THE UNIVERSITY OF ZAMBIA
SCHOOL OF ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING

MEC 3352 – STRENGTH OF MATERIALS II

Torsion of Non-Circular Shafts

Solid Circular Shafts

Introduction

Torsion is the twisting of a structural member, when it is loaded by a couple ***T*** that produce rotation about its longitudinal axis.

All cross sections of the shaft are subjected to the same internal torque *T*; therefore, the shaft is said to be in *pure torsion*.

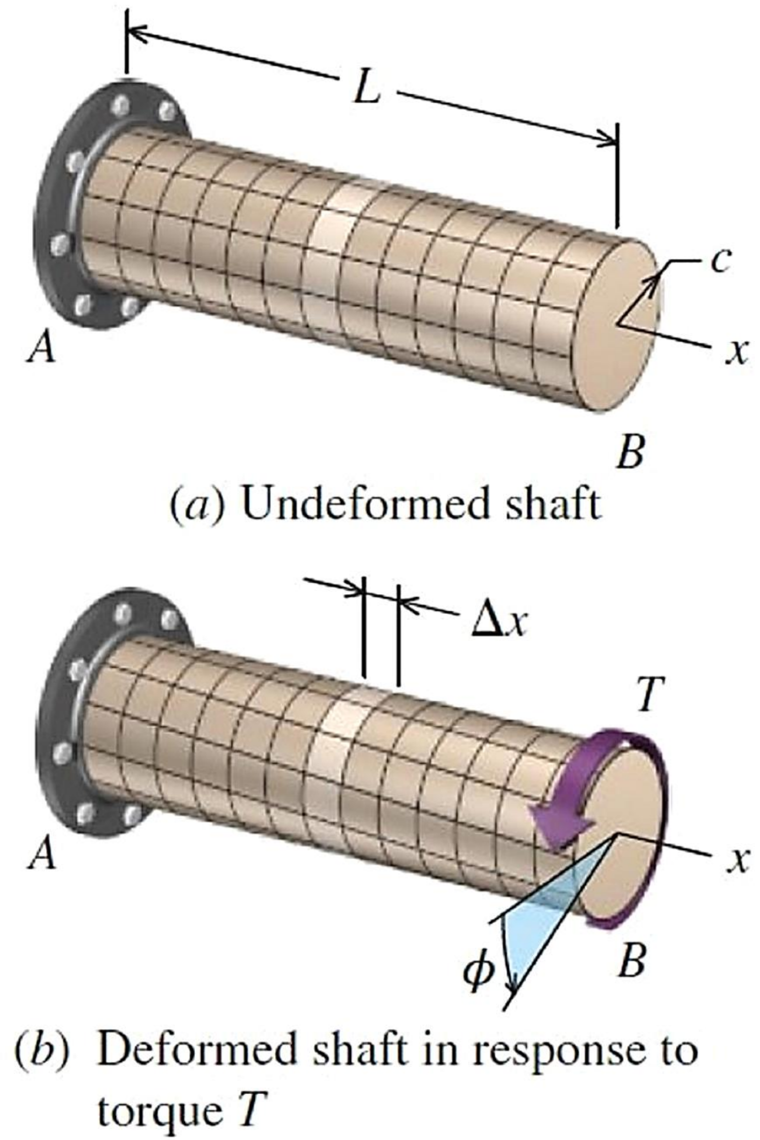


Fig.7.1 (a) and (b): Torsion in a circular shaft

Solid Circular Shafts

Introduction

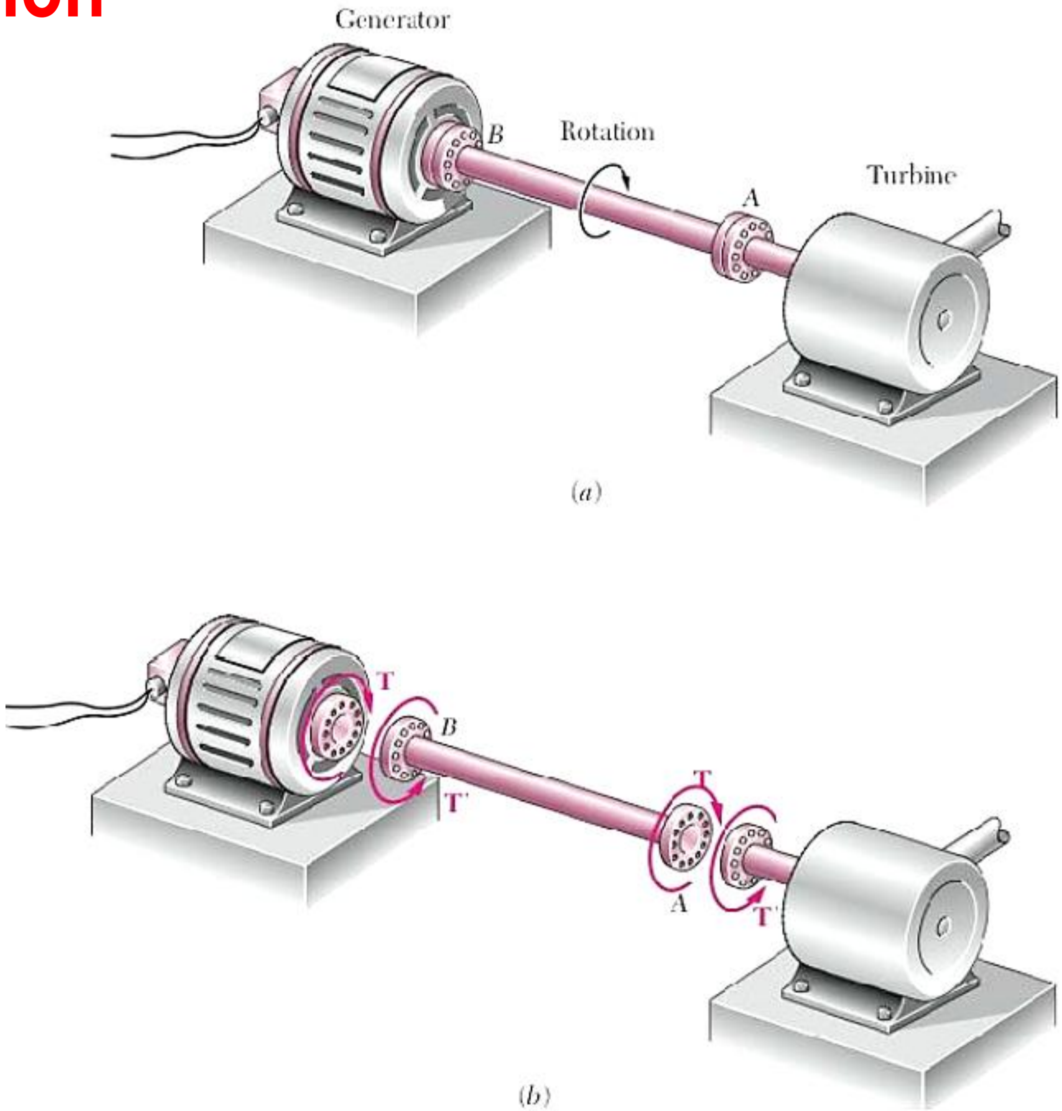


Fig.7.1 (c): Circular shaft subjected to pure torsion

Solid Circular Shafts

Introduction

When a torque is applied to a shaft having a circular cross section (***axisymmetric*** shaft) the shear strains vary linearly from zero at its center to a maximum at its outer surface.

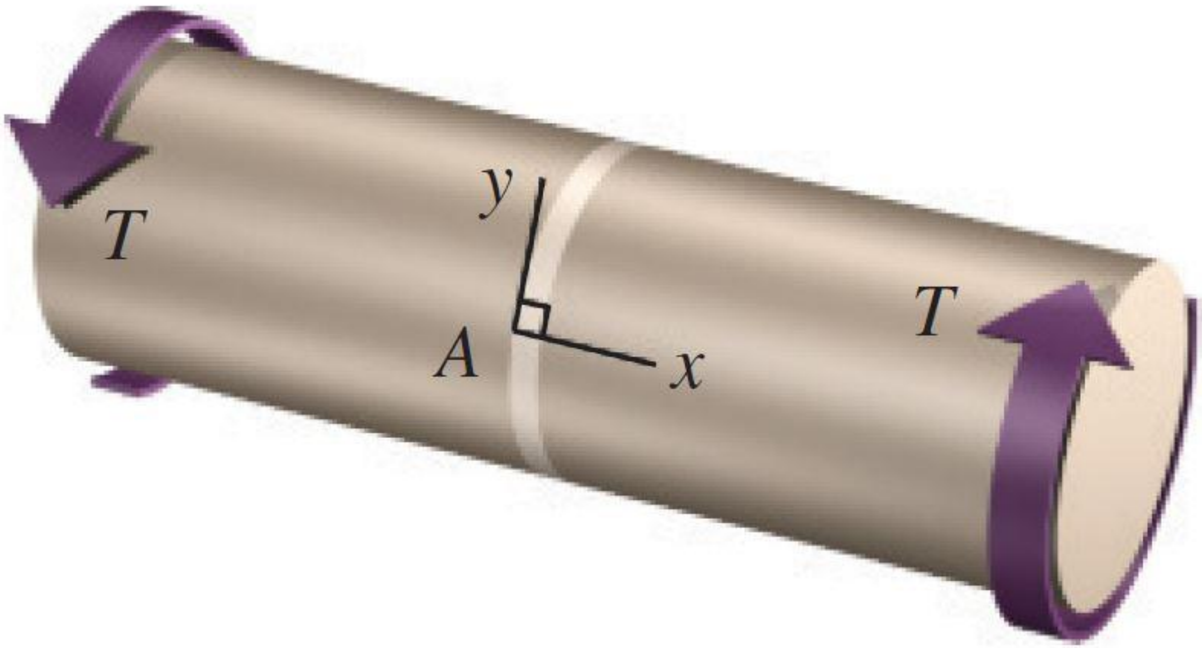


Fig.7.1 (d): Circular shaft subjected to pure torsion

Solid Circular Shafts

Introduction

Due to the uniformity of the shear strain at all points on the same radius, the cross sections do not deform, but rather remain plane after the shaft has twisted.

A bar or shaft of circular cross section twisted by a couple *T*, assume the left-hand end is fixed and the right-hand end will rotate a small angle Ø, called angle of twist.

Under twisting deformation, it is assumed that:

1. Plane sections remain plane
2. Radii remain straight and the cross sections remain plane and circular
3. if Ø is small, neither the length *L* nor the r radius will change

Solid Circular Shafts:

- A plane section before twisting remains plane after twisting. In other words, circular cross sections do not *warp* as they twist.
- Cross sections rotate about, and remain perpendicular to, the longitudinal axis of the shaft. (see figure 7.1 (b).
- Each cross section remains undistorted as it rotates relative to neighboring cross sections. In other words, the cross section remains circular and there is no strain in the plane of the cross section.
- Radial lines remain straight and radial as the cross section rotates.
- The distances between cross sections remain constant during the twisting deformation. In other words, no axial strain occurs in a round shaft as it twists.

Solid Non-circular Shafts

- Shafts that have a non-circular cross section, however, ***are not axisymmetric***, and so their cross sections will ***bulge*** or ***warp*** when the shaft is twisted.
- Evidence of this can be seen from the way grid lines deform on a shaft having a square cross section when the shaft is twisted, Fig. 7.2.
- As a consequence of this deformation, the torsional analysis of ***noncircular*** shafts becomes considerably more complicated.

