MEC 3352

BEAMS ON ELASTIC FOUNDATIONS

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- There are many problems in which a beam is supported on a compressible foundation exerting a distributed reaction on the beam.
- Reaction intensity is proportional to the compressibility.
- Sometimes foundations exerts upward forces only – therefore if beam is sufficiently long, it might lose contact with foundation,
- In others, pressures may be exerted either way.
- The support may not be truly continuous (e.g. holding down a railway line) but can be replaced by an equivalent support.

If y is the upward deflection of the foundation at any point, the rate of upward reaction is $-ky$, and

$$\frac{EId^4y}{dx^4} = -ky$$

or

$$\frac{d^4y}{dx^4} = -4\alpha^4 y \qquad (1)$$

where: $$\alpha^4 = \frac{k}{4EI}$$

A number of standard cases will now be considered.

(a) Long Beam Carrying a Central Load (Fig. 1(a))

Assuming that the foundation can exert upward forces only, let $2l$ be the length of beam in contact with the foundation, and take the origin o at the left hand end.

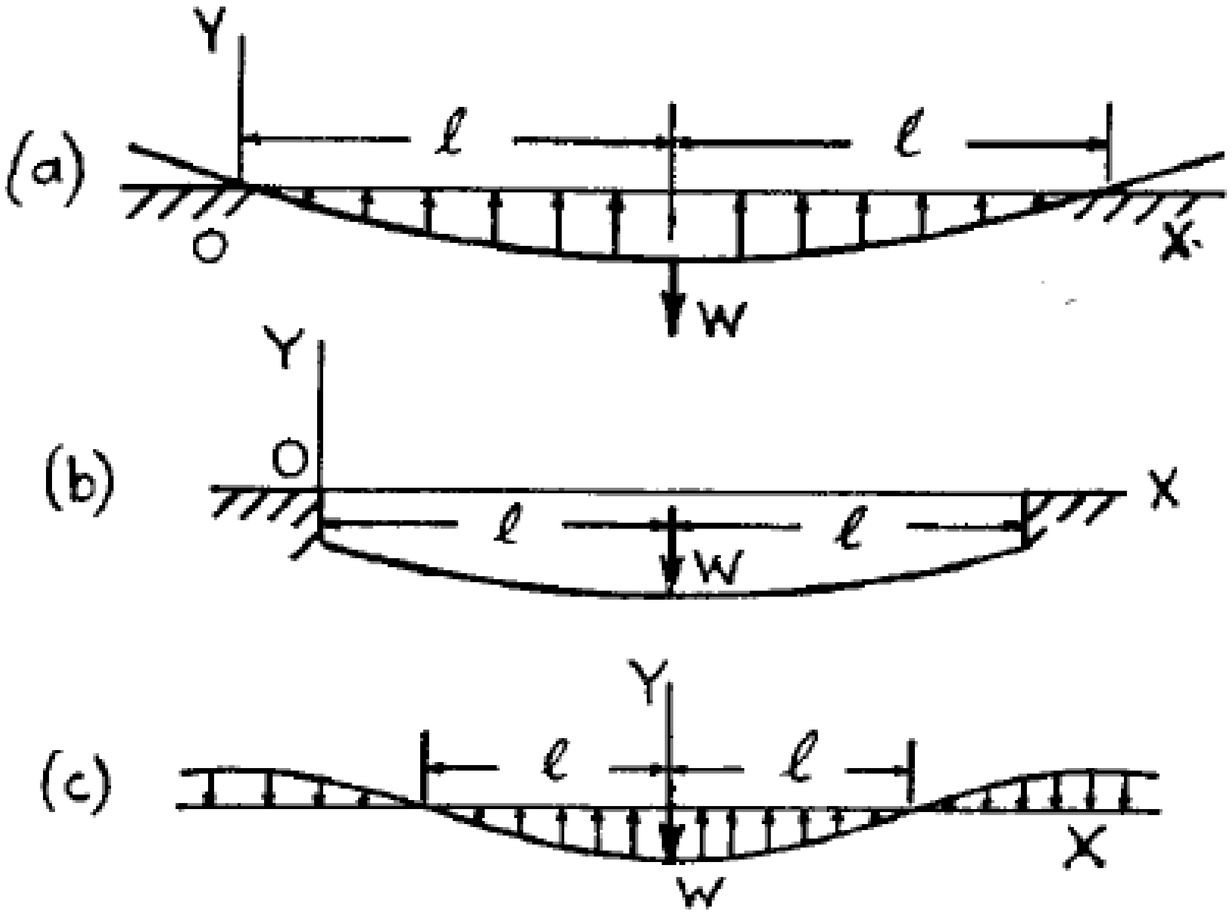

Figure 1

The solution to Equation (1) can be written as:

$$y = A \sin \alpha x \cdot \sinh \alpha x + B \cos \alpha x \cdot \sinh \alpha x$$
$$+C \sin \alpha x \cdot \cosh \alpha x + D \cos \alpha x \cdot \cosh \alpha x$$

At $x = 0$, $y = 0$. Therefore, $D = 0$

and $M = \frac{EId^2y}{dx^2} = 0$ therefore, A = 0

also $F = \frac{EId^3y}{dx^3} = 0$

giving
