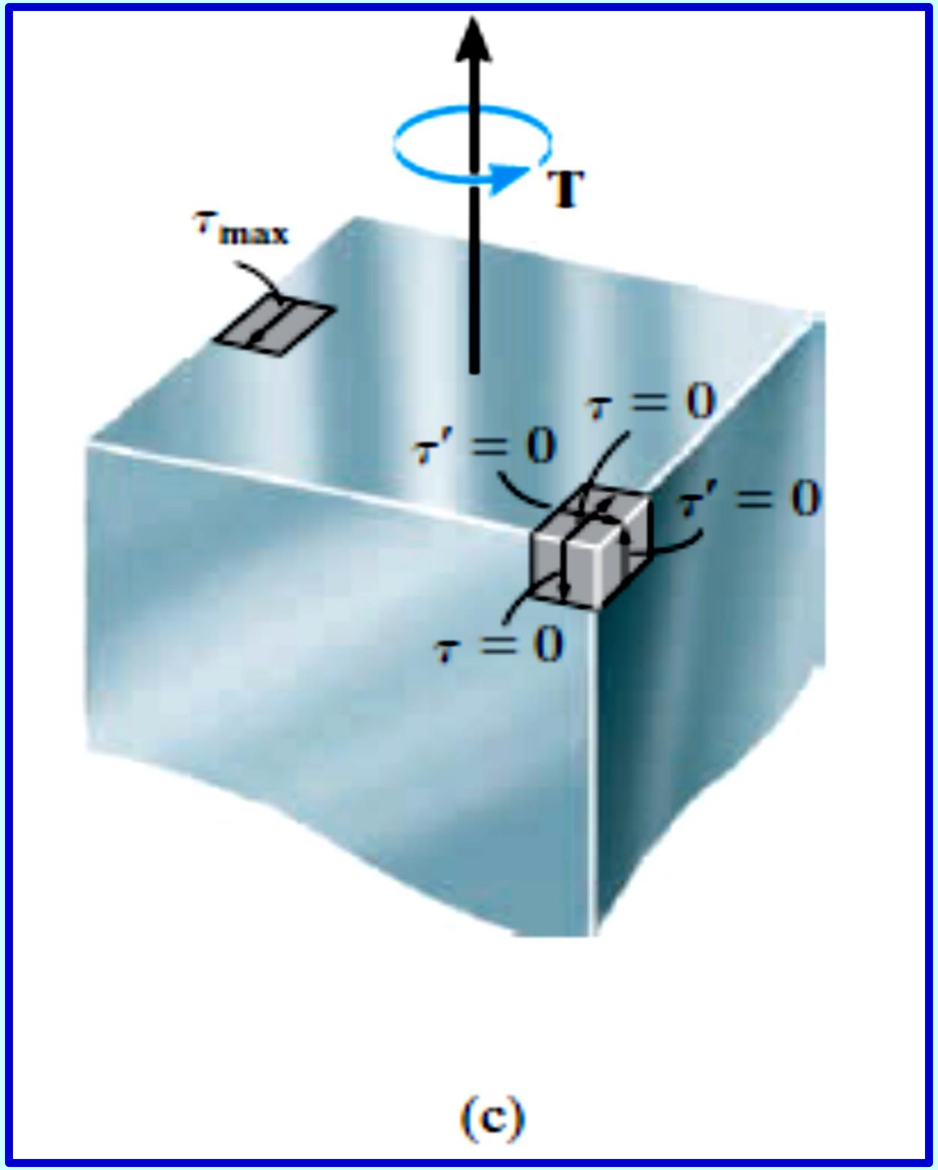
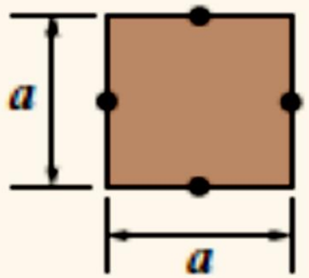
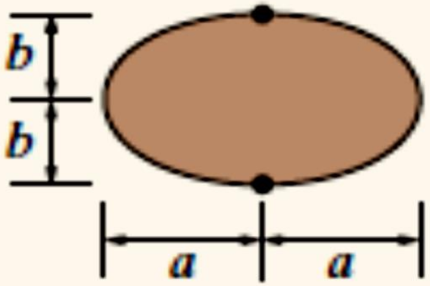


Fig. 1–2*c*.

- The results of the analysis for ***square cross sections***, along with other results from the theory of elasticity, for ***shafts having triangular and elliptical cross sections***, are reported in Table 1–1.

- In all cases the ***maximum shear stress occurs at a point on the edge of the cross section that is closest to the centre axis*** of the shaft. In Table 1–1 these points are indicated as "dots" on the cross sections.

Table 1–1

Shape of cross section	τ_{max}	ϕ
Square 	$\frac{4.81\,T}{a^3}$	$\frac{7.10\,TL}{a^4G}$
Equilateral triangle 	$\frac{20\,T}{a^3}$	$\frac{46\,TL}{a^4G}$
Ellipse 	$\frac{2\,T}{\pi ab^2}$	$\frac{(a^2+b^2)TL}{\pi a^3b^3G}$

- Also given are formulae for the angle of twist of each shaft.
- Extending these results to a shaft having an ***arbitrary*** cross section, it can also be shown that:
- A shaft having a ***circular*** cross section is most efficient, since it is subjected to both a ***smaller maximum shear stress*** and a ***smaller angle of twist*** than a corresponding shaft having a noncircular cross section and subjected to the same torque.

BREDT-BATHO EQUATIONS

Two very important equations in the treatment of shafts undergoing twisting load or torque are the ***Bredt-Batho equations*** for the ***average torque*** and ***angle of twist*** of the shaft.

- $$\tau = \frac{T}{2At}$$

  - This relation is known as ***Bredt's first formula*** (Rudolf Bredt, 1842–1900) or as torsion formula for thin-walled tubes.

- $$\theta = \frac{TL}{4A^2G}\int\frac{ds}{t}$$ ***OR*** $$\phi = \frac{TL}{4A_m^2G}\oint\frac{ds}{t}$$ is ***Bredt's second formula***.

Thin-Walled Tubes with Closed Cross Sections

- We will analyse the effects of applying a ***torque*** to a thin-walled tube having a ***closed*** cross section, that is, a tube that does not have any breaks or slits along its length.

- Such a tube, having a ***constant yet arbitrary cross-sectional shape***, and ***variable thickness t***, is shown in Fig. 1–3*a*.

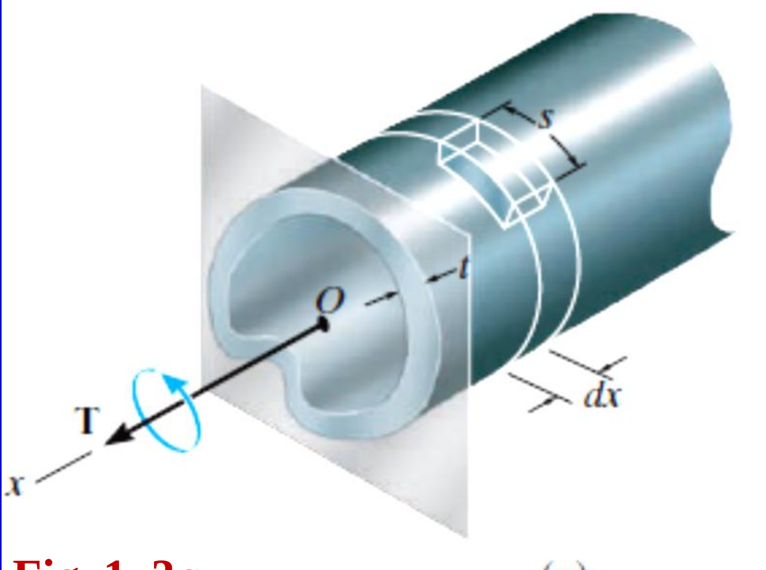


Fig. 1–3*a*.

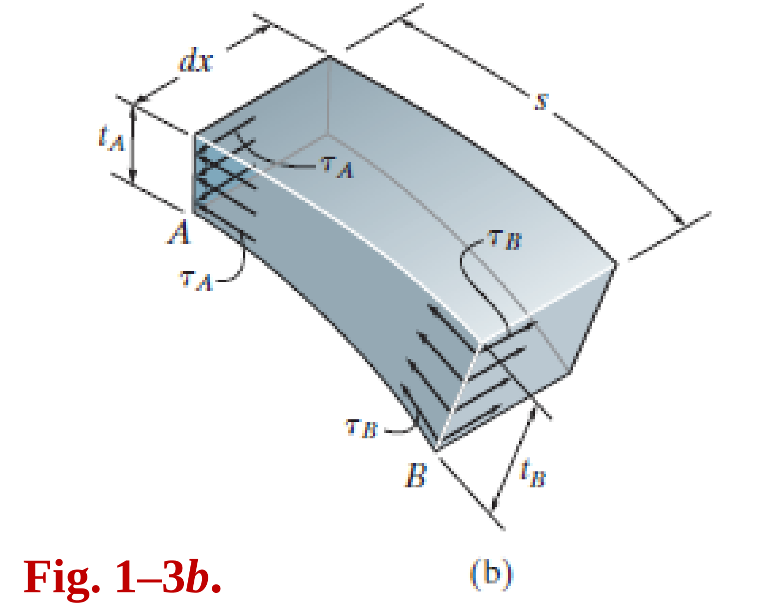


Fig. 1–3*b*.

Thin-Walled Tubes with Closed Cross Sections

- Since ***the walls are thin***, we will obtain the ***average shear stress*** by assuming that this stress is ***uniformly distributed*** across the thickness of the tube at any given point.

- Discussion of ***shear stress*** over the cross section is important to understanding concept.