Solid Non-circular Shafts

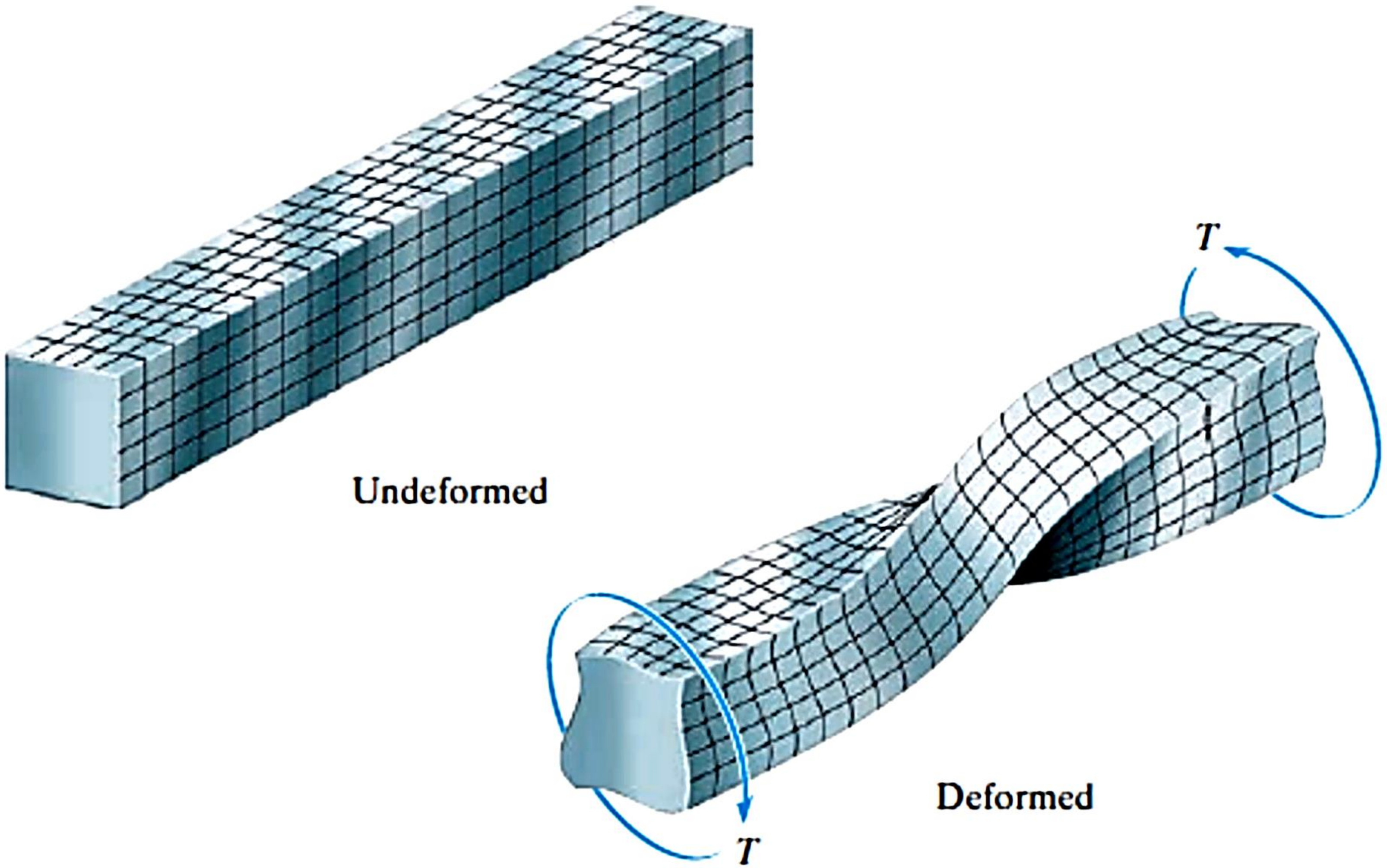


Fig. 7.2: Torsion in non-circular members

Solid Non-circular Shafts

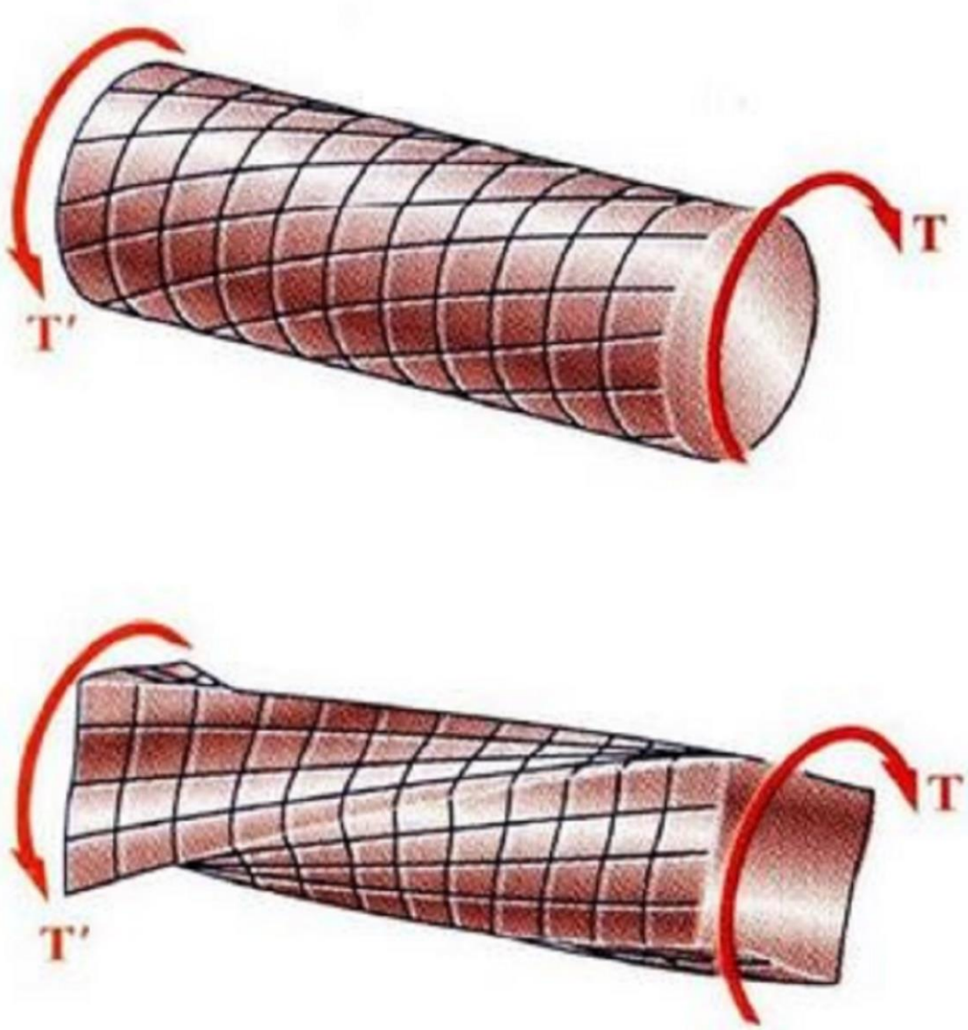


Fig. 7.2 (b): Comparison of Torsion in circular and non-circular shafts

Solid Non-circular Shafts

- Using a ***mathematical analysis*** based on the theory of elasticity, it is difficult to determine the ***shear-stress distribution*** within a shaft of ***square cross section***.
- Examples of how this shear stress varies along two radial lines of the shaft are shown in Figure 7.3 (*a*).
- Because these shear-stress distributions vary in a complex manner, the ***shear strains*** they create will ***warp the cross section*** as shown in Figure 7.3 (*b*).
- In particular notice that the ***corner points*** of the shaft must be subjected to zero shear stress and therefore ***zero shear strain***.

Solid Non-circular Shafts

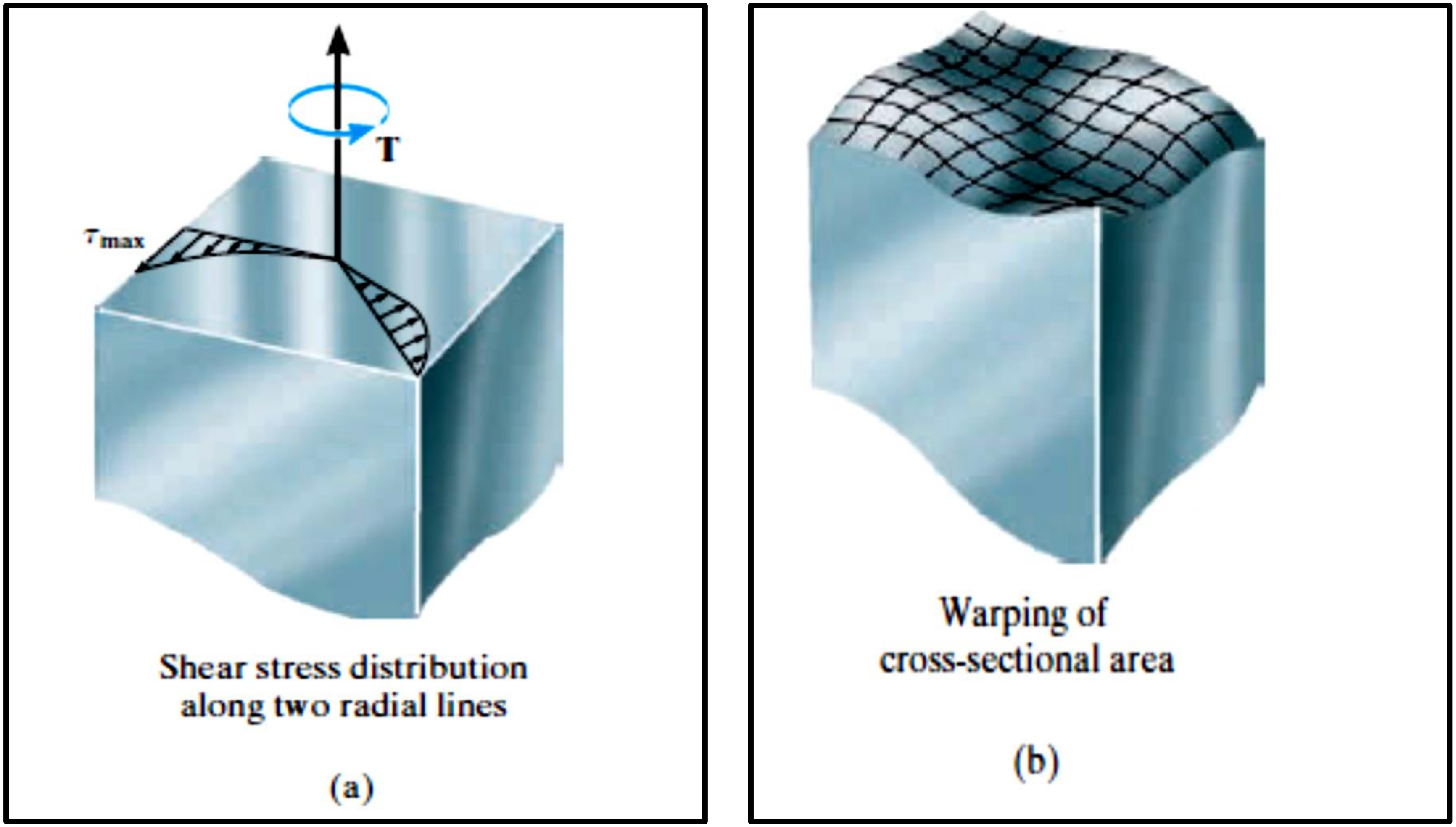


Fig. 7.3.1: Stress distribution in a square shaft

Solid Noncircular Shafts

- The reason for this can be shown by considering an element of material located at one of these points, Figure 7.3 (*c*).
- One would expect the top face of this element to be subjected to a shear stress in order to aid in resisting the applied torque **T**.
- This, however, cannot occur since the **complementary shear stresses τ and τ', acting on the *outer surface* of the shaft, must be zero.**