$$EI \cdot 2\alpha^3 B(-\cos 0 \cdot \cosh 0 - \sin 0 \cdot \sinh 0)$$
$$+ C(-\sin 0 \cdot \sinh 0 + \cos 0 \cdot \cosh 0)] = 0$$

i.e.: $C = B$

The equation is now reduced to:

$$y = B(\cos \alpha x \cdot \sinh \alpha x + \sin \alpha x \cdot \cosh \alpha x)$$

At $x = l$, $\quad \frac{dy}{dx} = 0$

Therefore:

$$B\alpha \cos \alpha l \cdot \cosh \alpha l = 0$$

The least solution of this is $\alpha l = \frac{\pi}{2}$ which determines the length in contact with the ground.

The value of the constant B is obtained from the condition that the shear force at the centre is $\frac{W}{2}$, since by symmetry it must be numerically the same on either side of the load and it must change by an amount W on passing through the load. Hence:

$$\frac{W}{2} = EI\frac{d^3y}{dx^3} = -EI \cdot 4\alpha^3 B \sin \alpha l \ \cdot \sinh \alpha l$$

$$B = -\frac{W\alpha}{2k} \sinh \frac{\pi}{2}$$

The maximum deflection and bending moment at the centres, $\alpha x = \frac{\pi}{2}$,

$$\hat{y} = -\left(\frac{W\alpha}{2k}\coth\frac{1}{2}\pi\right)$$

$$\widehat{M} = EI\left(\frac{W\alpha^3}{k}\right)\coth\frac{1}{2}\pi$$

$$= \left(\frac{W}{4\alpha}\right)\coth\frac{1}{2}\pi$$

(b) Short Beam Carrying Central Load W (Fig. 1(b))

If $\alpha l < \frac{\pi}{2}$ in case (a), the beam will sink below the unstressed level of the foundation at all points.

Again taking the origin at the left-hand end and the overall length of beam as $2l$, the following are obtained for the constants of integration of the general solution of the previous paragraph.