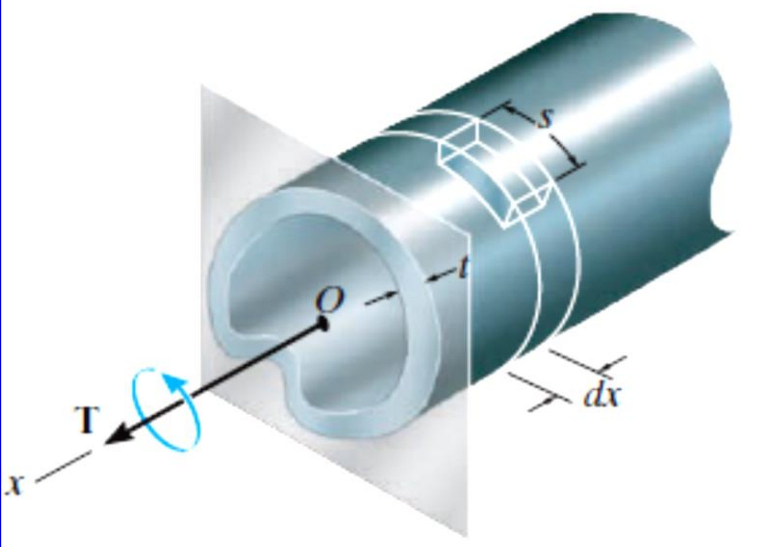


Fig. 1–3*a*. (a)

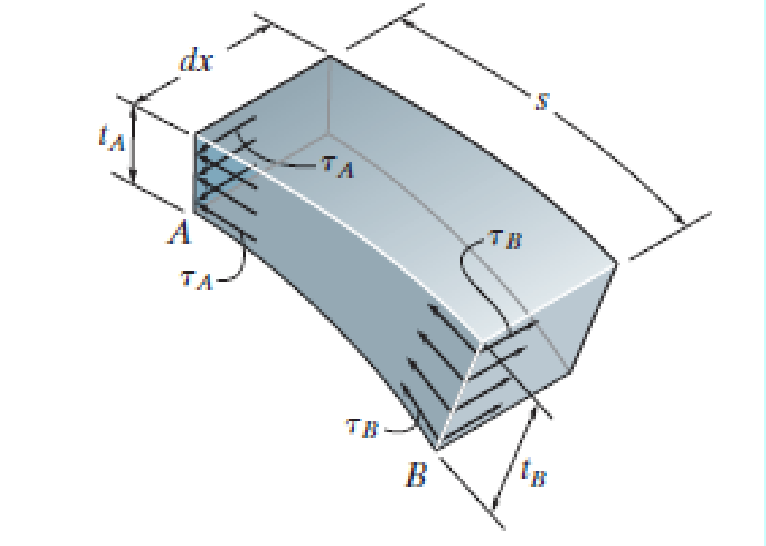


Fig. 1–3*b*. (b)

Concepts of shear stress over the cross section.

Shear Flow.

- Shown in Figs. 1–3*a* and1–3*b* is a small element of the tube having a finite length ***s*** and differential width ***dx***. At one end the element has a thickness t_A and at the other end the thickness is t_B.

- Due to the ***internal torque T***, shear stress is developed on the front face of the element. Specifically, at end *A* the shear stress is τ_A and at end *B* it is τ_B.

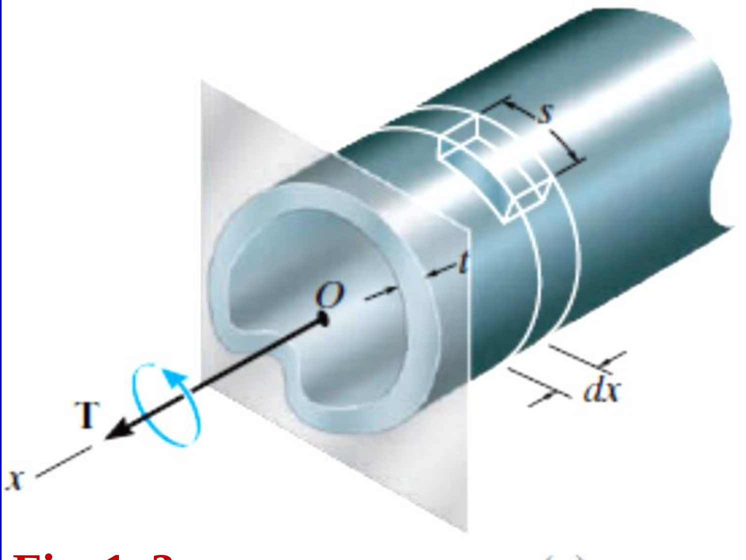


Fig. 1–3*a*.

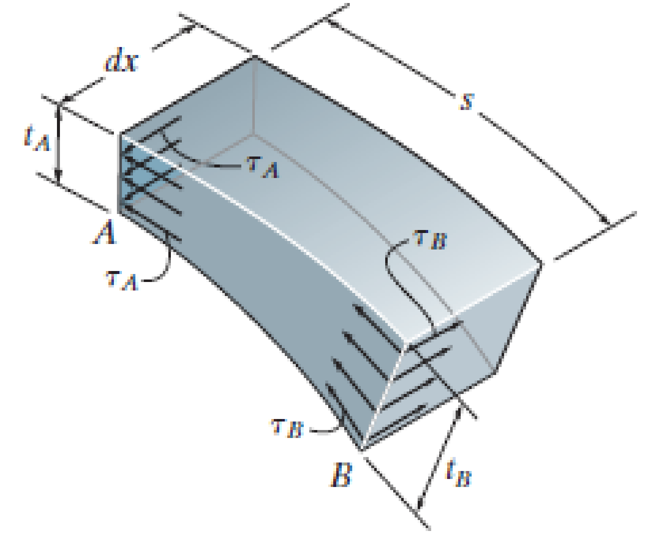


Fig. 1–3*b*.

Concepts of shear stress over the cross section.

- These stresses can be related by noting that equivalent shear stresses τ_A and τ_B must also act on the longitudinal sides of the element.

- Since these sides have a ***constant width dx***, the forces acting on them are $dF_A = \tau_A(t_A dx)$ and $dF_B = \tau_B(t_B dx)$.

- Equilibrium requires these forces to be of equal magnitude but opposite direction, so that:

$$\tau_A t_A = \tau_B t_B$$

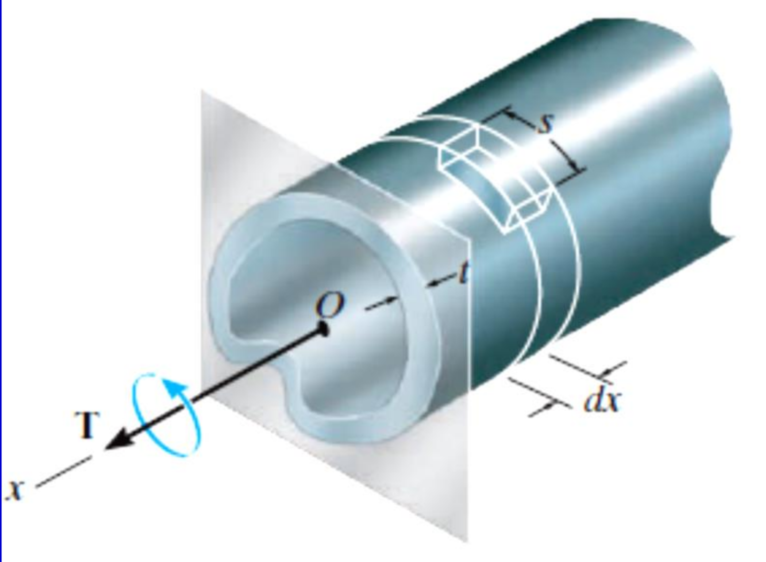


Fig. 1–3*a*.

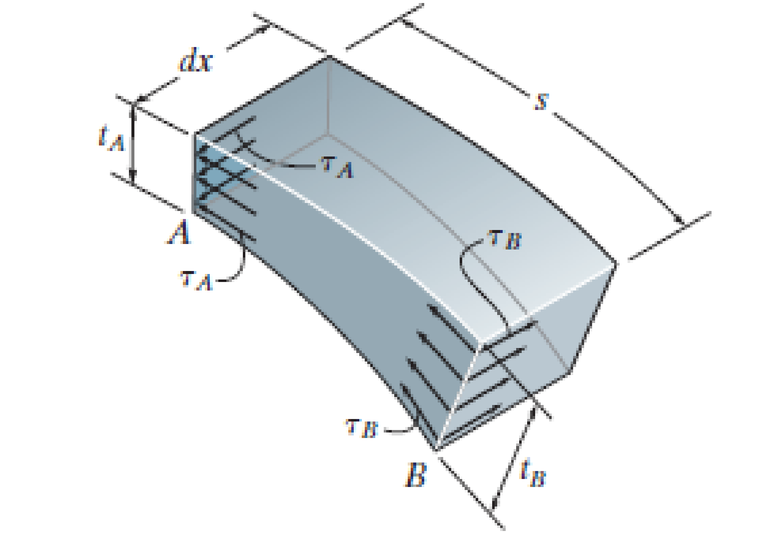


Fig. 1–3*b*.

Shear Flow

- This important result states that ***"the product of the average shear stress times the thickness of the tube is the same at each point on the tube's cross-sectional area"***. This product is called ***shear flow********, ***q***, and in general terms we can express it as

$$q = \tau_{avg} t$$

- Since q is constant over the cross section, the ***largest* average shear stress must occur where the tube's thickness is the *smallest*.**

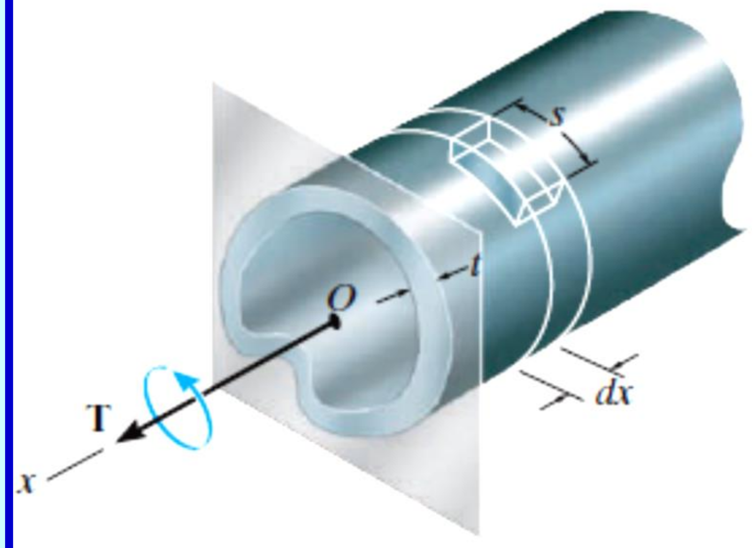


Fig. 1–3*a*. (a)

