MEC 3352
STRENGTH OF MATERIALS II

BENDING OF CURVED BEAMS/BARS
Part II

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(c) Circular Cross-Section

- Following the method already established in the “Trapezium” Cross-Section,

$$e = R_m - \frac{A}{\int \frac{dA}{R_m + z}}$$

from (7) under trapezoidal section.

where from Fig. 6,

$$\int \frac{dA}{R_m + z} = 2\int_{-r}^{r} \frac{\sqrt{(r^2 - z^2)}}{R_m + z}\, dz$$

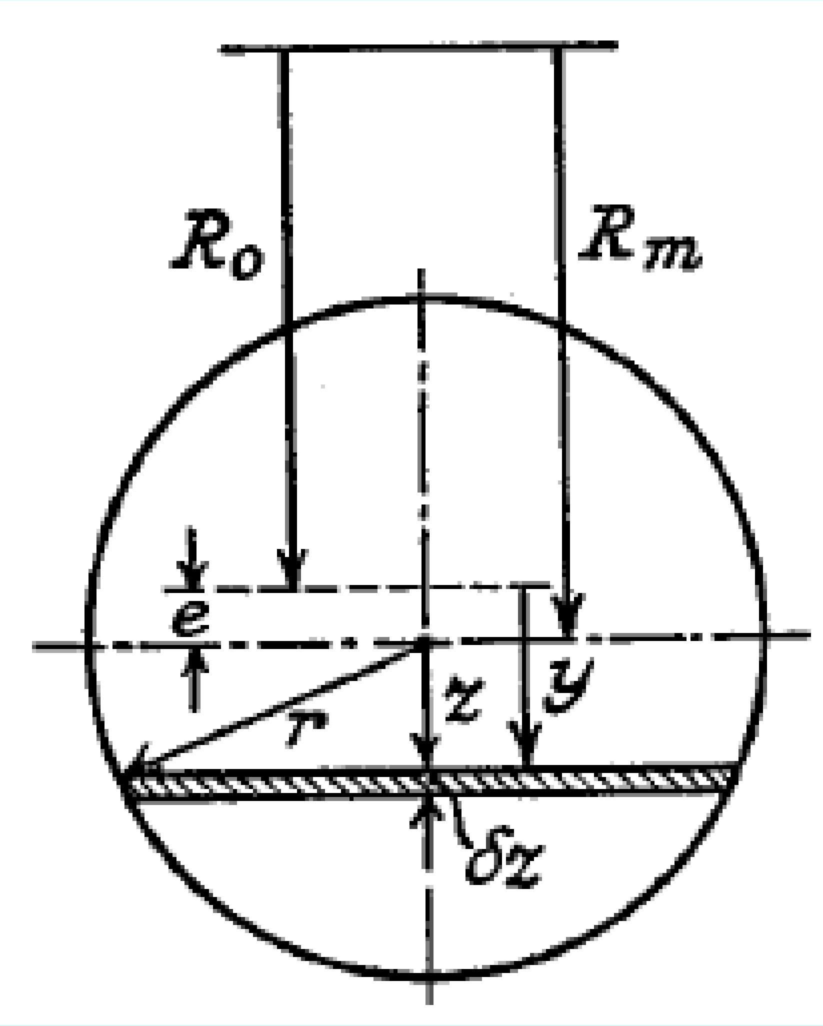


Fig. 6

$$\int \frac{dA}{R_m + z} = 2\pi \left[R_m - \sqrt{(R_m^2 - r^2)} \right]$$ from calculus methods