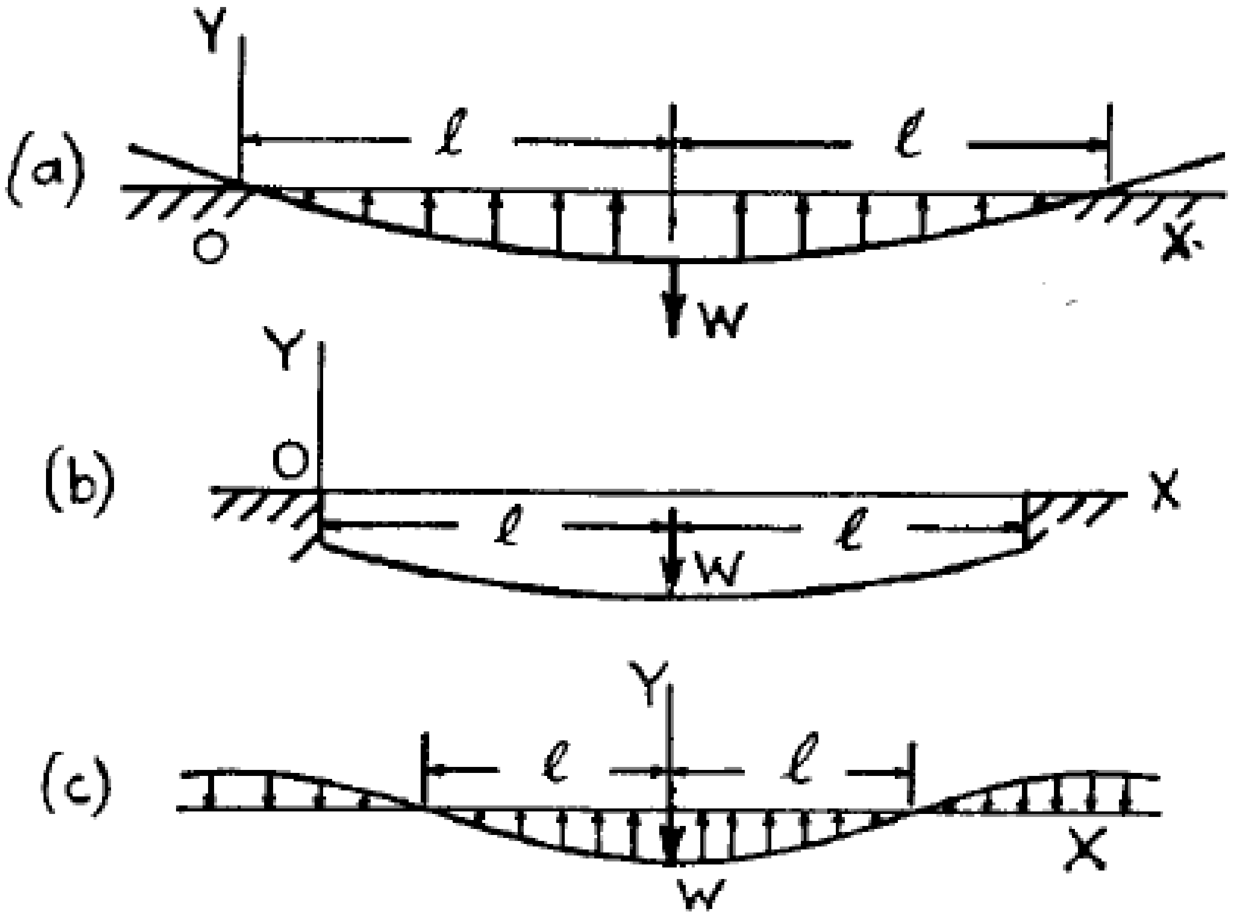

Figure 1

At $x = 0$, $\qquad \dfrac{d^2y}{dx^2} = 0.$

therefore: $\quad A = 0$

and $\quad \dfrac{d^3y}{dx^3} = 0.$

therefore: $\quad B = C$

And

$$y = B\left(\cos \alpha x \cdot \sinh \alpha x + \sin \alpha x \cdot \cosh \alpha x\right) + D \cos \alpha x \cdot \cosh \alpha x$$

At $x = l$, $\dfrac{dy}{dx} = 0$,

giving

$$B \cdot 2\cos\alpha l \cdot \cosh\alpha l + D(-\sin\alpha l \cdot \cosh\alpha l + \cos\alpha l \cdot \sinh\alpha l) = 0$$

and $EI\dfrac{d^3y}{dx^3} = \dfrac{W}{2}$

giving

$$-B \cdot 2\sin\alpha l \cdot \sinh\alpha l - D(\sin\alpha l \cdot \cosh\alpha l + \cos\alpha l \cdot \sinh\alpha l)$$

$$= \frac{W}{4EI\alpha^3} = \frac{W\alpha}{k}$$

Solving for B and D gives:

$$B = -\frac{W\alpha}{k} \cdot \frac{\sin \alpha l \cdot \cosh \alpha l + \cos \alpha l \cdot \sinh \alpha l}{\sin 2\alpha l + \sinh 2\alpha l}$$

$$D = -\frac{2W\alpha}{k} \cdot \frac{\cos \alpha l \cdot \cosh \alpha l}{\sin 2\alpha l + \sinh 2\alpha l}$$

The complete solution for y is now known, the maximum deflection and bending moment being under the load.