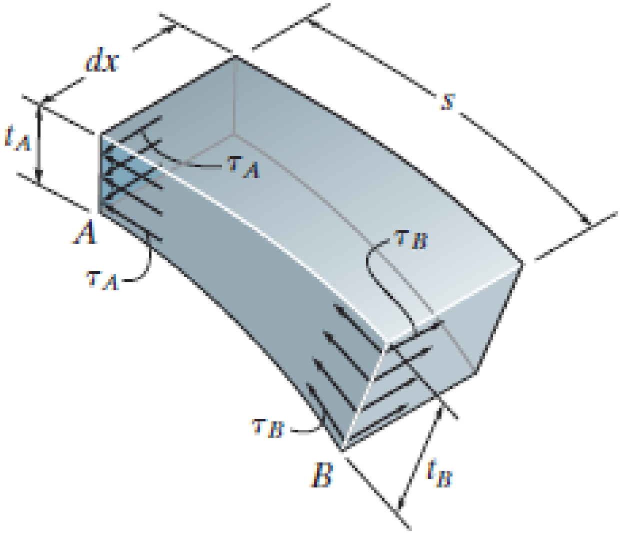
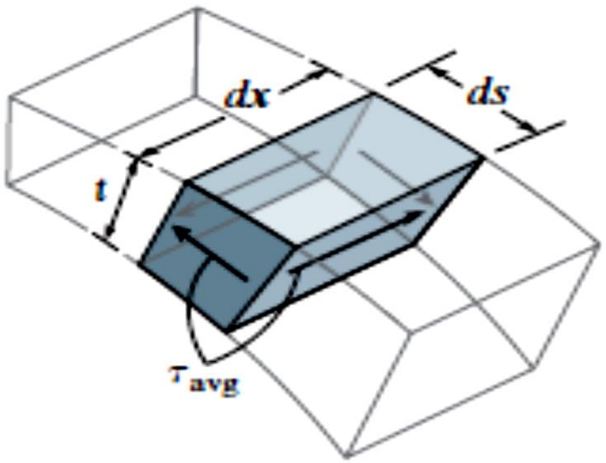


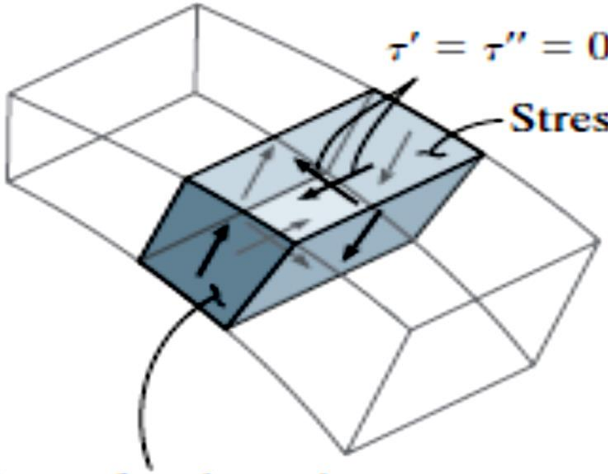
Fig. 1–3*b*. (b)

- This is because the top and bottom faces of the element are at the inner and outer walls of the tube, and these boundaries must be free of stress.

- Instead, as noted above, the applied torque causes ***the shear flow and the average stress to always be directed tangent to the wall of the tube, such that it contributes to the resultant internal torque T.***



(c) **Fig. 1–3c**



(d) **Fig. 1–3d**

Concepts: Average Shear stress.

- The average shear stress can be related to the torque ***T*** by considering the torque produced by this shear stress about a selected point *O* within the tube's boundary, Fig. 1–3*e*.

- As shown, the shear stress develops a force $dF = \tau_{avg} dA = \tau_{avg}(tds)$ on an element of the tube. This force acts tangent to the centerline of the tube's wall, and if the moment arm is ***h***, the torque is

$$dT = h(dF) = h(\tau_{avg} tds)$$

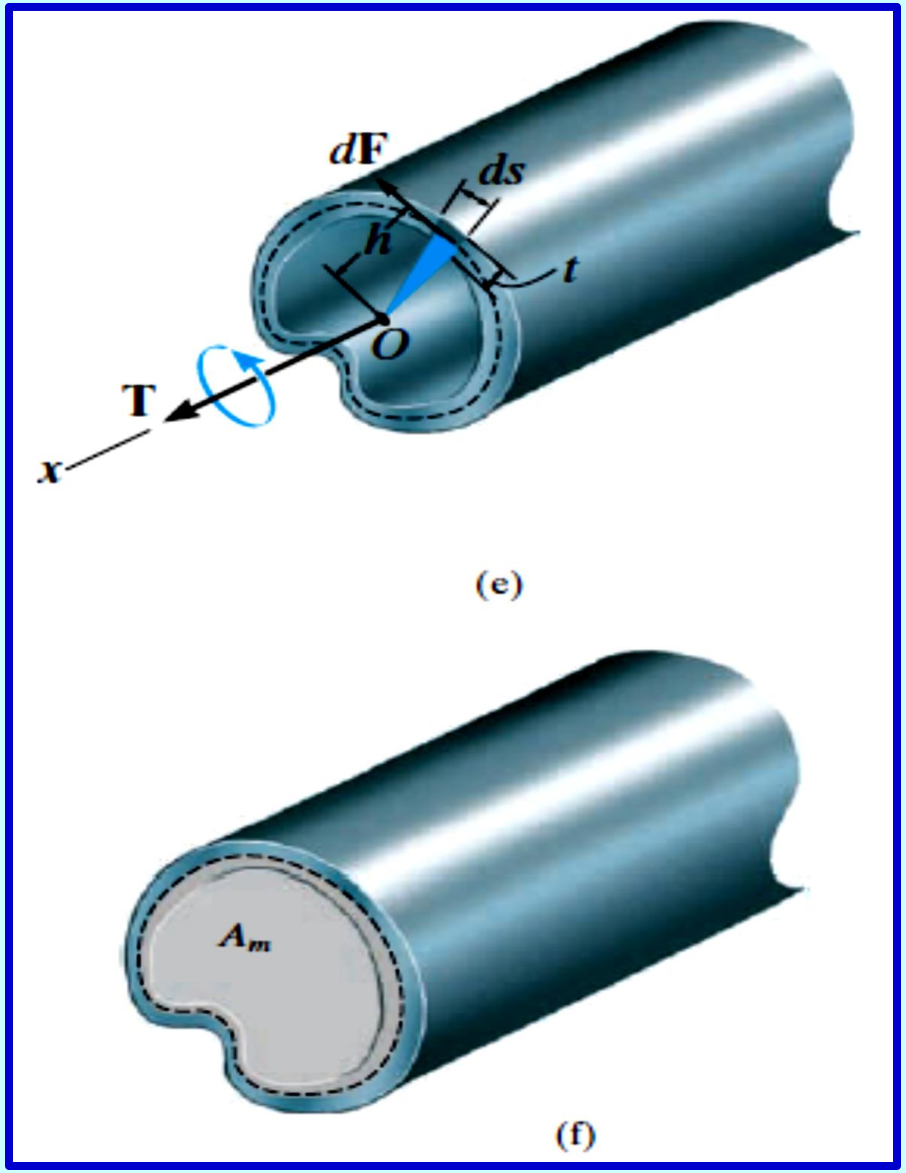


Fig. 1–3*e* and Fig. 1–3*f*

- For the entire cross section, we require

$$T = \oint h\tau_{avg}\, t\, ds$$

- Here the ***"line integral"*** indicates that integration must be performed ***around*** the entire boundary of the area.

- Since the shear flow $q = \tau_{avg} t$ is *constant*, it can be factored out of the integral, so that

$$T = \tau_{avg} t \oint h\, ds$$

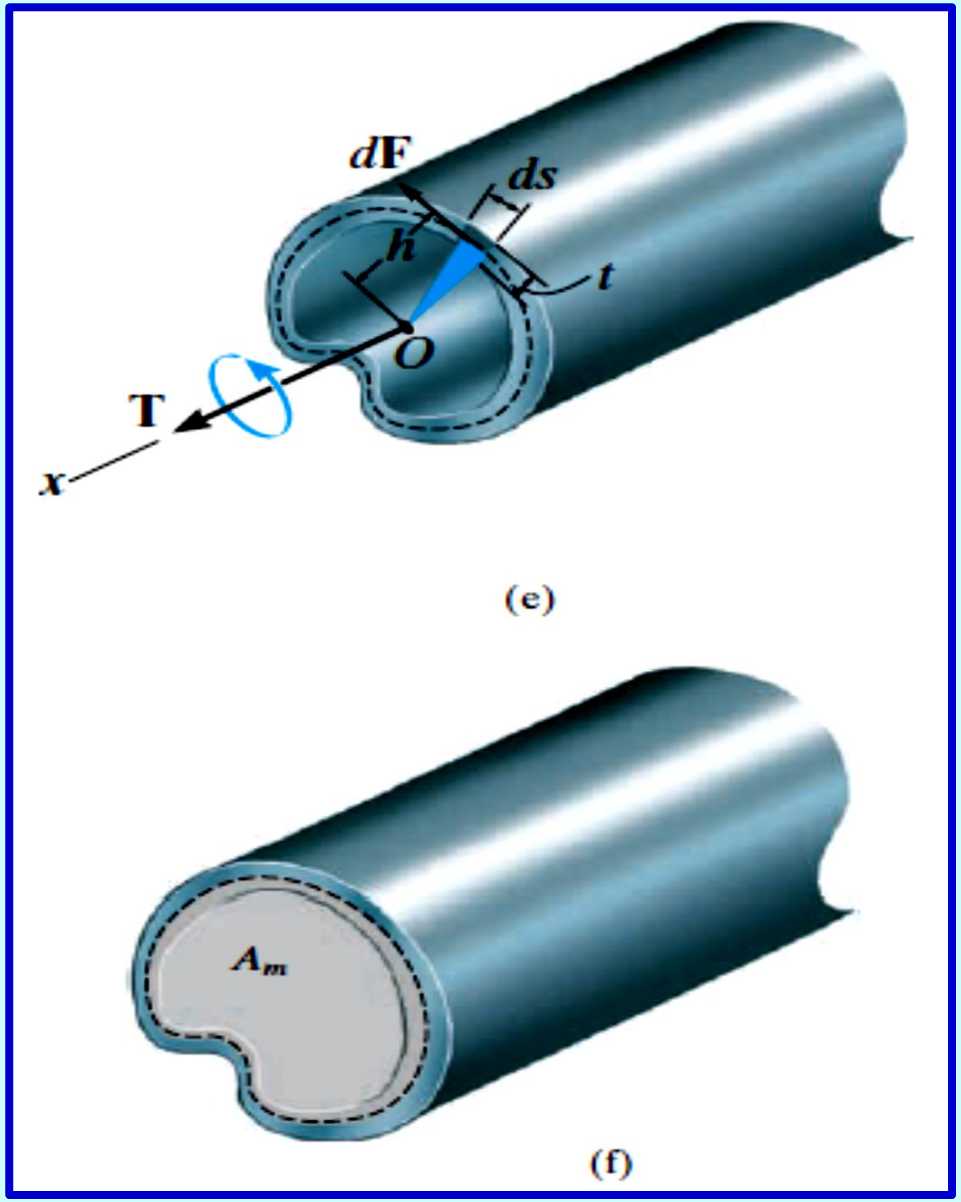


Fig. 1–3*e* and Fig. 1–3*f*

- A graphical simplification can be made for evaluating the integral by noting that the ***mean area***, shown by the blue colored triangle in Fig. *1–3e, is* $dA_m = (1/2)hds$. Thus,