Hence:

$$\boldsymbol{e = R_m - \frac{r^2}{2\left[R_m - \sqrt{\left(R_m^2 - r^2\right)}\right]}} \tag{1}$$

$$= R_m - \frac{r^2}{2} \cdot \frac{R_m + \sqrt{(R_m^2 - r^2)}}{R_m^2 - (R_m^2 - r^2)} = \frac{1}{2}\left[R_m - \sqrt{(R_m^2 - r^2)} \right]$$

$$= \frac{1}{2}\left[R_m - R_m + \frac{1}{2} R_m \left(\frac{r^2}{R_m^2} \right) + \frac{1}{8} R_m \left(\frac{r^4}{R_m^4} \right) + \cdots \right]$$

$$\boxed{e = \frac{1}{4} R_m \left(\frac{r^2}{R_m^2}\right)\left[1 + \frac{1}{4}\left(\frac{r^2}{R_m^2}\right) + \frac{1}{8}\left(\frac{r^4}{R_m^4}\right) + \cdots\right]} \tag{2}$$

and

$$\boxed{\sigma = \frac{My}{Ae(R_0 + y)}} \quad \text{as before.} \tag{3}$$

3. Deflection of Curved Bars (Direct Method)

- Consider Fig. 7.

- If a length δs of an initially curved bean is acted upon by a bending moment M, it follows from Equation (3) in Section 1 that:

$$\frac{M\delta s}{EI} = \delta s\left(\frac{1}{R} - \frac{1}{R_0}\right)$$

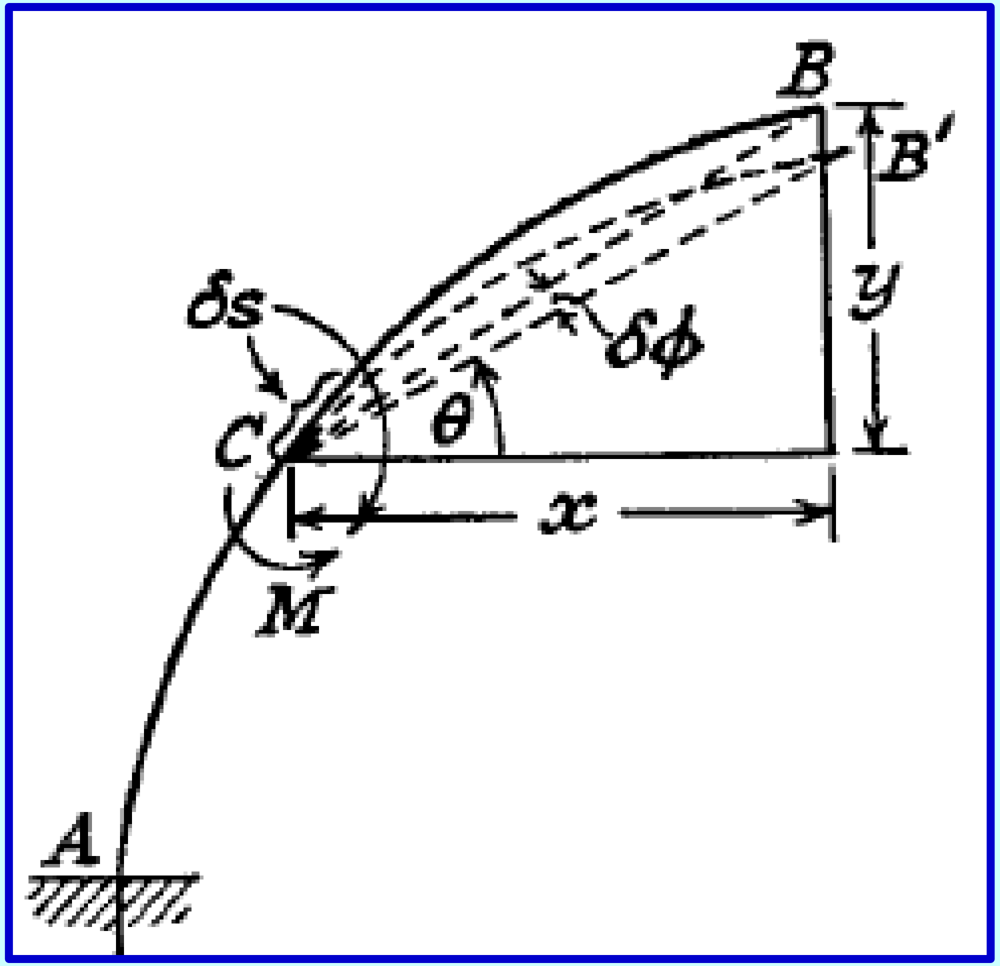


Fig. 7.