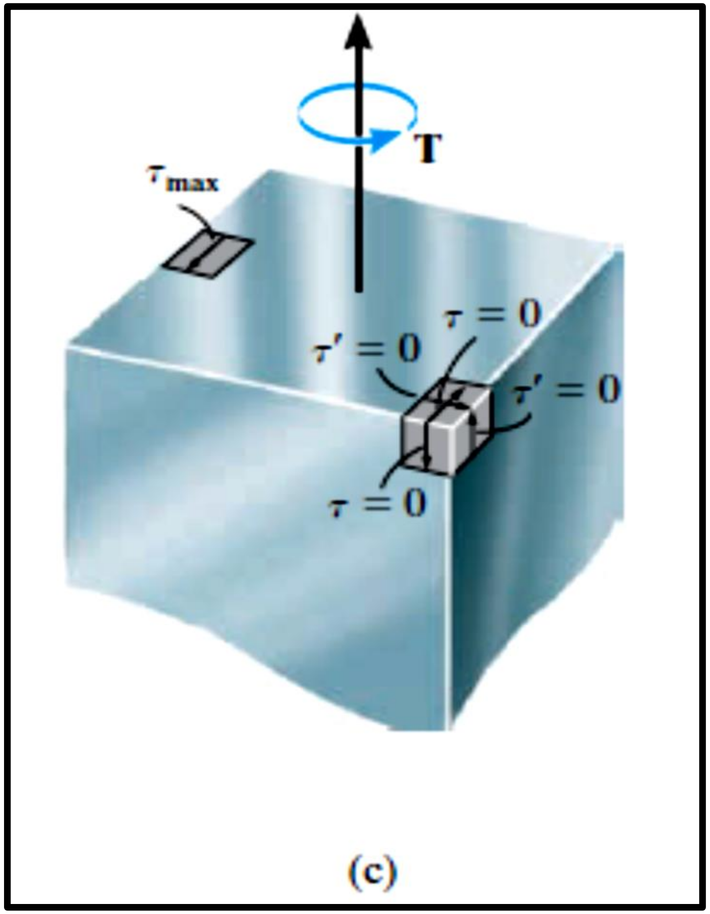


Fig. 7.3.1 (c): Stress distribution in a square shaft

Solid Noncircular Shafts

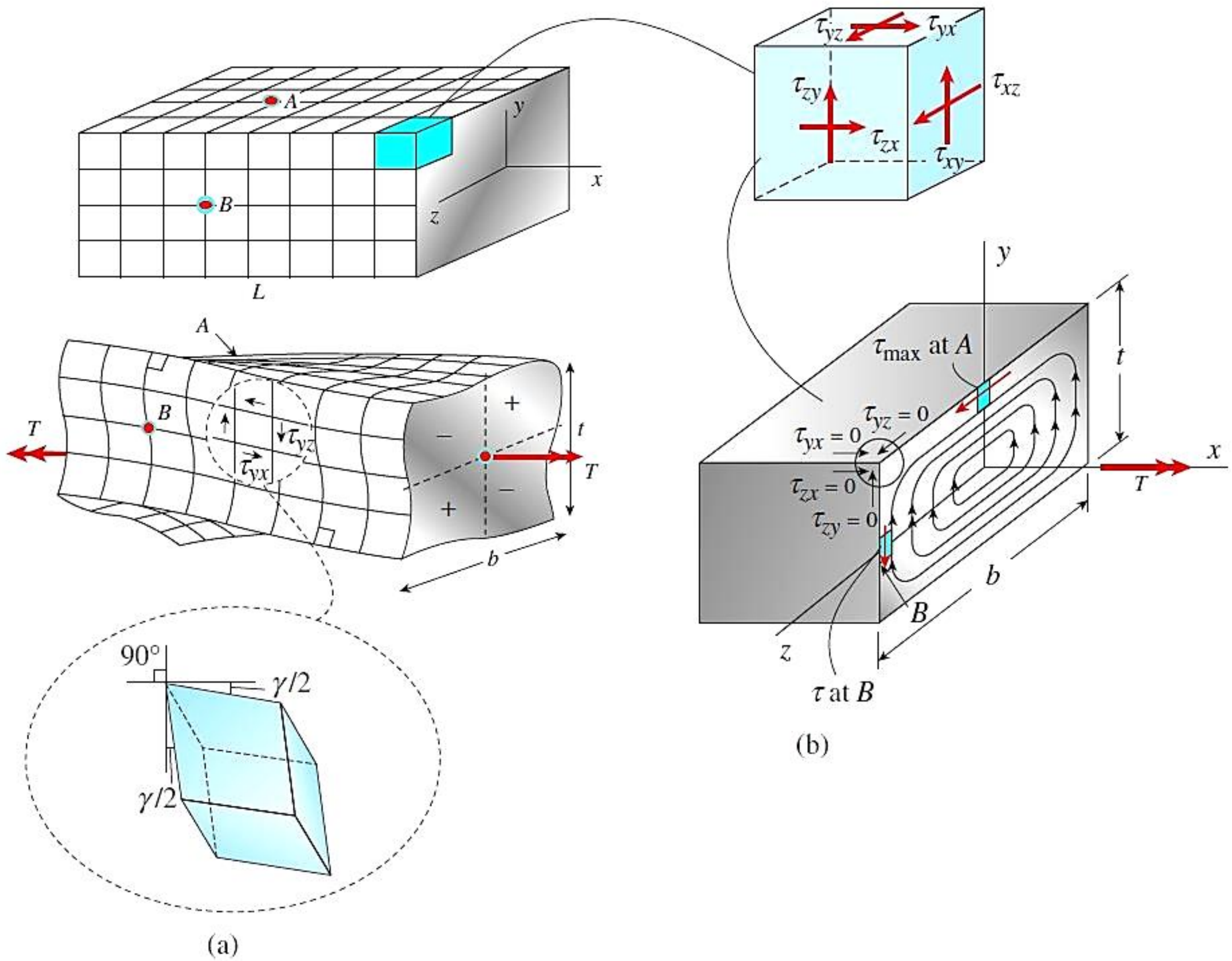


Figure 7.3.2 (a) Torsion of a bar of rectangular cross section and (b) shear stress distribution for a bar of rectangular cross section acted upon by torsional moment *T*

Solid Noncircular Shafts

Shear Stress Distribution and Angle of Twist

- We now consider the basic relations between applied torsional moment *T* and three key items of interest for a variety of noncircular cross sections.
  1) The location and value of the maximum shear stress τ_{max} in the cross section
  2) The torsional rigidity *GJ*
  3) The angle of twist Ø of a prismatic bar of length *L*
  4) Constant *G* is the shearing modulus of elasticity of the material, and variable *J* is the *torsion constant* for the cross section.
- Note that only for a circular cross section the torsion constant *J* become the polar moment of inertia I_p.

Solid Noncircular Shafts

Some common and basic shapes

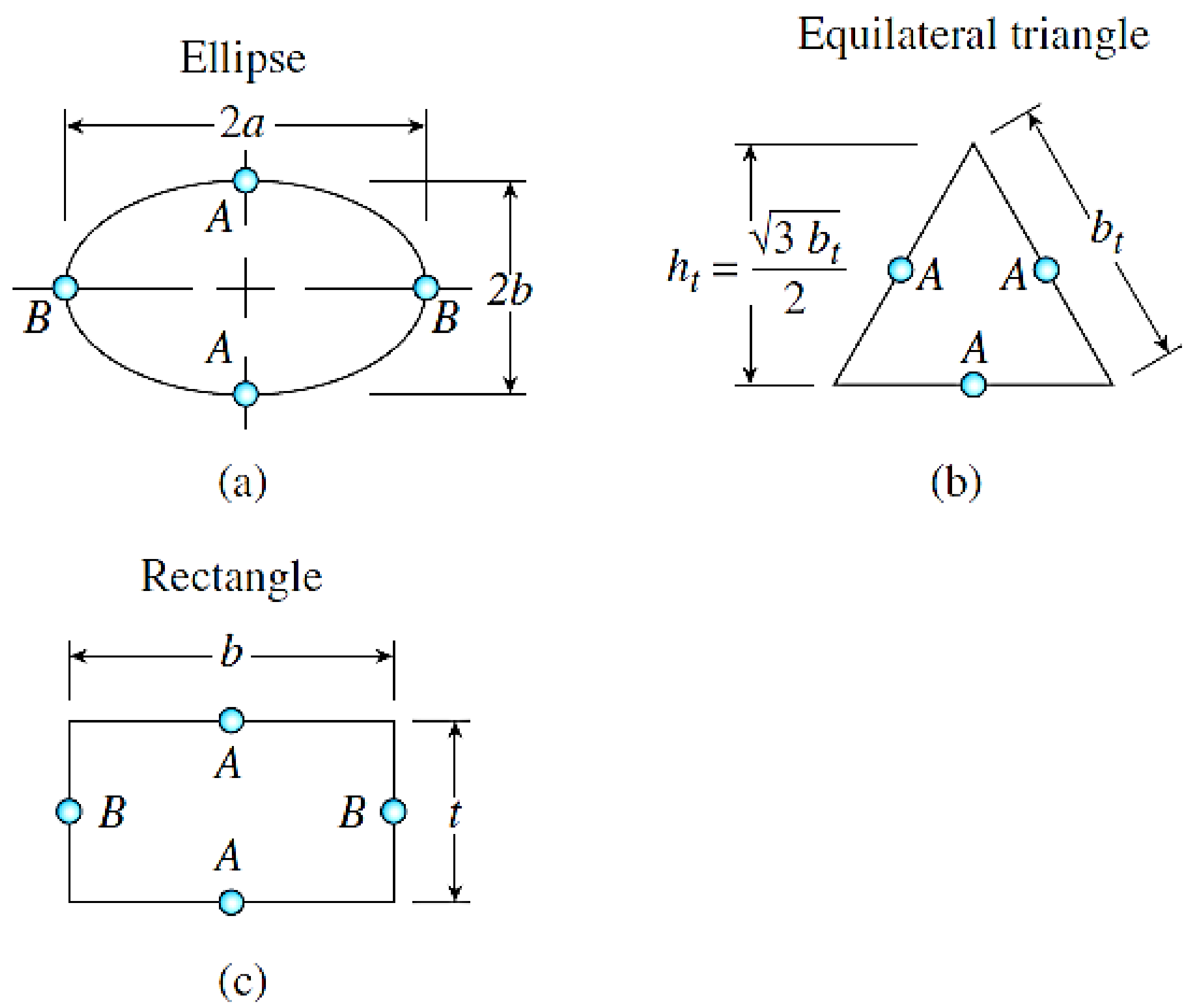


Figure 7.3.3 : Solid elliptical, triangular, and rectangular cross-sectional shapes

Solid Noncircular Shafts

Elliptical Cross Sections

- The shear stress distribution for a bar with an *elliptical cross section* (2*a* along major axis, 2*b* along minor axis, area $A = \pi ab$) is shown in Figure 7.3.4.