$$T = 2\tau_{avg}t\int dA_m = 2\tau_{avg}\,tA_m$$

- Solving for τ_{avg} we have

- $\tau_{avg} = \dfrac{T}{2tA_m}$ **(Eqn 1-1)**

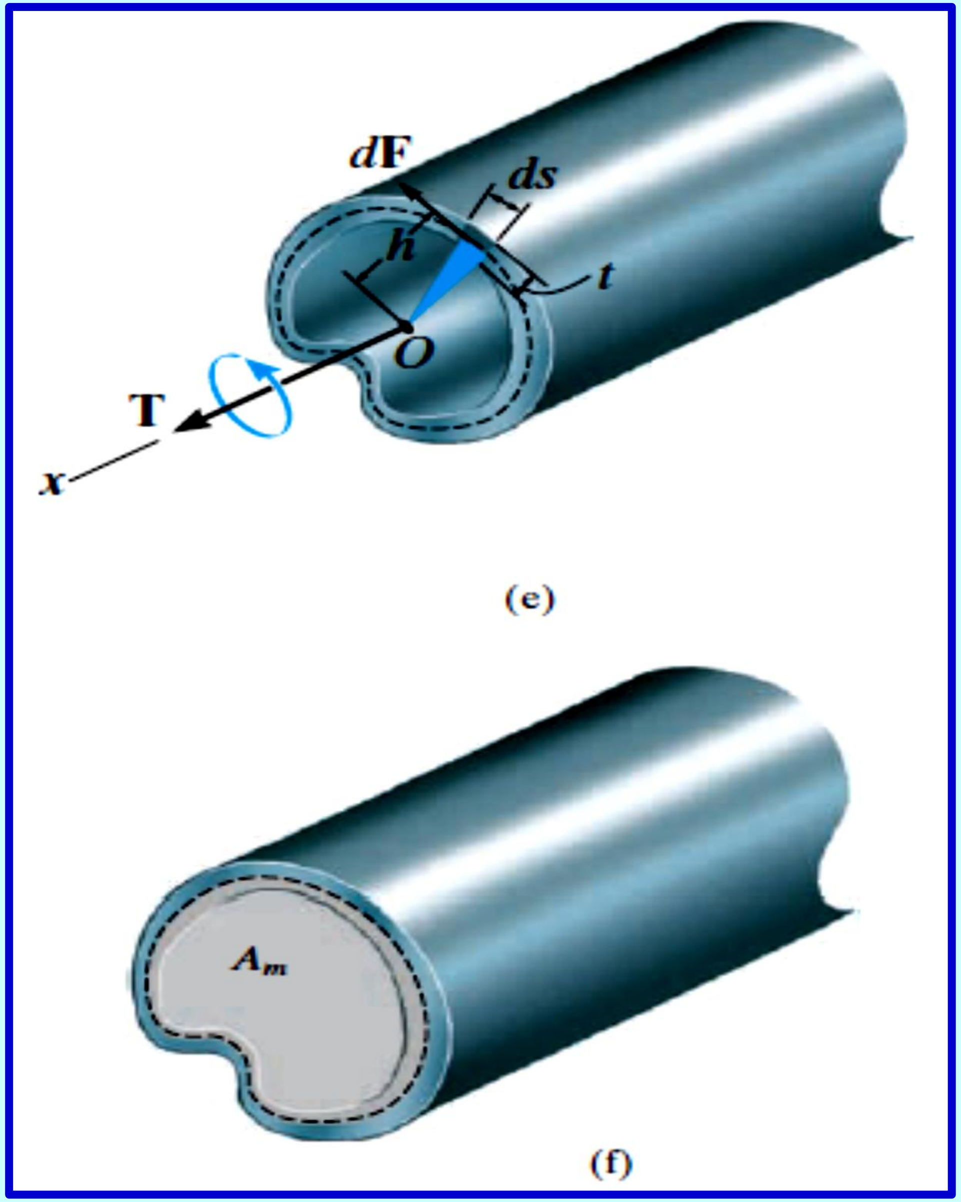


Fig. 1–3*e and Fig. 1–3f*

- Here
  - τ_{avg} = the average shear stress acting over a particular thickness of the tube the resultant internal torque at the cross section
  - T = resultant internal torque at the Cross section;
  - t = the thickness of the tube where τ_{avg} is to be determined;

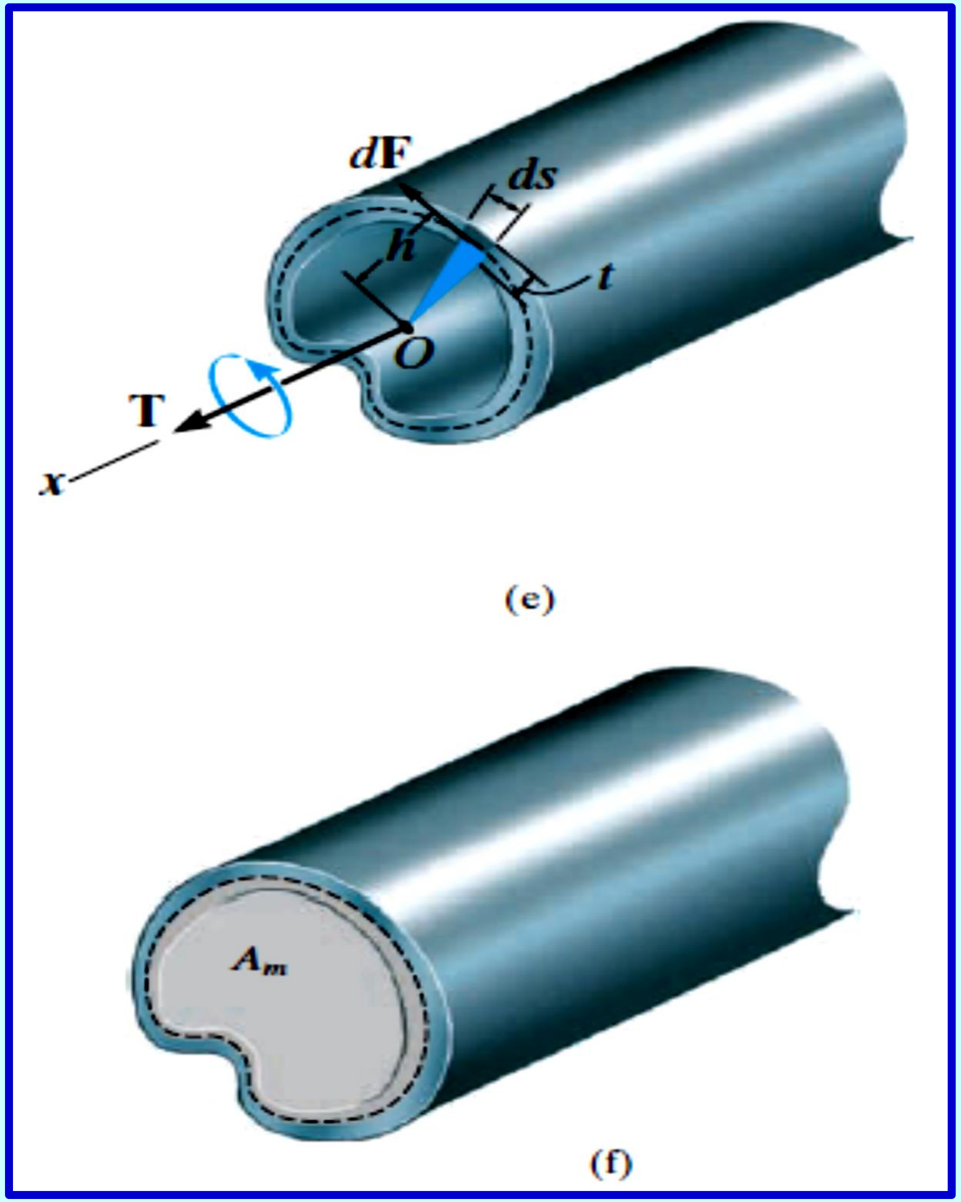


Fig. 1–3e and Fig. 1–3f

- A_m = the mean area enclosed within the boundary of the centerline of the tube's thickness. A_m is shown shaded in Fig 1-3f.