(c) Infinite Beam Carrying Load W (Fig. 1(c))

- Assuming that the support can exert pressure either upwards or downwards;
- taking the *Y* axis through the load and the *X* axis at the undeformed level;
- a solution of equation (1) can be written in the form:

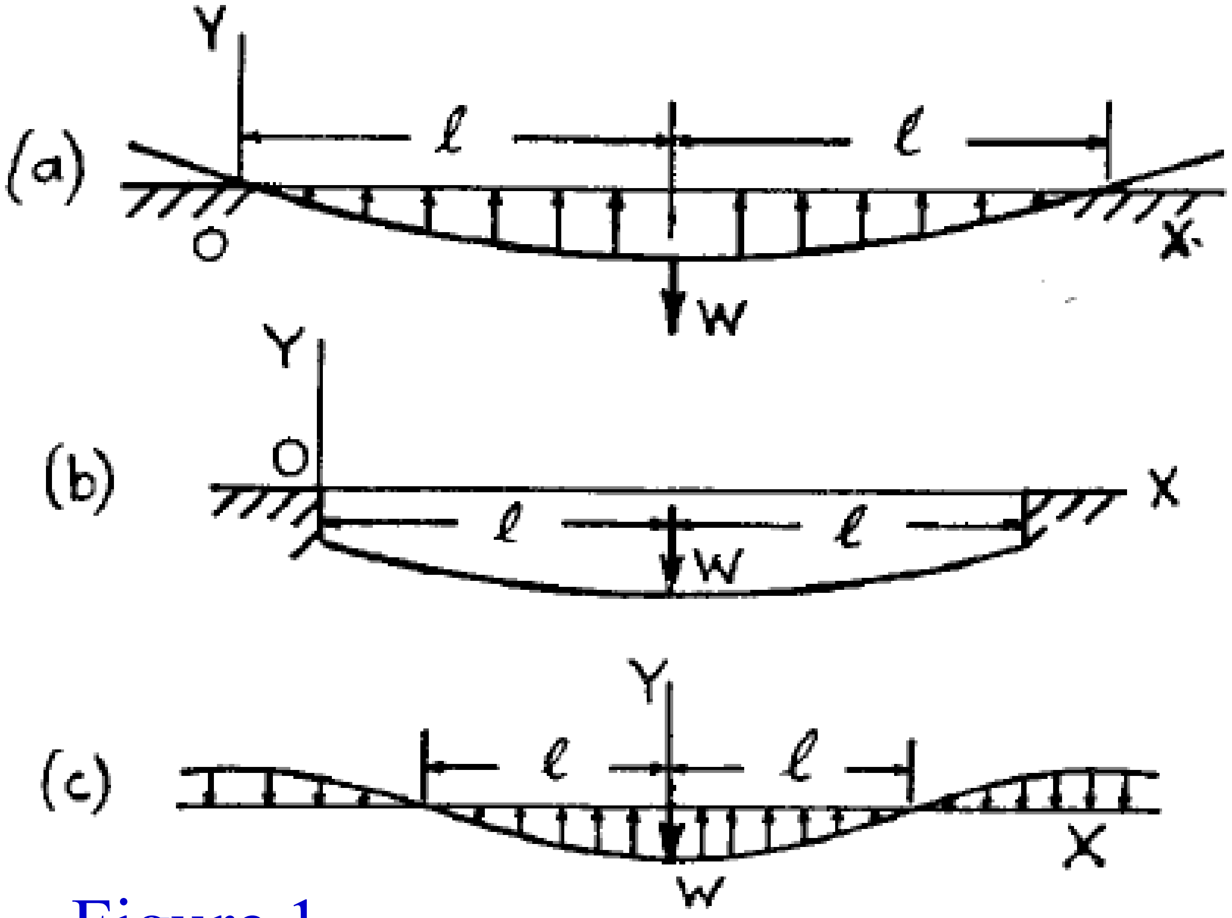

Figure 1

$$y = e^{\alpha x}(A \sin \alpha x + B \cos \alpha x) + e^{-\alpha x}(C \sin \alpha x + D \cos \alpha x)$$