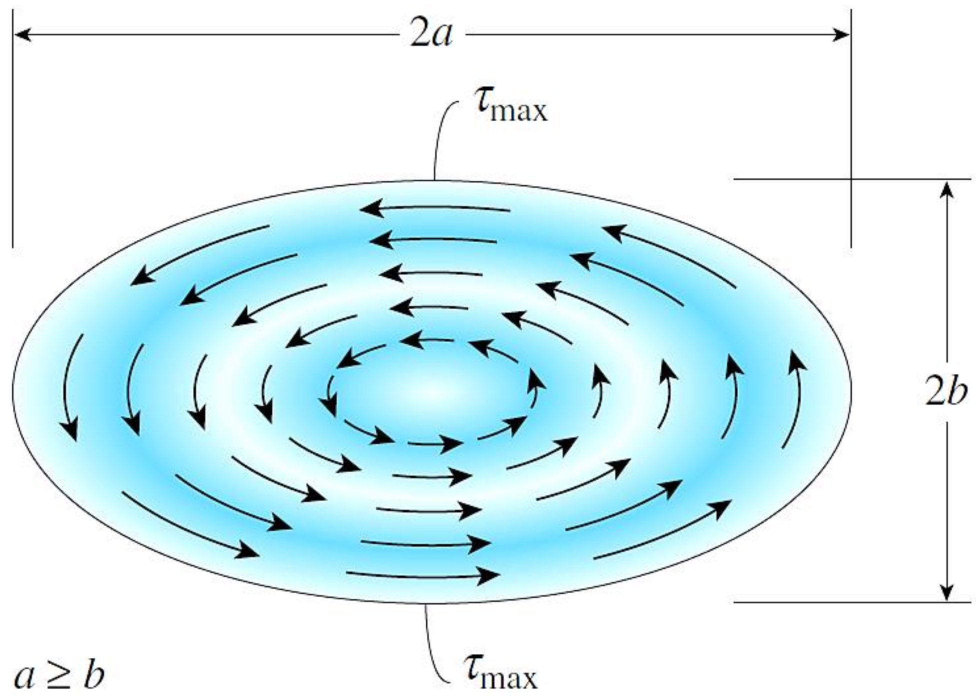


Figure 7.3.4: Shear Stress distribution in an elliptical bar

Solid Noncircular Shafts

Elliptical Cross Sections

- The maximum shear stress in an elliptical shaft is at the ends of the *minor axis* and may be computed using

$$\tau_{\max} = \frac{2T}{\pi a b^2} \qquad \textbf{7.10}$$

- The angle of twist Ø of a prismatic shaft of length *L* with an elliptical cross section is expressed as

$$\phi = \frac{TL}{GJ_e} \qquad \textbf{7.11}$$

- The torsion constant is given by

$$J_e = \frac{\pi a^3 b^3}{a^2 + b^2} \qquad \textbf{7.12}$$

Solid Noncircular Shafts

Triangular Cross Sections

- The maximum shear stress in an equilateral triangular shaft is at the mid point of each side, and may be computed using

$$\tau_{max} = \frac{T\left(\frac{h_t}{2}\right)}{J_t} = \frac{15\sqrt{3}T}{2h_t^3} \qquad \textbf{7.13}$$

- The angle of twist Ø of a prismatic shaft of length *L* with an equilateral triangular cross section is expressed as

$$\phi = \frac{TL}{GJ_t} = \frac{15\sqrt{3}TL}{Gh_t^4} \qquad \textbf{7.14}$$

- The torsion constant is given by

$$J_t = \frac{h_t^4}{15\sqrt{3}} \qquad \textbf{7.15}$$

Solid Noncircular Shafts

Rectangular Cross Sections

- The maximum shear stress in a rectangular shaft with an aspect ratio (b/t) is at the mid point of side A (Figure 7.33), and may be computed using

$$\tau_{\max} = \frac{T}{k_1 b t^2} \qquad \textbf{7.16}$$

- The angle of twist Ø of a prismatic shaft of length *L* with a rectangular cross section is expressed as

$$\phi = \frac{TL}{(k_2 b t^3)G} = \frac{TL}{GJ_r} \qquad \textbf{7.17}$$

- The torsion constant is given by

$$J_r = k_2 b t^3 \qquad \textbf{7.18}$$

- The dimensionless coefficients k_1 and k_2 are listed in Table 7.1.

Solid Noncircular Shafts

Rectangular Cross Sections

Table 7.1: Dimensionless coefficients rectangular members

b/t	1.00	1.50	1.75	2.00	2.50	3.00	4	6	8	10	∞
k_1	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.298	0.307	0.312	0.333
k_2	0.141	0.196	0.214	0.229	0.249	0.263	0.281	0.298	0.307	0.312	0.333

Example 7.1

The two rectangular solid polymer bars whose cross-sections measures (25 x 64 mm) and (48 x 32 mm) respectively are each subjected to a torque *T* = 25Nm. For each bar, determine

(a) the maximum shear stress.

(b) the rotation angle at the free end if the bar has a length of 305 mm. Assume that *G* = 350MPa for the polymer material.

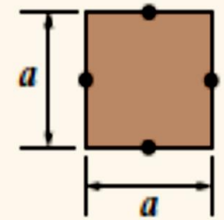
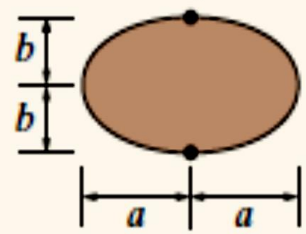
Example 7.1

Solution 7.1

Solid Non-circular Shafts

- The results of the analysis for square cross sections, along with other results from the theory of elasticity, for ***shafts having square, triangular and elliptical cross sections,*** are reported in Table 7.2.
- In all cases the *maximum shear stress* occurs at a point on the edge of the cross section that is *closest to* the center axis of the shaft. In Table 7.2 these points are indicated as “dots” on the cross sections.

Table 7.2: Shear Stress in non-circular members

Shape of cross section	τ_{max}	ϕ
Square 	$\dfrac{4.81\,T}{a^3}$	$\dfrac{7.10\,TL}{a^4G}$
Equilateral triangle 	$\dfrac{20\,T}{a^3}$	$\dfrac{46\,TL}{a^4G}$
Ellipse 	$\dfrac{2\,T}{\pi ab^2}$	$\dfrac{(a^2+b^2)TL}{\pi a^3b^3G}$

Solid Noncircular Shafts

- Also given in Table 7.2 are formulas for the angle of twist of each shaft.
- Comparing these results to a shaft having an **arbitrary** cross section, it can be shown that a shaft having a **circular** cross section is most efficient.
- Note that the circular cross section is subjected to both a **smaller maximum shear stress** and a **smaller angle of twist** than a corresponding shaft having a non-circular cross section **and subjected to the same torque T**.

SAINT VENANT'S THEORY

- Results from Saint-Venant's analysis of torsion of prismatic bars of non-circular cross section generally indicate that, when twisted:
  1) every section will warp
  2) every section will not remain plane

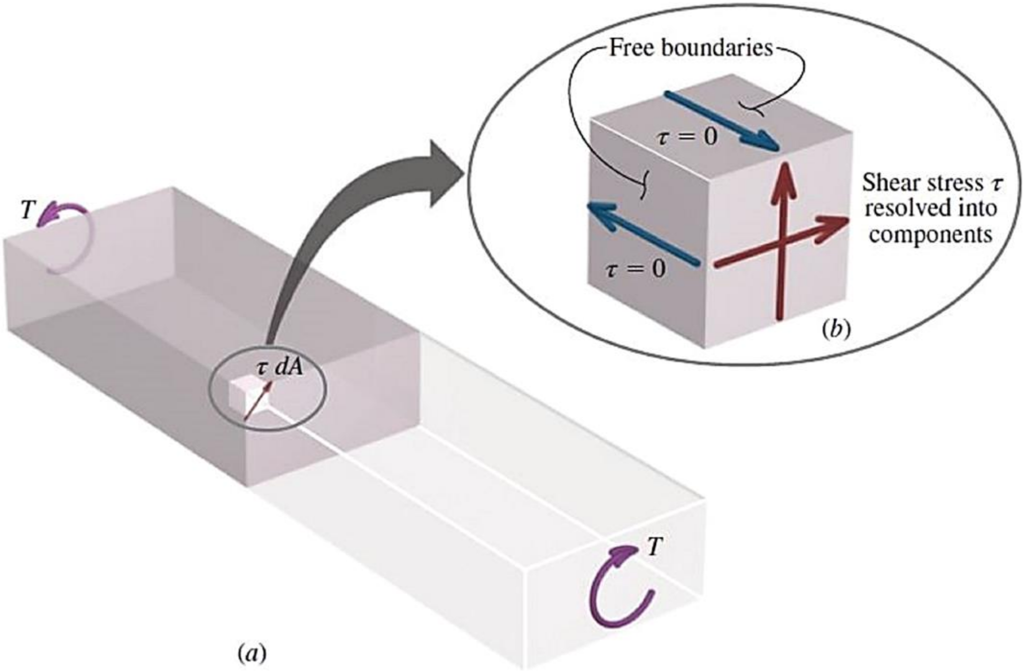


Fig. 7.4:Torsional shear stresses in a rectangular bar.

SAINT VENANT'S THEORY

- For a square shafts, distortion of the small squares :
  1) is greatest at the midpoint of a side of the cross section
  2) disappears at the corners of the cross section
- Since this distortion is a measure of shear strain, Hooke's law requires that the shear stress:
  1) be largest at the midpoint of a side of the cross section
  2) be zero at the corners of the cross section
- Equations for the maximum shear stress and the angle of twist for a rectangular section obtained from Saint-Venant's theory are:

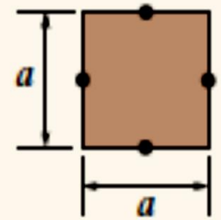
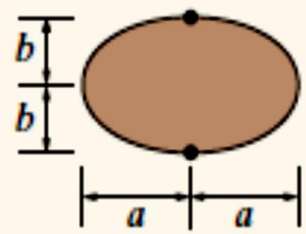
$$\tau_{max} = \frac{T}{\alpha a^2 b} \qquad 7.19$$

$$\emptyset = \frac{TL}{\beta a^3 bG} \qquad 7.20$$

Solid Non-circular Shafts

- The *maximum shear stress* occurs at a point on the edge of the cross section
- Point with maximum shear stress is *closest to* the centre of the axis of the shaft.
- In Table 7.2 these points are indicated as “dots” on the cross sections.