  Since $q = \tau_{avg} t$, then the shear flow throughout the cross section becomes

- $q = \dfrac{T}{2A_m}$ (Eqn 1–2)

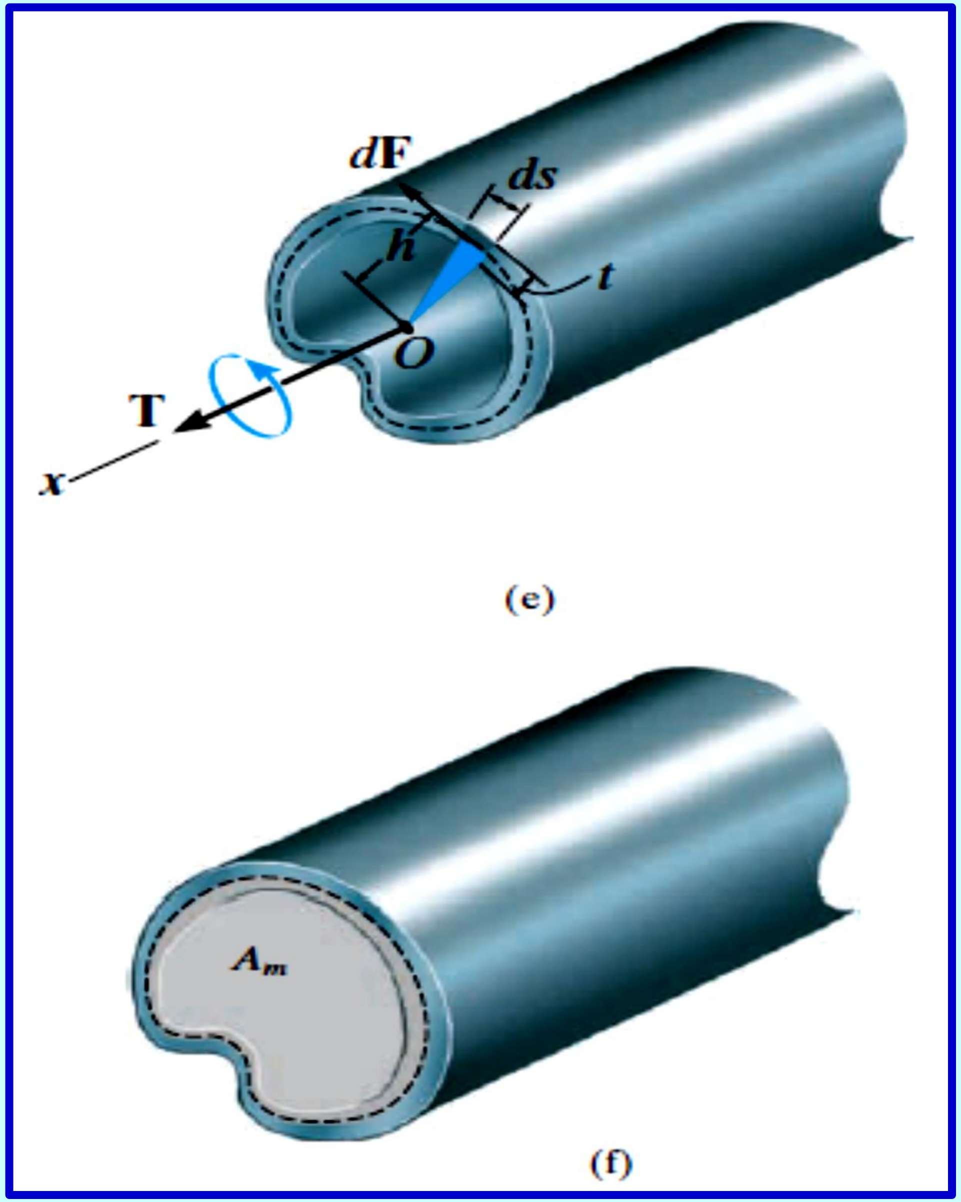


Fig. 1–3e and Fig. 1–3f

Angle of Twist

- The angle of twist of a thin-walled tube of length L can be determined using energy methods, If the material behaves in a linear elastic manner and G is the shear modulus, then this angle ϕ given in radians, can be expressed as

$$\phi = \frac{TL}{4A_m^2 G} \oint \frac{ds}{t} \qquad \text{(Eqn. 1-3)}$$

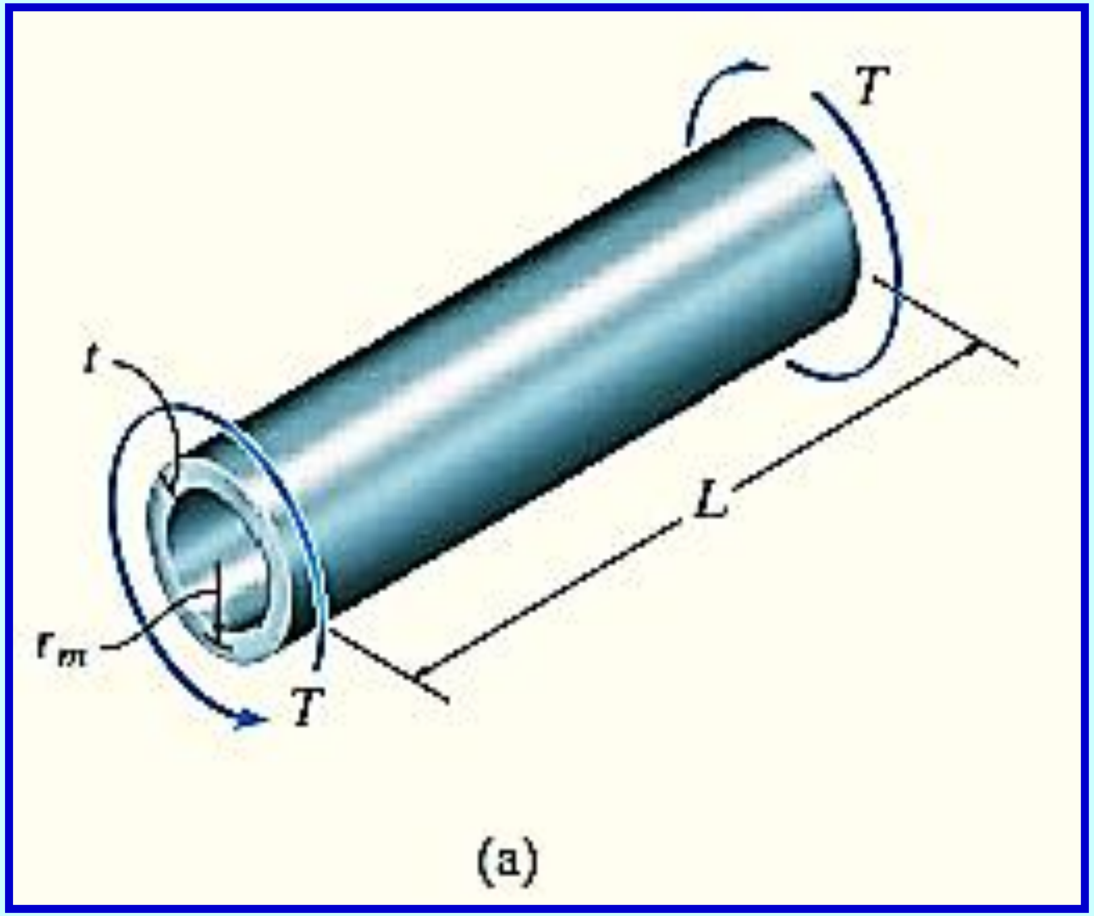
- Here again the integration must be performed around the entire boundary of the tube's cross-sectional area.

EXAMPLES/QUIZ

EXAMPLE 1

Calculate the average shear stress in a thin-walled tube (see figure overleaf) having a circular cross section of mean radius r_m and thickness t, which is subjected to a torque T, Fig. 1–4a. Also, what is the relative angle of twist if the tube has a length L?

QUIZ


T
t
L
r_m
T
(a)

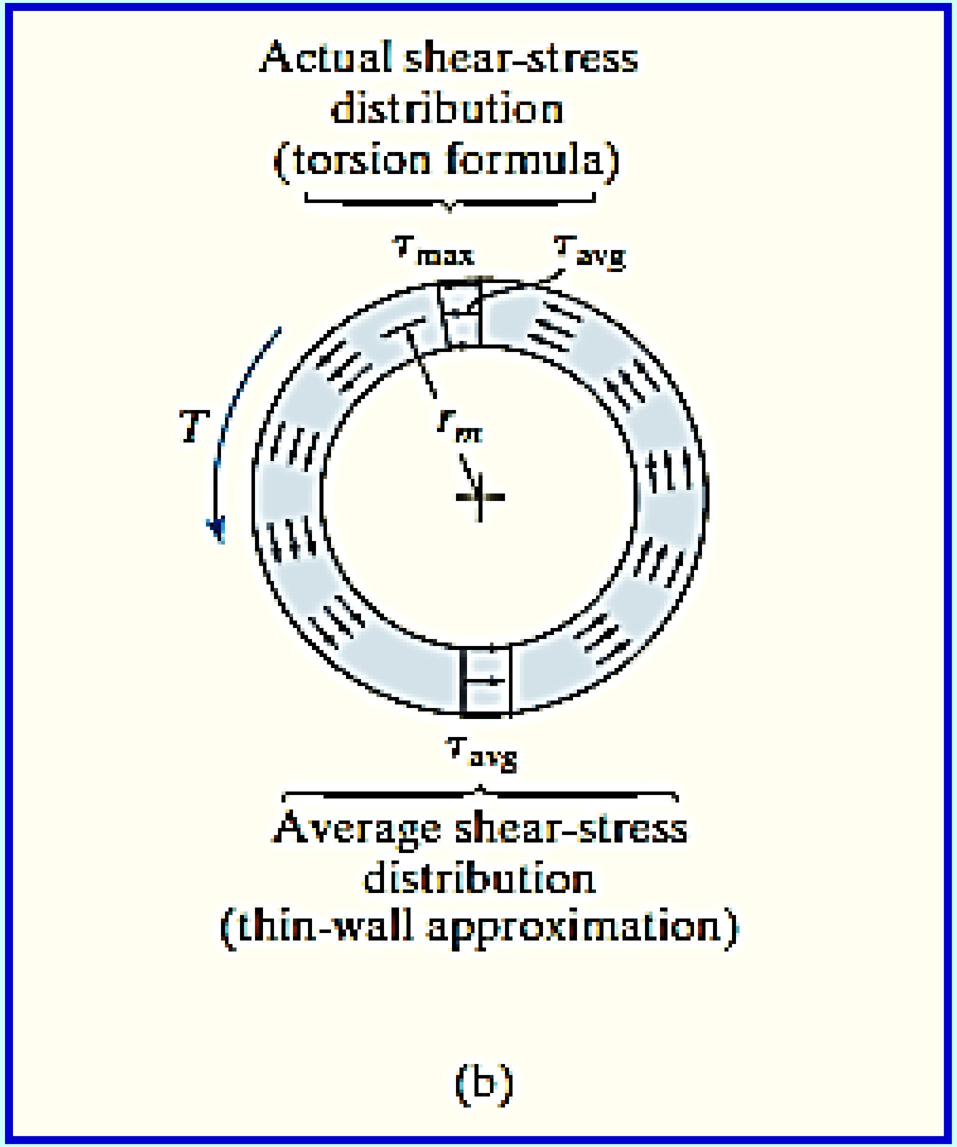

Actual shear-stress
distribution
(torsion formula)
τ_{max}
τ_{avg}
T
r_m
τ_{avg}
Average shear-stress
distribution
(thin-wall approximation)
(b)