For the length to the right of W, since $y \to 0$ as $x \to \infty$, $\quad A = B = 0$

At $x = 0$, $\quad \dfrac{dy}{dx} = 0$, $\quad$ Therefore $\quad C = D$

and $\quad EI\dfrac{d^3y}{dx^3} = -\dfrac{W}{2}$

Giving $\quad C = -\dfrac{W}{8\alpha^3 EI} = -\dfrac{W\alpha}{2k}$

and $\quad y = -\left(\dfrac{W\alpha}{2k}\right)e^{-\alpha x}(\sin \alpha x + \cos \alpha x)$

The distance from the load at which $y = 0$ is given by

$$\sin \alpha l + \cos \alpha l = 0$$

The least solution being $\alpha l = \frac{3\pi}{4}$

The maximum deflection and bending moment at the centres, $x = 0$,

$$\hat{y} = -\frac{W\alpha}{2k}$$

$$\widehat{M} = EI\frac{W\alpha^3}{k} = \frac{W}{4\alpha}$$

Example:

A steel railway track is supported on timber sleepers which exert an equivalent load of 2800 N/m length of rail per mm deflection from its unloaded position. For each rail $I = 12 \times 10^6$ mm^4, $Z = 16 \times 10^4$ mm^3 and E = 205,000 N/mm^2. if a point load of 100 kN acts on each rail, find the length of rail over which the sleepers are depressed and the maximum bending stress in the rail.

Solution:

Given:

Equivalent load, k = 2800 N/m per mm deflection

$I = 12 \times 10^6$ mm^4

$Z = 16 \times 10^4$ mm^3

E = 205,000 N/mm^2

W = 100 kN

Required:

- Length of rail over which the sleepers are depressed
- Maximum bending stress in the rail

Solving the problem:

$$\alpha^4 = \frac{k}{4EI} = \frac{2800}{4 \times 10^3 \times 205{,}000 \times 12 \times 10^6}$$

giving

$$\alpha = 0.731 \times 10^{-3} \text{mm}^{-1}.$$

Each rail can be treated as an infinitely long beam, for which the length over which downward deflection occurs is given by scenario (c).

$$2l = \frac{3\pi}{2\alpha} = \frac{3\pi \times 10^3}{2} \times 0.731$$

$$= 6440 \text{ mm} = 6.44 \text{ m}$$

and

$$\hat{M} = \frac{W}{4\alpha} = \frac{100 \times 10^3}{4 \times 0.731} = 34{,}200 \text{ Nm}$$

$$\sigma = \frac{\hat{M}}{Z} = \frac{34{,}200}{16 \times 10^4 \times 10^{-9}} = 213.75 \times 10^6 \text{ N/m}^2$$

$$= 214 \text{ N/mm}^2$$

Questions

1. A beam rests on three supports *A*, *B* and *C*. *A* and *C* are rigid, but *B* compresses 0.0005 mm per kg of load carried. If *AB* = *BC* = 4.5 m, what is the deflection at *B* when the beam is loaded with 16 kN/m run? What is the maximum bending moment and where does it occur? *E* = 204 GPa; *I* = 9350 cm^4.
   (*Ans.:* 4.2 mm; 28.5 kNm, 1.85 m.)

2. A timber beam 15 cm wide and 10 cm deep, rests om compressible ground which exerts an upward pressure of 7000 N/m^2 per mm compression. It supports a load of 1000 kg at its mid-point. Compute the maximum bending stresses when the beam is (a) 1.8 m long, (b) 3 m long. *E* = 10 GPa.
   (*Ans.:* $\alpha = 0.0012\ \mathrm{mm}^{-1}$, (a) 12.6 MPa; (b) 9.3 MPa.)

The End