Table 7.2: Shear Stress in non-circular members

Shape of cross section	τ_{max}	ϕ
Square 	$\frac{4.81\,T}{a^3}$	$\frac{7.10\,TL}{a^4G}$
Equilateral triangle 	$\frac{20\,T}{a^3}$	$\frac{46\,TL}{a^4G}$
Ellipse 	$\frac{2\,T}{\pi ab^2}$	$\frac{(a^2+b^2)TL}{\pi a^3b^3G}$

SAINT VENANT'S THEORY

$$\tau_{max} = \frac{T}{\alpha a^2 b} \qquad 7.19$$

$$\emptyset = \frac{TL}{\beta a^3 b G} \qquad 7.20$$

- Where ***a*** and ***b*** are the lengths of the short and long sides of the rectangle, respectively.
- Numerical constants ***α*** and ***β*** can be obtained from Table 7.3

Table 7.3: Constants for Torsion of a Rectangular Bar

Ratio b/a	α	β
1.0	0.208	0.1406
1.2	0.219	0.166
1.5	0.231	0.196
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

SAINT VENANT'S THEORY

- For aspect ratios $b/a \geq 5$, the coefficients α and β that respectively appear in Equations (7.19) and (7.20) can be calculated from the following equation:

$$\alpha = \beta = \frac{1}{3}\left[1 - 0.630\left(\frac{a}{b}\right)\right]$$

- As a practical matter, an aspect ratio $b/a \geq 21$ is sufficiently large that values of $\alpha = \beta = 0.333$ can be used to calculate maximum shear stresses and deformations in narrow rectangular bars within an accuracy of 3 %.
- Accordingly, equations for the maximum shear stress and angle of twist in narrow rectangular bars can be expressed as:

SAINT VENANT'S THEORY

$$\tau_{max} = \frac{3T}{a^2 b} \qquad 7.21$$

$$\emptyset = \frac{3TL}{a^3 bG} \qquad 7.22$$

- For narrow rectangular sections, we have the following:

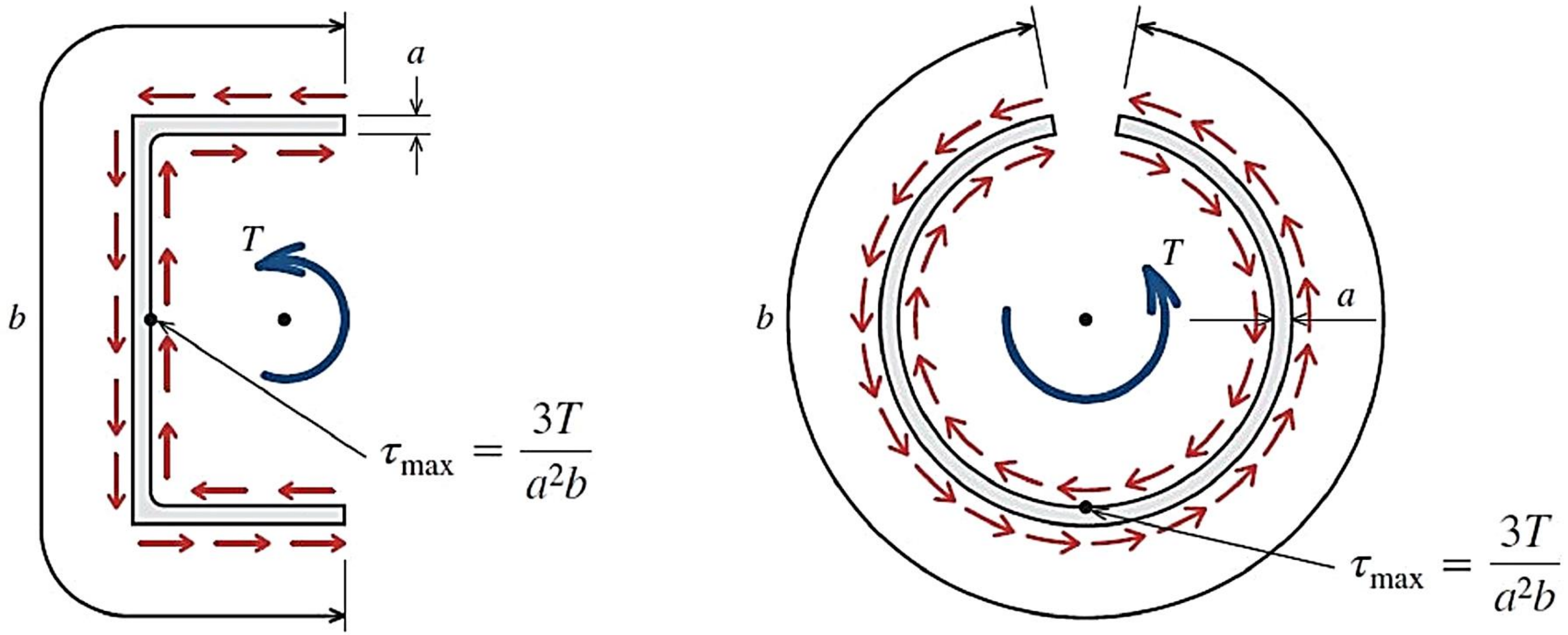


Fig 7.5: Equivalent narrow rectangular sections with shear stress distribution.

Example 7.2

The two rectangular solid polymer bars whose cross-sections measures (25 x 64 mm) and (48 x 32 mm) respectively are each subjected to a torque T = 25Nm. For each bar, determine

(a) the maximum shear stress.

(b) the rotation angle at the free end if the bar has a length of 305 mm. Assume that G = 350MPa for the polymer material.

Use St. Venant's Theory

Example 7.2

Solution 7.2

- The procedure is similar to the one in **Solution 7.1**
- Use equations 7.19 and 7.20

$$\tau_{max} = \frac{T}{\alpha a^2 b} \quad 7.19$$

$$\emptyset = \frac{TL}{\beta a^3 bG} \quad 7.20$$

- Table 7.3 will be useful
- Compare result with what was obtained in Solution 7.1

Table 7.3: Constants for Torsion of a Rectangular Bar

Ratio b/a	α	β
1.0	0.208	0.1406
1.2	0.219	0.166
1.5	0.231	0.196
2.0	0.246	0.229
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3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

Thin-Walled Tubes with Closed Cross Sections

- We now look at the effects of applying a **torque** to a thin-walled tube having a **closed** cross section.
- Such a tube that does not have any breaks or slits along its length.
- Since **the walls are thin**, we will obtain the **average shear stress** by assuming that this stress is **uniformly distributed** across the thickness of the tube at any given point.
- We also discuss **shear stress distribution** over the cross section.

Thin-Walled Tubes with Closed Cross Sections

Shear Flow

- A useful concept associated with the analysis of thin-walled sections is the **shear flow** q,
- The shear flow is defined as the internal shearing force per unit of length of the thin section.
- The SI unit for shear flow is the newton per meter.
- In terms of stress, q equals $\tau \times t$, where τ is the average shear stress across the thickness t.
- We can demonstrate that the shear flow on a cross section is constant even though the wall thickness of the section may vary.

Thin-Walled Tubes with Closed Cross Sections

Shear Flow

Figure 7.6 *b* shows a block cut from the member of Figure 7.6 *a* between *A* and *B*.

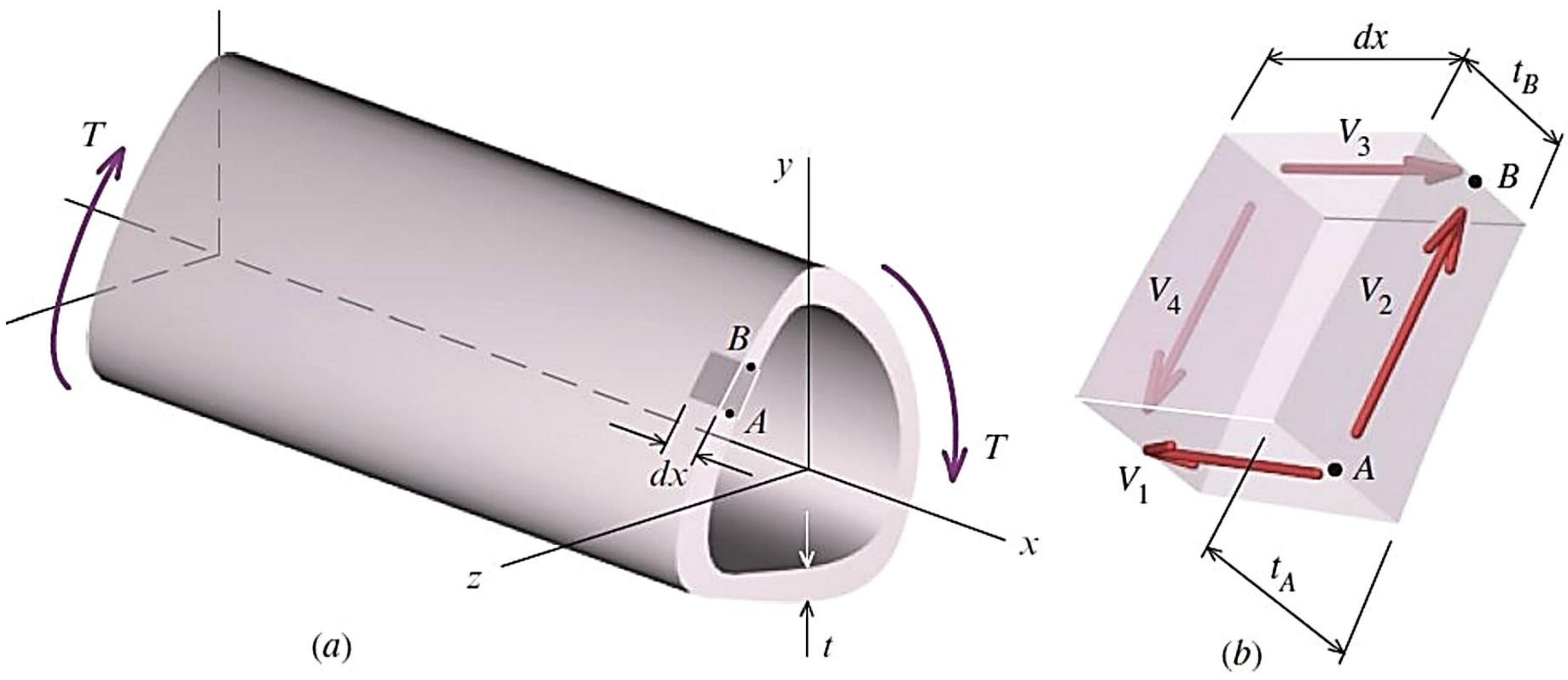


Fig. 7.6: Shear flow in thin-walled tubes.

Thin-Walled Tubes with Closed Cross Sections

Shear Flow

- Since the member is subjected to pure torsion, the shear forces V_1, V_2, V_3, and V_4 alone are necessary and sufficient for equilibrium (i.e., no normal forces are involved).
- Summing forces in the x direction gives

$$V_1 = V_3$$

$$q_A dx = q_B dx$$

$$q_A = q_B$$

and, since $q = \tau \times t$,

$$\tau_A t_A = \tau_B t_B \qquad 7.23$$

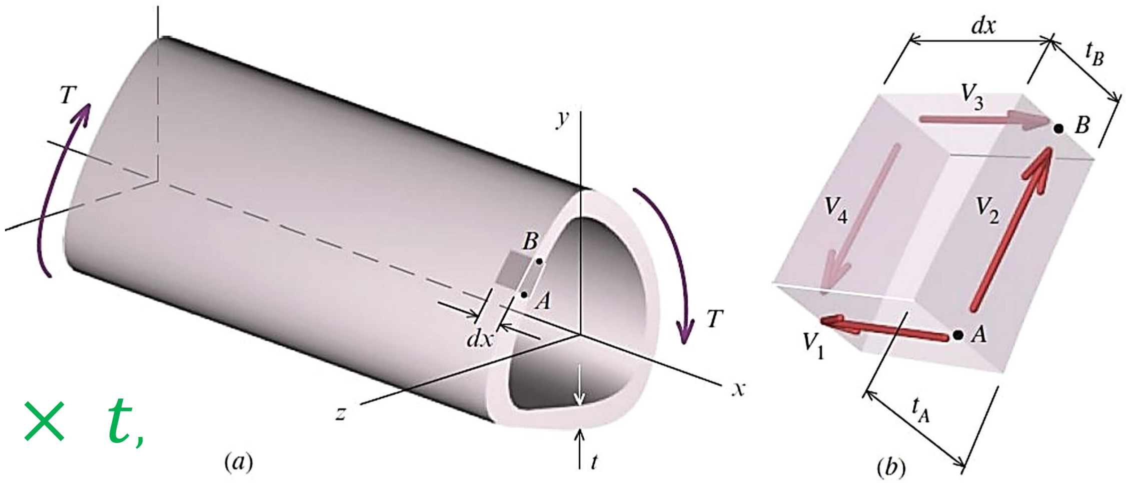


Fig. 7.6: Shear flow in thin-walled tubes.