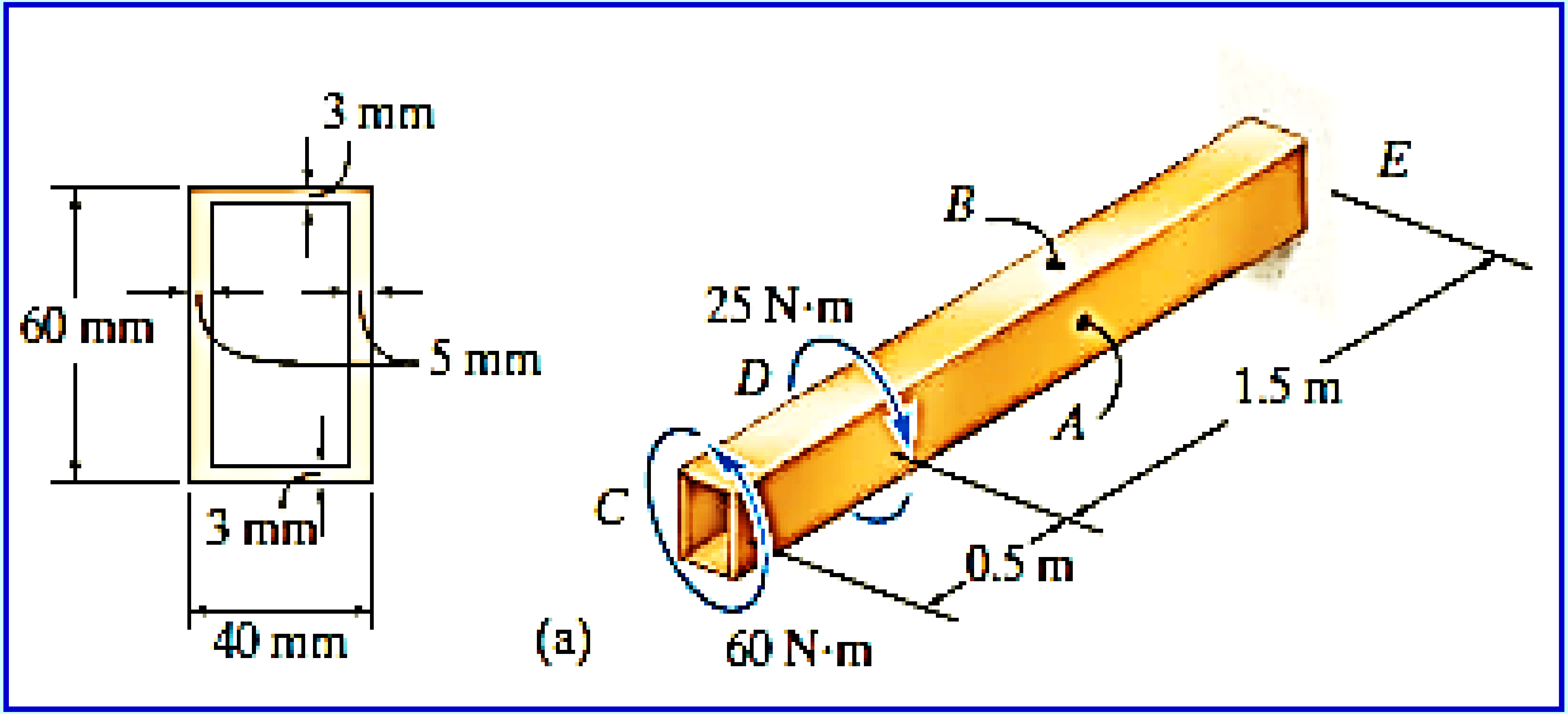
Fig. 1–4*a*

EXAMPLE 2

EXAMPLE 2

The tube, in the figure overleaf, is made of C86100 bronze and has a rectangular cross section as shown in Fig. 1–5*a*. If it is subjected to the two torques, determine the average shear stress in the tube at points *A* and *B*. Also, what is the angle of twist of end *C*? The tube is fixed at *E*.

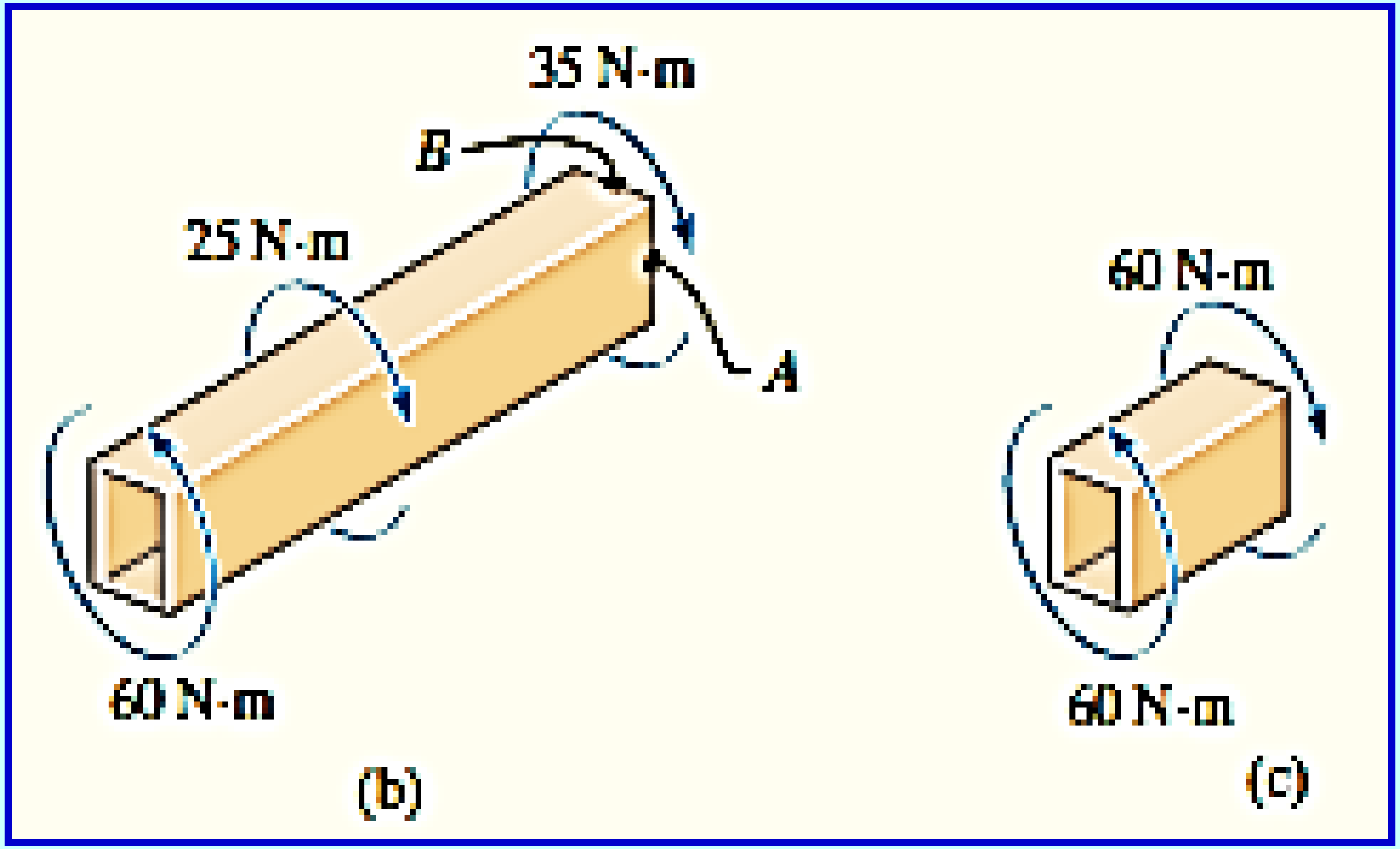
Fig for EXAMPLE 2



SOLUTION

Average Shear Stress.

- If the tube is sectioned through points A and B, the resulting free-body diagram is shown in Fig.Q1b.
- The internal torque is 35 N.m.



- As shown in Fig Q1d, the mean area is

$$A_m = (0.035\ m)(0.057m) = \mathbf{0.00200m^2}$$

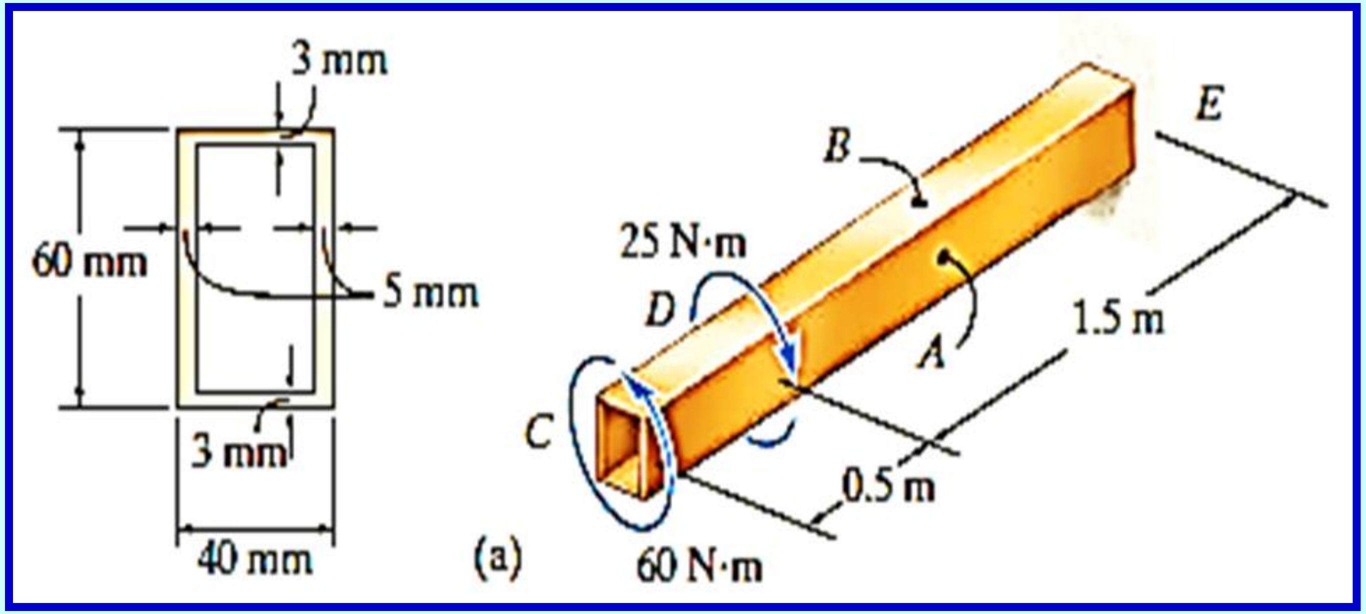
- Applying Eq.