- Note that the shear flow and the shear stress always act tangent to the wall of the tube.

SHEAR STRESS OVER THE CROSS SECTION

- Shown in **Figures 7.7 *a*** and **7.7 *b*** is a small element of the tube having a finite length ***s*** and differential width ***dx*.**
- At one end the element has a thickness t_A and at the other end the thickness is t_B.
- Due to the ***internal torque T***, shear stress is developed on the faces of the element.
- Specifically, shear stresses are at *A* and B as τ_A and τ_B respectively.

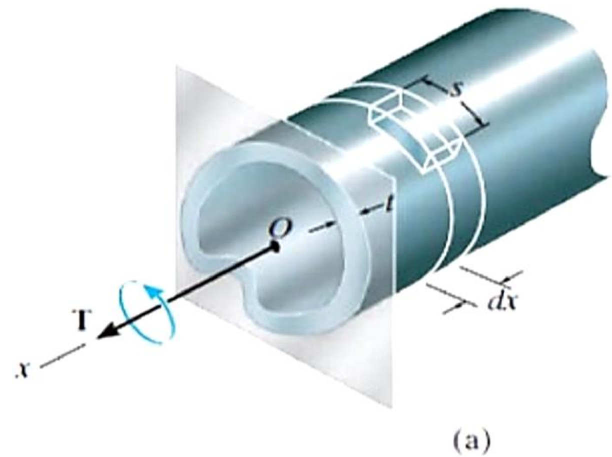


Fig. 7.7 *a*.

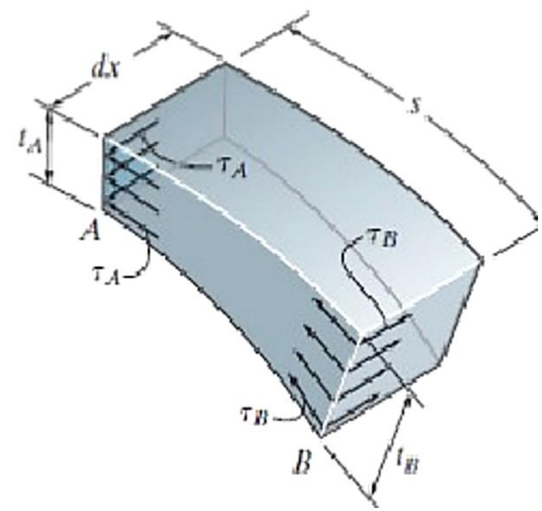


Fig. 7.7 *b*.

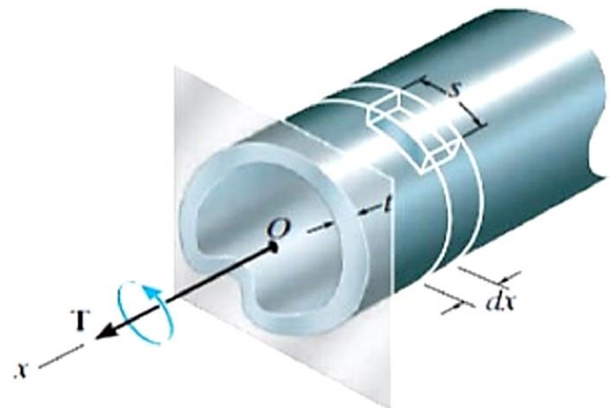
Shear Flow

It follows that ***"the product of the average shear stress and the thickness of the tube is the same at each point on the tube's cross-sectional area".***

This product is called ***shear flow,**** *q*, and in general terms we can express it as

$$q = \tau_{avg} t \qquad 7.24$$

Since *q* is constant over the cross section, the ***largest* average shear stress must occur where the tube's thickness is the *smallest*.**



(a)

Fig. 7.7 *a*.

It follows that ***"the product of the average shear stress and the thickness of the tube is the same at each point on the tube's cross-sectional area".***

This product is called ***shear flow,**** *q*, and in general terms we can express it as

$$q = \tau_{avg} t$$

Since *q* is constant over the cross section, the ***largest* average shear stress must occur where the tube's thickness is the *smallest*.**

Fig. 7.7 *b*.

RELATIONSHIP BETWEEN INTERNAL TORQUE (T) AND SHEAR STRESS (τ)

- Consider the force dF acting through the centre of a differential length of perimeter ds, as shown in Figure 7.8.
- The differential moment produced by dF about the origin O is given by:

$$\rho \times dF$$

- Where ρ is the mean radial distance from the perimeter element to the origin.

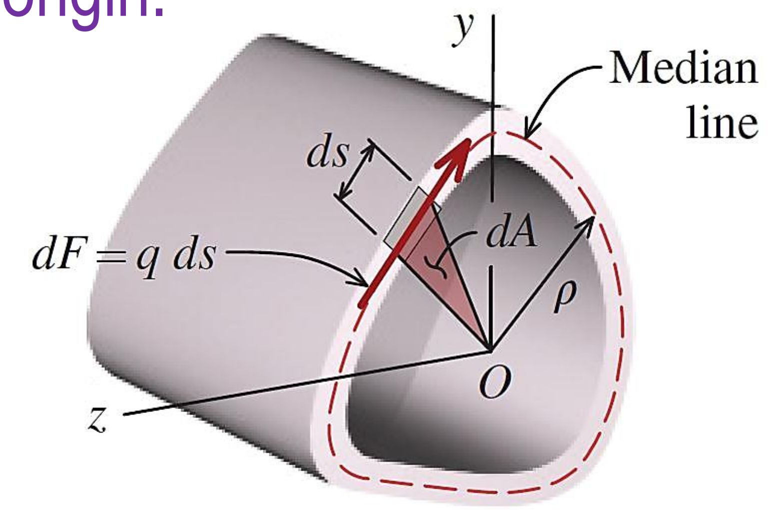


Fig. 7.8 : Relationship between internal torque and shear stress in a thin-walled section.

RELATIONSHIP BETWEEN INTERNAL TORQUE (T) AND SHEAR STRESS (τ)

- The internal torque equals the resultant of all of the differential moments; that is,

$$T = \int (dF)\,\rho = \int (q\,ds)\,\rho = q\int \rho\,ds$$

- This integral may be difficult to integrate by formal calculus; however, the quantity $\rho.ds$ is twice the area of the triangle shown shaded in Figure 7.8,
- Thus, the integral is equal to twice the area A_m *enclosed by the median line.*
- A_m is the mean area enclosed within the boundary of the tube wall *centreline*.
- The resulting expression relates the torque *T* and shear flow *q as follows*:

RELATIONSHIP BETWEEN INTERNAL TORQUE (T) AND SHEAR STRESS (τ)

BREDT-BATHO EQUATIONS

$$T = q(2.A_m) \qquad 7.25$$

and, since $q = \tau \times t$,

$$T = (\tau * t)(2.A_m)$$

$$\tau = \frac{T}{2.A_m.t} \qquad 7.26$$

- This relation (equation 7.26) is known as the *Bredt's first formula* (Rudolf Bredt, 1842– 1900)
- Also known as torsion formula for thin-walled tubes or simply the **1st BREDT-BATHO formula**

BREDT-BATHO EQUATIONS

1st BREDT-BATHO formula

- The variable t represents the thin-walled component's wall thickness.
- The enclosed area A_m lies within the centre line of the tube and is also called the hollow area.
- The shear stress τ resulting from the Torsion (T), i.e internal torque, is constant over the entire wall thickness t,
- And which means that the shear flow q also remains constant in the circumferential.

BREDT-BATHO EQUATIONS

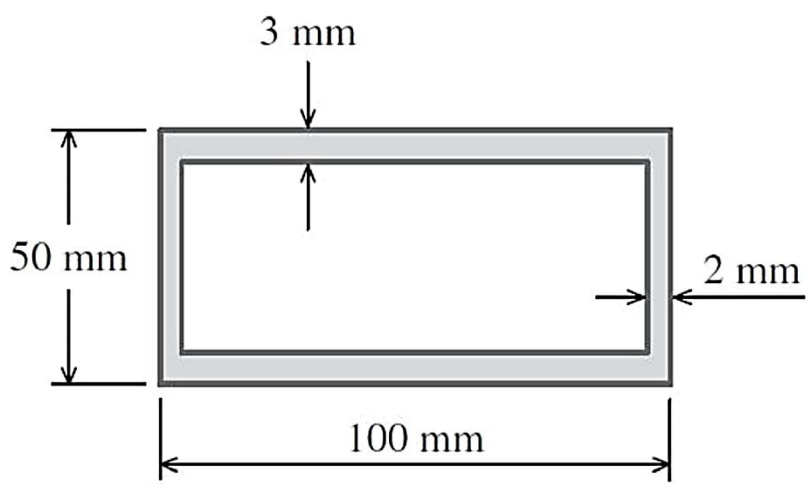
2nd Bredt-Batho formula

$$\theta = \frac{TL}{4{A_m}^2 G}\int \frac{ds}{t} \quad or \quad \phi = \frac{TL}{4A_m^2 G}\oint \frac{ds}{t} \qquad 7.27$$

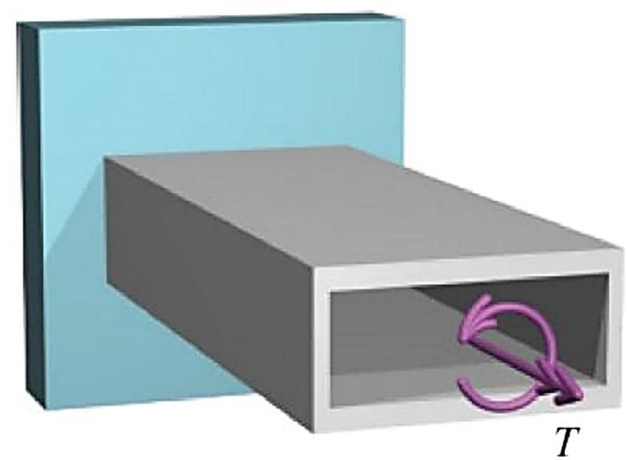
- Equation 7.27 is Bredt's second formula.
- The 2nd Bredt-Batho formula indicates the component's twisting θ, which depends on the material's shear modulus G.
- The Bredt-Batho formulae apply only to torsion acting on closed hollow tubes with an axis of Rotation that lies on the shear centre.
- Note that Equation (7.26 and 7.27) applies only to "closed" sections - that is, sections with a continuous periphery , with no slits.

Example 7.2

A rectangular box section of aluminum alloy has outside dimensions of 100 mm by 50 mm. The plate thickness is 2 mm for the 50 mm sides and 3 mm for the 100 mm sides. If the maximum shear stress must be limited to 95 MPa, determine the maximum torque *T* that can be applied to the section.



Cross-sectional dimensions.