- $\tau_{avg} = \frac{T}{2tA_m}$ for point A,

  t_A= 5 mm, so that

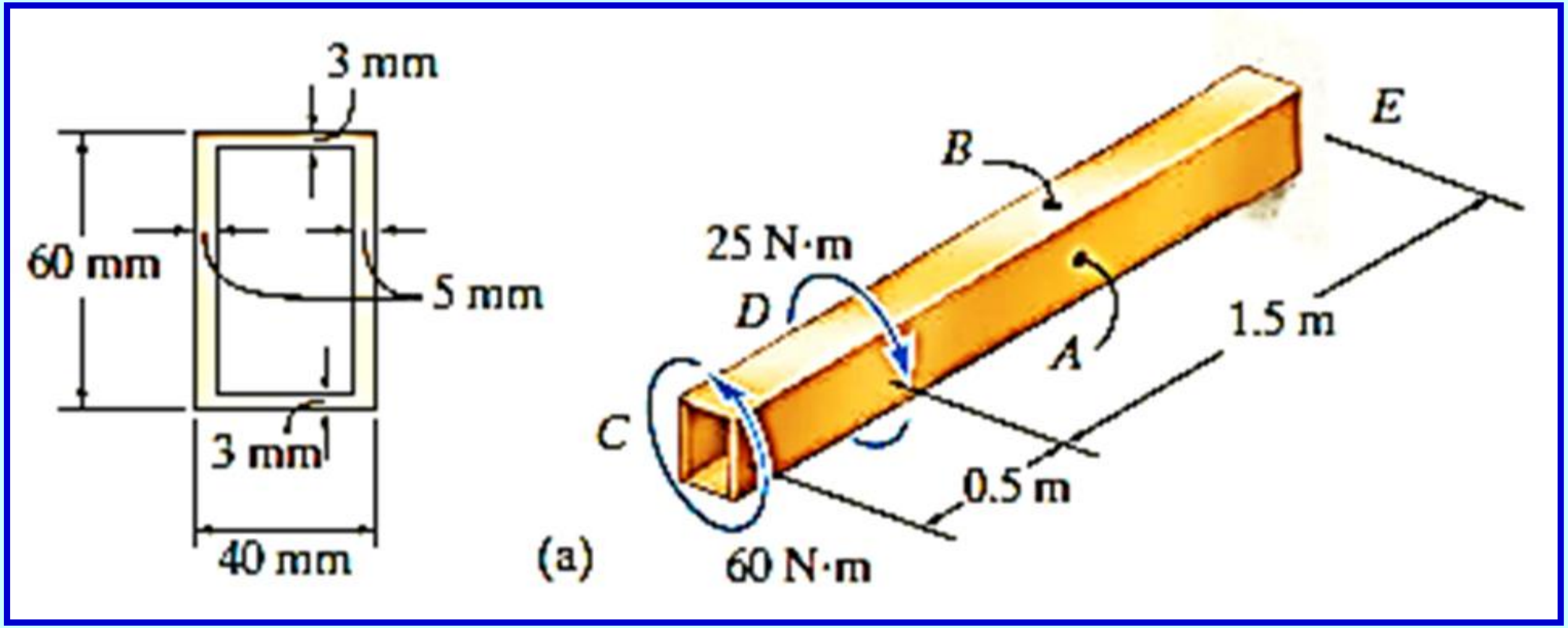
- $\boldsymbol{\tau_A} = \frac{\mathbf{35\ N.m}}{\mathbf{2(0.005\ m)(0.00200m^2)}}$

  = 1.75MPa ***Ans***



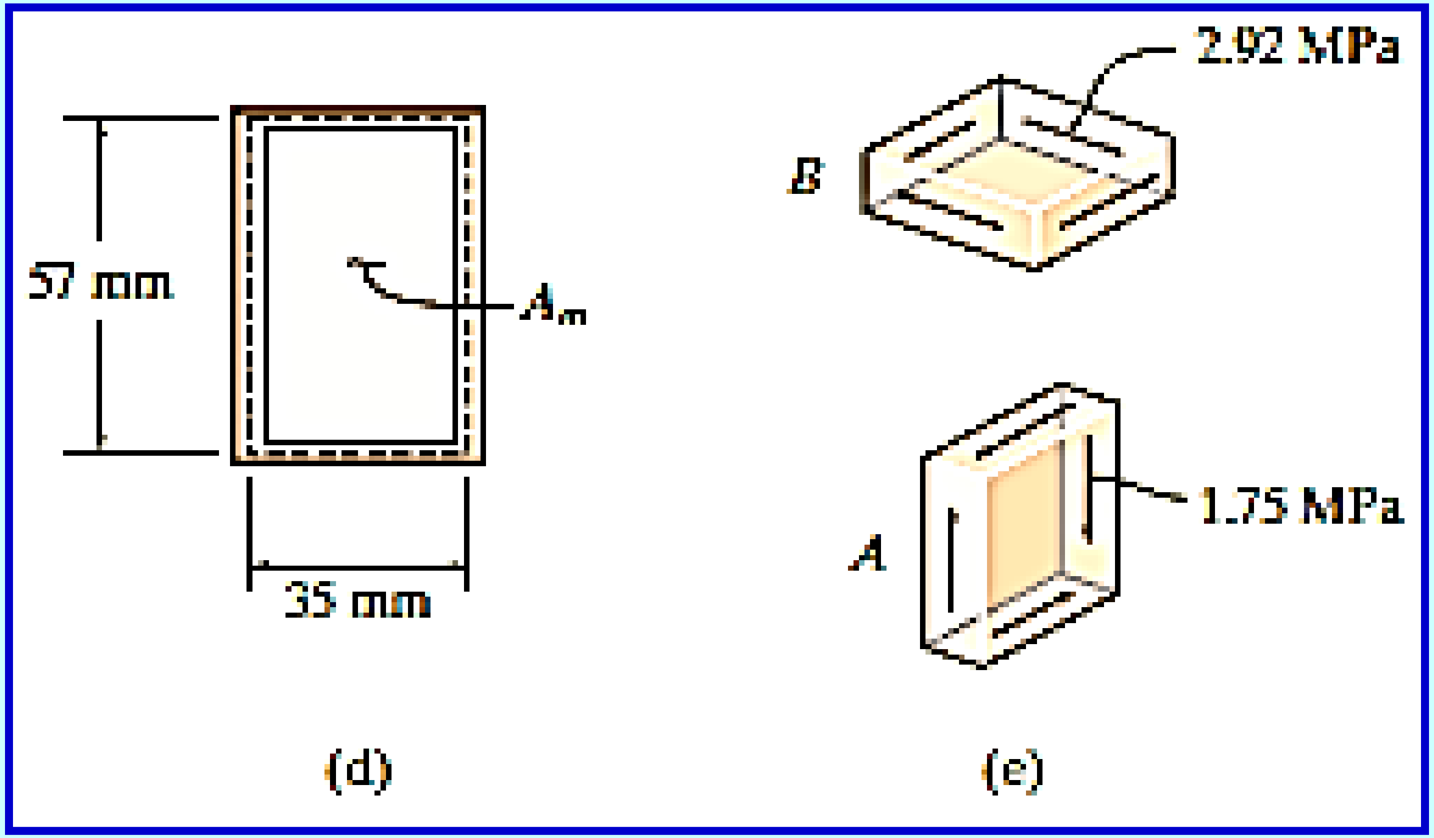
- And for point B, t_B= 3 mm, and therefore

- $$\tau_B = \frac{T}{2tA_m} = \frac{35\ N.m}{2(0.003\ m)(0.00200m^2)} = 2.92\ \text{MPa}$$ ***Ans.***

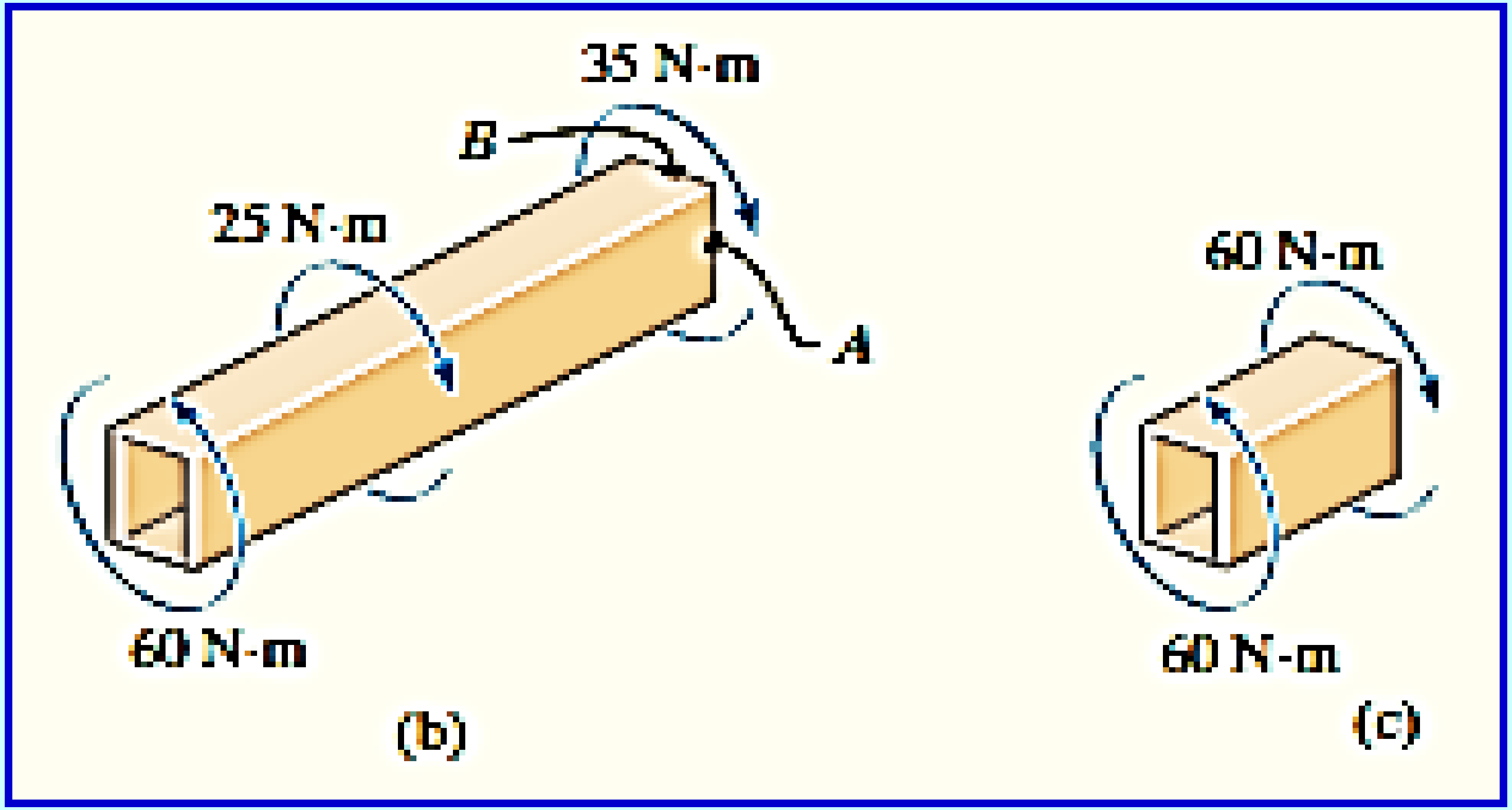


- These results are shown on elements of material located at points *A* and *B*, Fig. Q1*e*.