Example 7.2

Solution

The maximum shear stress will occur in the thinnest plate; therefore, the critical shear flow q is

$$q = \tau t = (95\ N/mm^2)(2\ mm) = 190\ N/mm$$

The area enclosed by the median line is

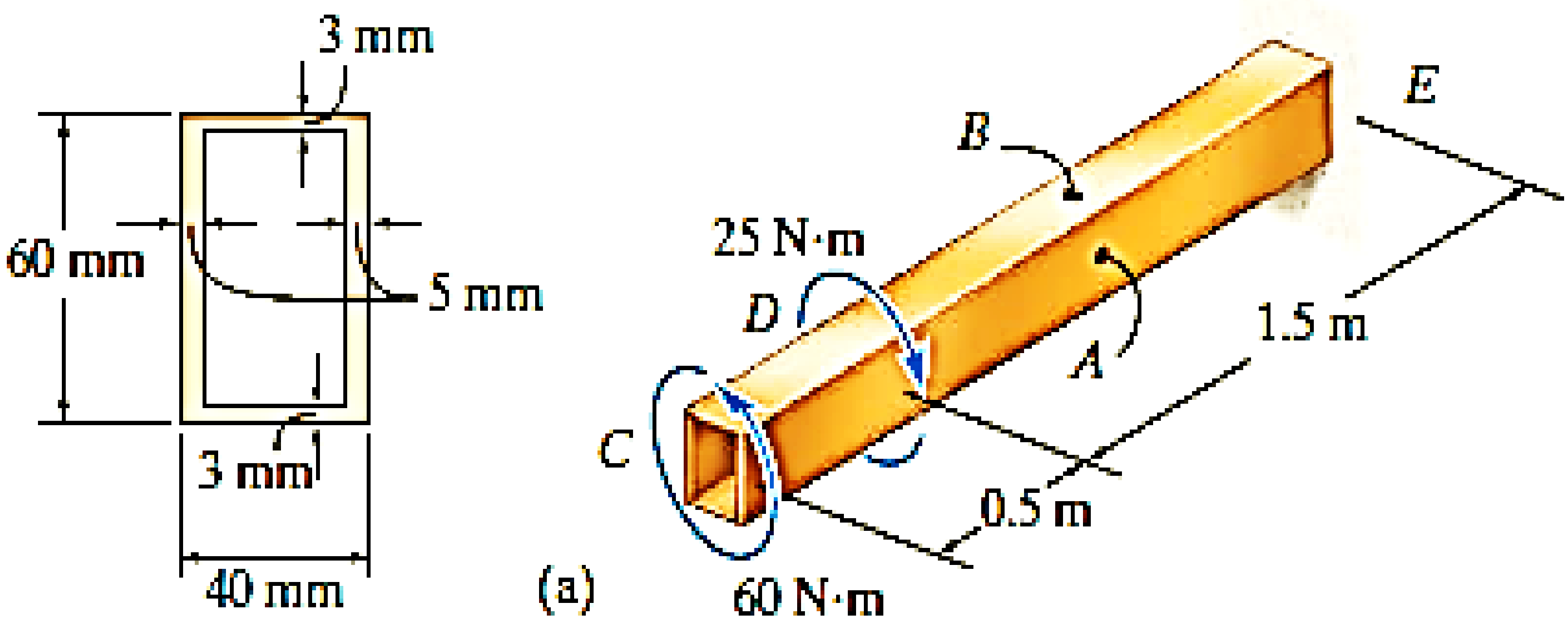
$$A_m = (100\ mm - 2\ mm)(50\ mm - 3\ mm) = 4{,}606\ mm^2$$

The torque that can be transmitted by the section is computed from Equation 7.26 as follows:

$$T = q(2.A_m)$$
$$= (190\ N/mm)(2)(4{,}606\ mm^2)$$
$$= 1{,}750{,}280\ N\ mm$$
$$= 1{,}750\ N\ m$$

Example 7.3

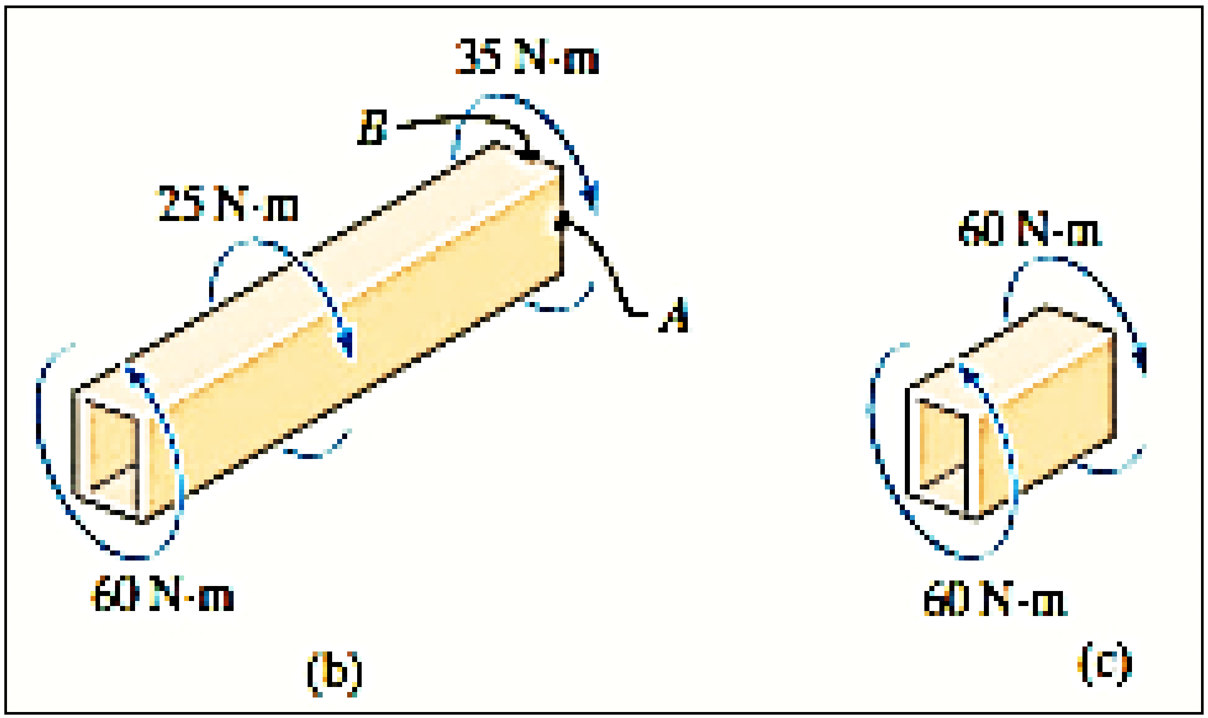
The tube is made of C86100 bronze and has a rectangular cross section as shown in the figure. If it is subjected to the two torques, determine the average shear stress in the tube at points *A* and *B*. Also, what is the angle of twist of end *C*? The tube is fixed at *E*.



Solution

Average Shear Stress.

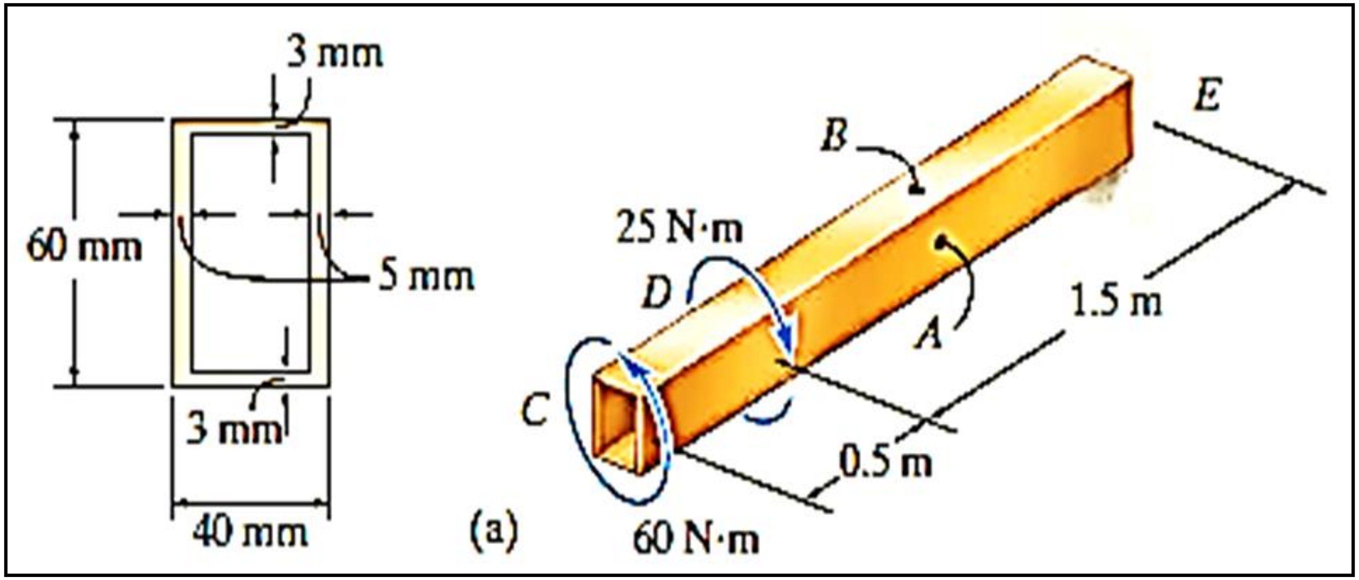
- If the tube is sectioned through points A and B, the resulting free-body diagram is shown in (b) and (c);
- The internal torque is 35 N.m.



$$A_m = (0.035\ m)(0.057m) = \mathbf{0.00200m^2}$$

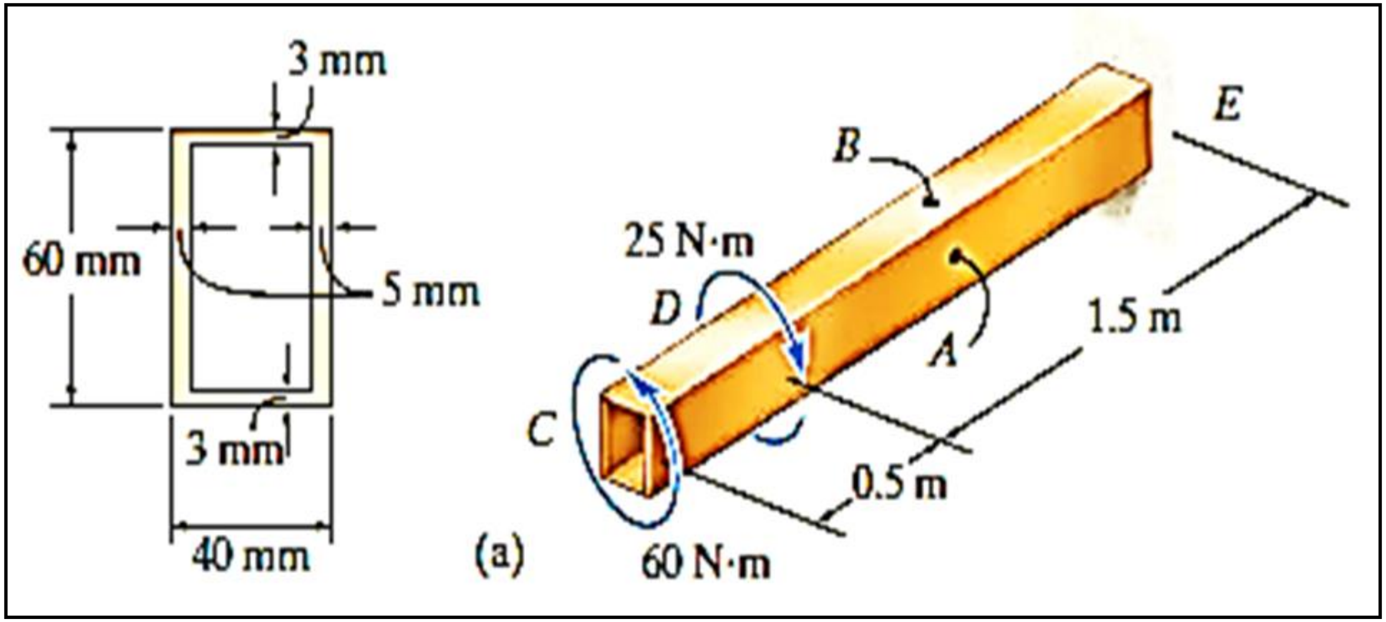
Solution

$A_m = (0.035\ m)(0.057m) = \mathbf{0.00200m^2}$



- Applying Equation 7.26.
- $\tau_{avg} = \dfrac{T}{2tA_m}$ for point A, t_A= 5 mm, so that
- $\boldsymbol{\tau_A}$ = $\dfrac{\mathbf{35\ N.m}}{\mathbf{2(0.005\ m)(0.00200m^2)}}$ = **1.75MPa**

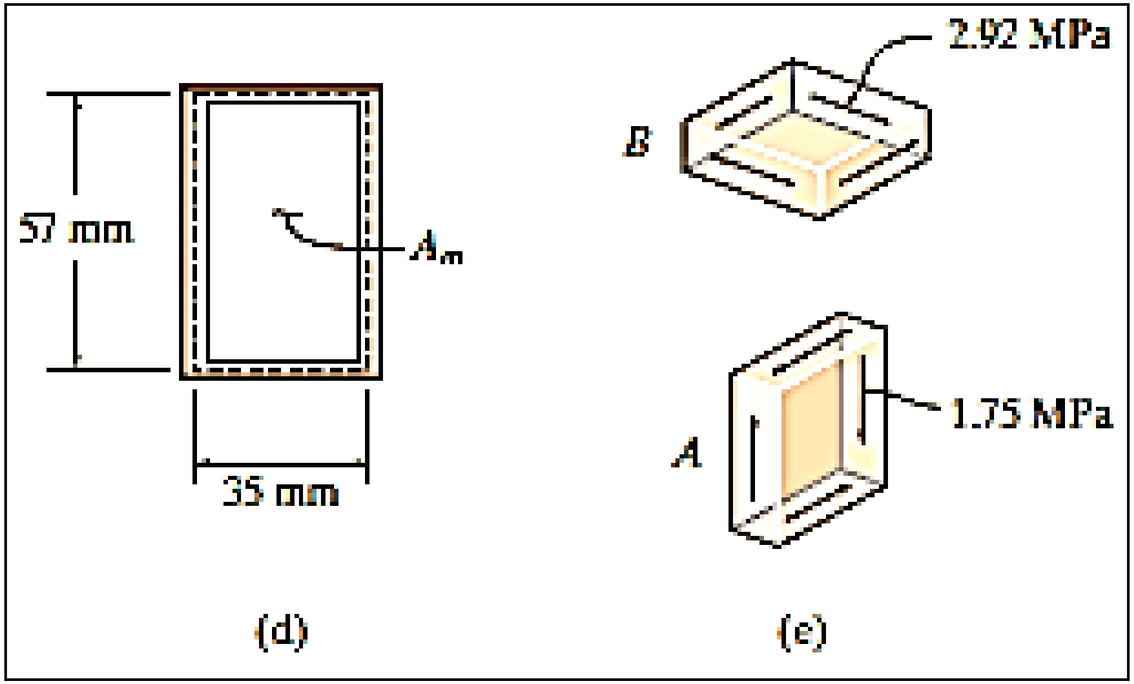
Solution



- And for point B, t_B= 3 mm,
- $\tau_B = \dfrac{T}{2tA_m}$ = $\dfrac{35\ N.m}{2(0.003\ m)(0.00200m^2)}$ = **2.92 MPa**

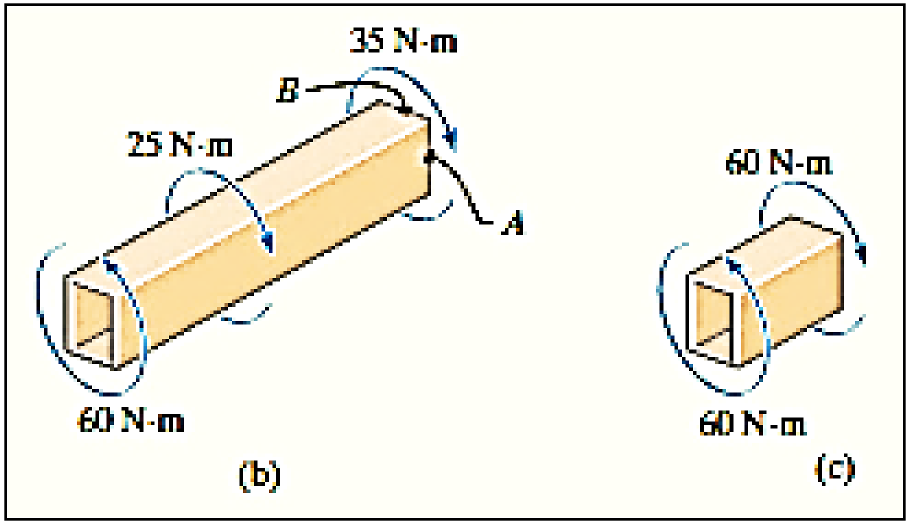
Solution

- These results are shown on elements of material located at points *A* and *B*, Figure e below.
- Note carefully how the torque in Figure *b* creates these stresses on the back sides of each element.



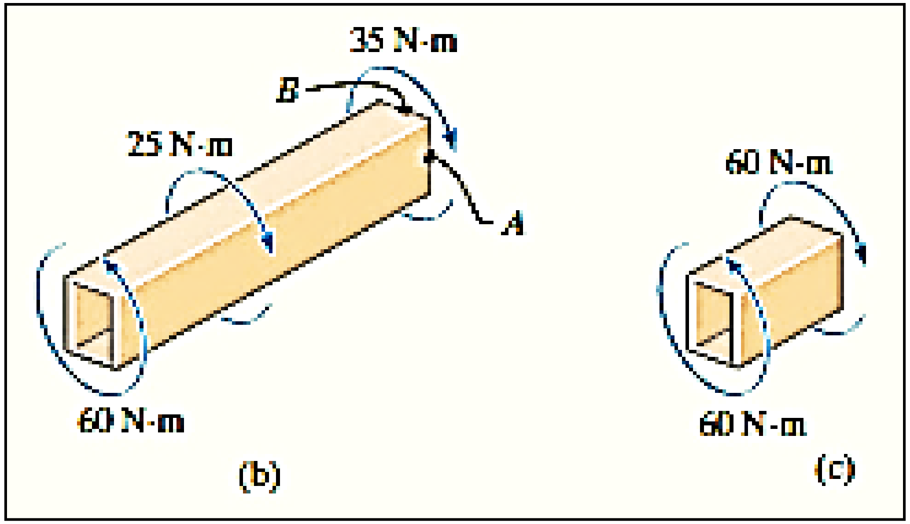
Solution

- **Angle of Twist**. From the free-body diagrams in Figures (*b*) and (*c*), the internal torques in regions *DE* and *CD* are 35 N.m and 60 N.m respectively.
- By standard convention, these torques are both positive.



- The Eqn. $\phi = \frac{TL}{4A_m^2G}\oint\frac{ds}{t}$ becomes; $\phi = \sum\frac{TL}{4A_m^2G}\oint\frac{ds}{t}$

Solution



- $$\phi = \sum \frac{\boldsymbol{TL}}{\boldsymbol{4A_m^2G}} \oint \frac{\boldsymbol{ds}}{\boldsymbol{t}} = \frac{60\ N.m(0.5\ m)}{4(0.00200\boldsymbol{m^2})^2\ (38\ (10^9)\boldsymbol{N/m^2})}\left[2\left(\frac{57mm}{5mm}\right) + 2\left(\frac{35mm}{3mm}\right)\right]$$

$$+ \quad \frac{35\ N.m(1.5\ m)}{4(0.00200\boldsymbol{m^2})^2\ (38\ (10^9)\boldsymbol{N/m^2})}\left[2\left(\frac{57mm}{5mm}\right) + 2\left(\frac{35mm}{3mm}\right)\right]$$

= 6.29(10^{-3}) rad

CONCLUSION

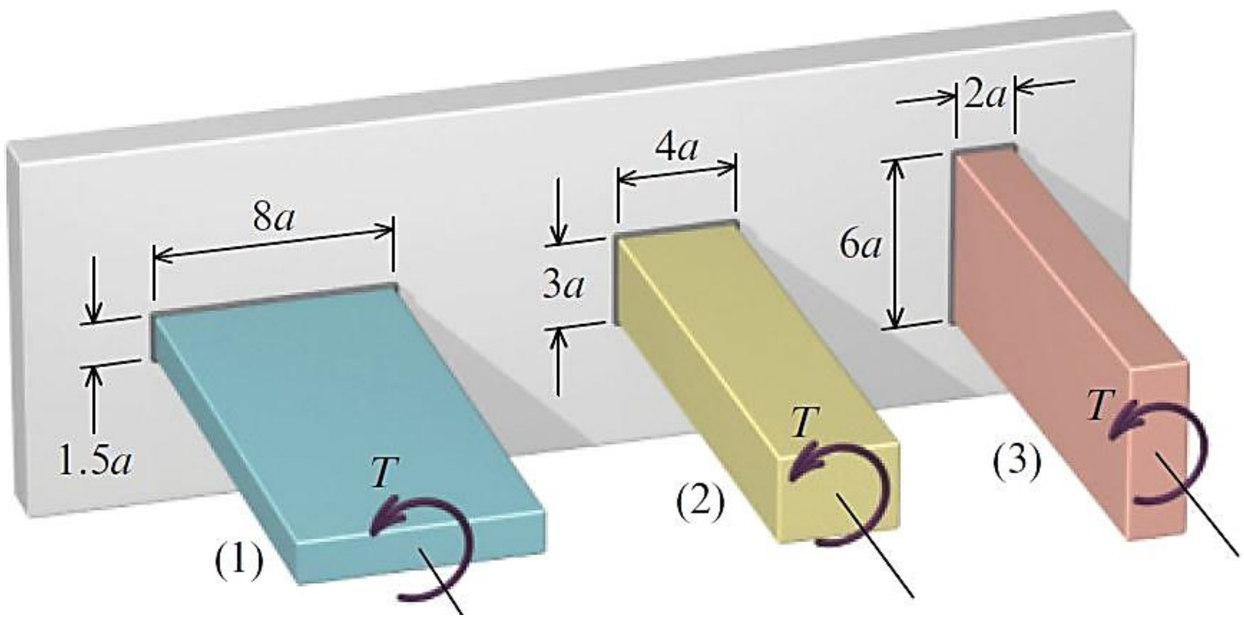
- **Important Points**
- Shear flow ***q*** is the product of the tube's thickness and the average shear stress.
- This value is the same at all points along the tube's cross section, analogous to the continuity equation in fluid mechanics.
- As a result, ***the largest average shear stress on the cross section occurs where the thickness is smallest.***
- Both ***shear flow*** and the ***average shear stress*** act ***tangentially*** to the wall of the tube at all points and in a direction so as to contribute to the resultant ***internal torque***.

Question 7.10

The bars shown in Figure have equal cross-sectional areas, and they are each subjected to a torque T = 550 Nm. Using a = 10 mm, determine

(a) the maximum shear stress in each bar.

(b) the rotation angle at the free end if each bar has a length of 900 mm. Assume that G = 28GPa.



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