- Note carefully how the torque in Fig.Q1*b* creates these stresses on the back sides of each element.

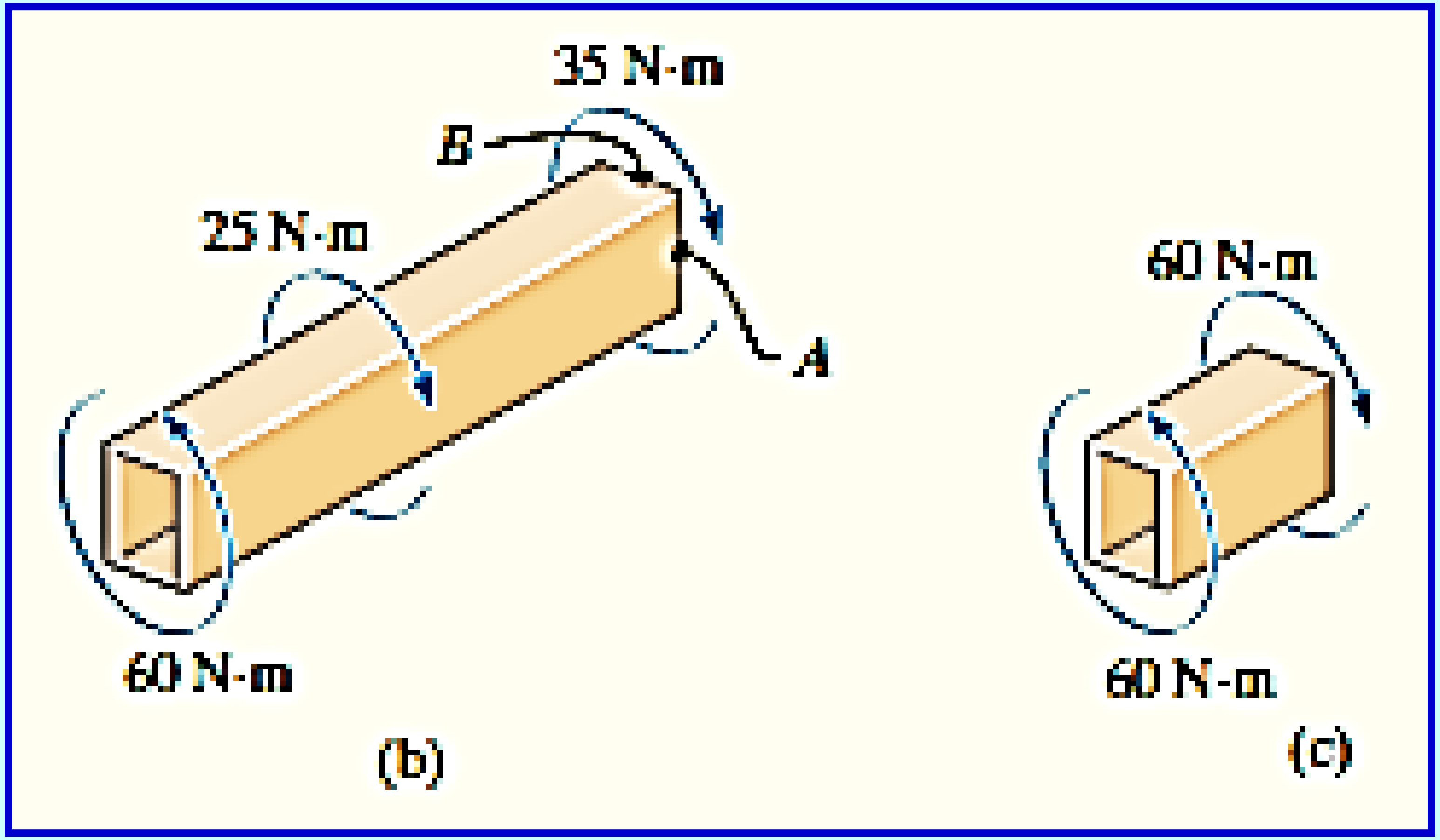


- **Angle of Twist**. From the free-body diagrams in Fig.Q1*b* and Q1*c*, the internal torques in regions *DE* and *CD* are 35 N.m and 60 N.m respectively.
- Following the sign convention outlined, these torques are both positive.



- Thus, Eqn. $\phi = \frac{TL}{4A_m^2 G} \oint \frac{ds}{t}$ becomes:

$$\phi = \sum \frac{TL}{4A_m^2 G} \oint \frac{ds}{t}$$

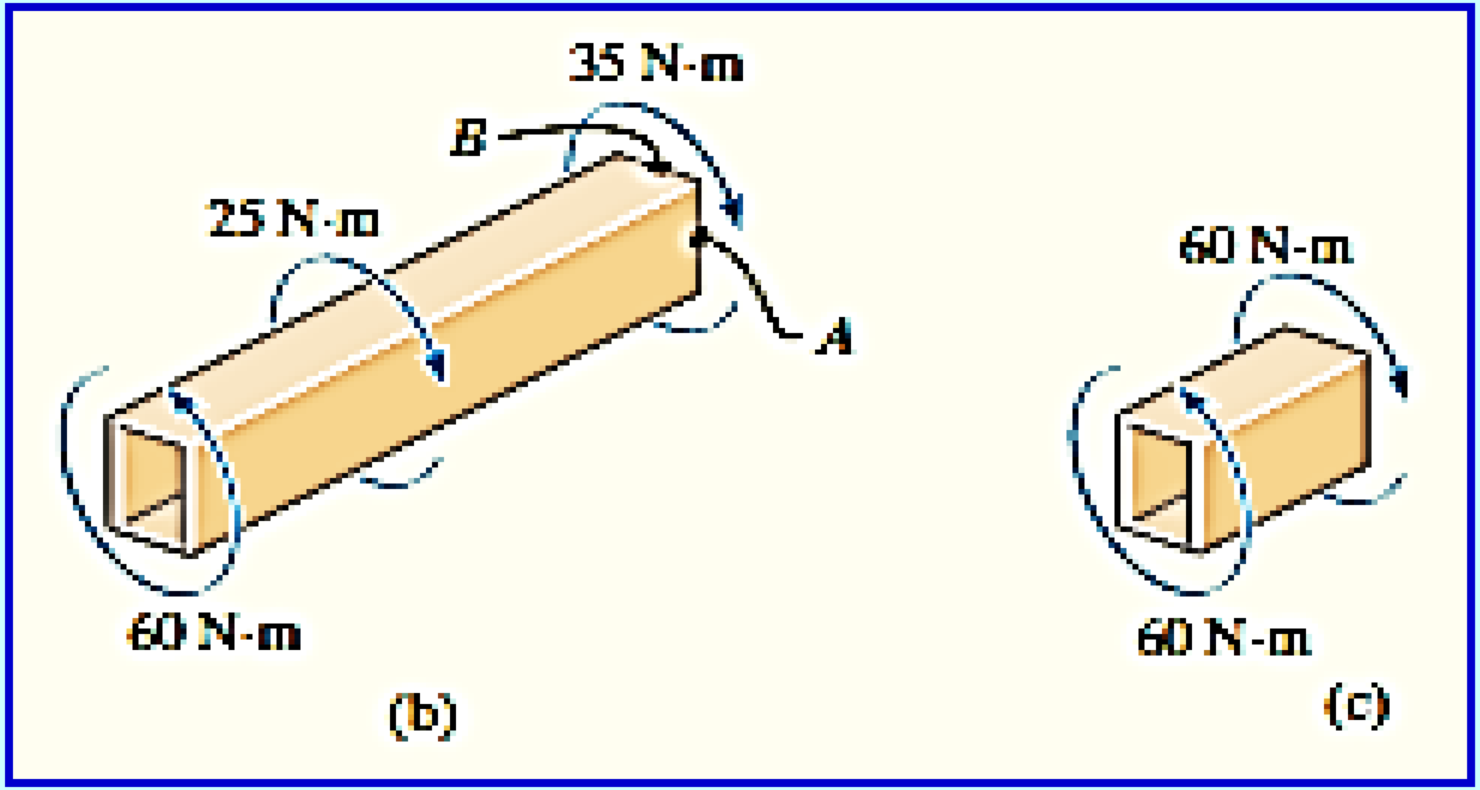


$$\phi = \sum \frac{TL}{4A_m^2 G} \oint \frac{ds}{t} = \frac{60\ N.m(0.5\ m)}{4(0.00200\boldsymbol{m^2})^2\ (38\ (10^9)\boldsymbol{N/m^2})}\left[2\left(\frac{57mm}{5mm}\right) + 2\left(\frac{35mm}{3mm}\right)\right]$$

$$+ \quad \frac{35\ N.m(1.5\ m)}{4(0.00200\boldsymbol{m^2})^2\ (38\ (10^9)\boldsymbol{N/m^2})}\left[2\left(\frac{57mm}{5mm}\right) + 2\left(\frac{35mm}{3mm}\right)\right]$$

$$= 6.29(10^{-3})\ \text{rad}$$

Ans.



CONCLUSION

Important Points

- Shear flow ***q*** is the product of the tube's thickness and the average shear stress.
- This value is the same at all points along the tube's cross section.
- As a result, ***the largest average shear stress on the cross section occurs where the thickness is smallest***.
- Both ***shear flow*** and the ***average shear stress act tangentially*** to the wall of the tube at all points and in a direction so as to contribute to the resultant ***internal torque***.

The End