- But $\frac{\delta s}{R} - \frac{\delta s}{R_0}$ is the change of angle, $\delta\phi$, subtended by δs at the centre of curvature, and consequently is the angle through which the tangent at one end of the element rotates relative to the tangent at the other end, i.e.

$$\boxed{\boldsymbol{\delta\phi = \frac{M\delta s}{EI}}} \qquad (1)$$

- Fig 7 shows a loaded bar AB which is fixed in direction at A, and it is required to find the deflection at the other end B.

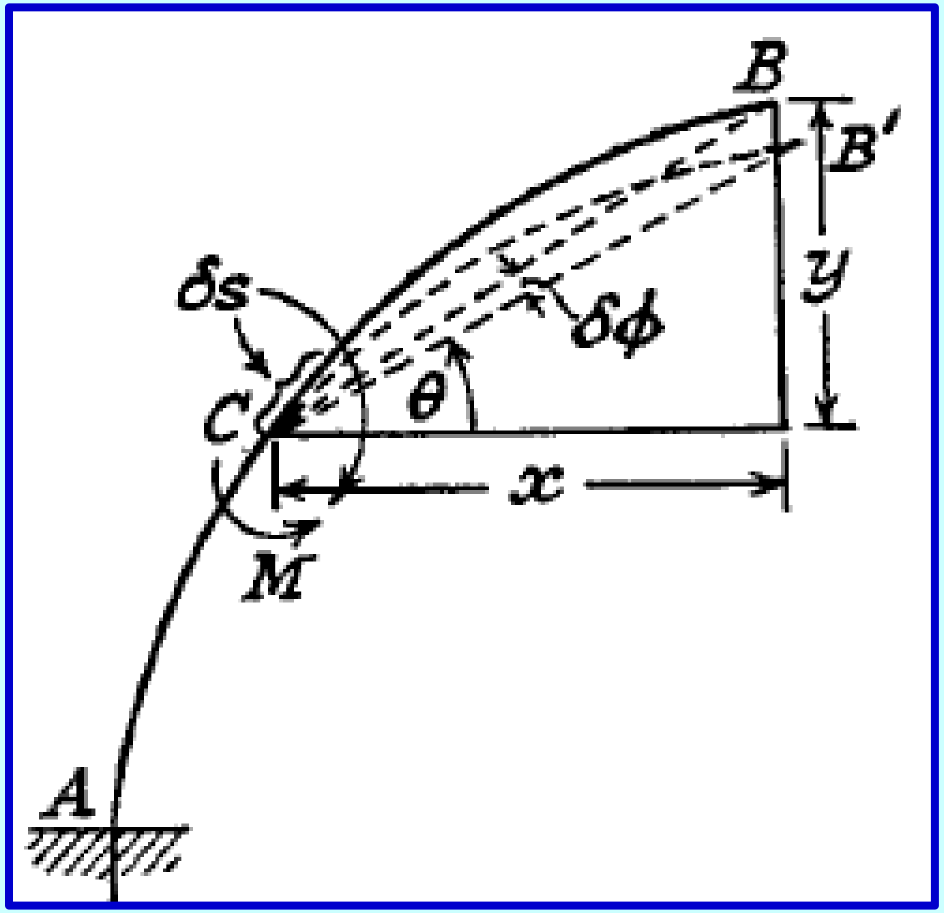


Fig. 7.

- Due to the action of M on δs at C only, the length CB is rotated through an angle

$$\delta\phi = \frac{M\delta s}{EI}$$

- B moves to B', where $BB' = CB \cdot \delta\phi$.

- The vertical deflection of $B = BB' \cdot \cos\theta$

$$= CB \cdot \cos\theta \cdot \delta\phi$$

Vertical Deflection of $B = x \cdot \delta\phi$

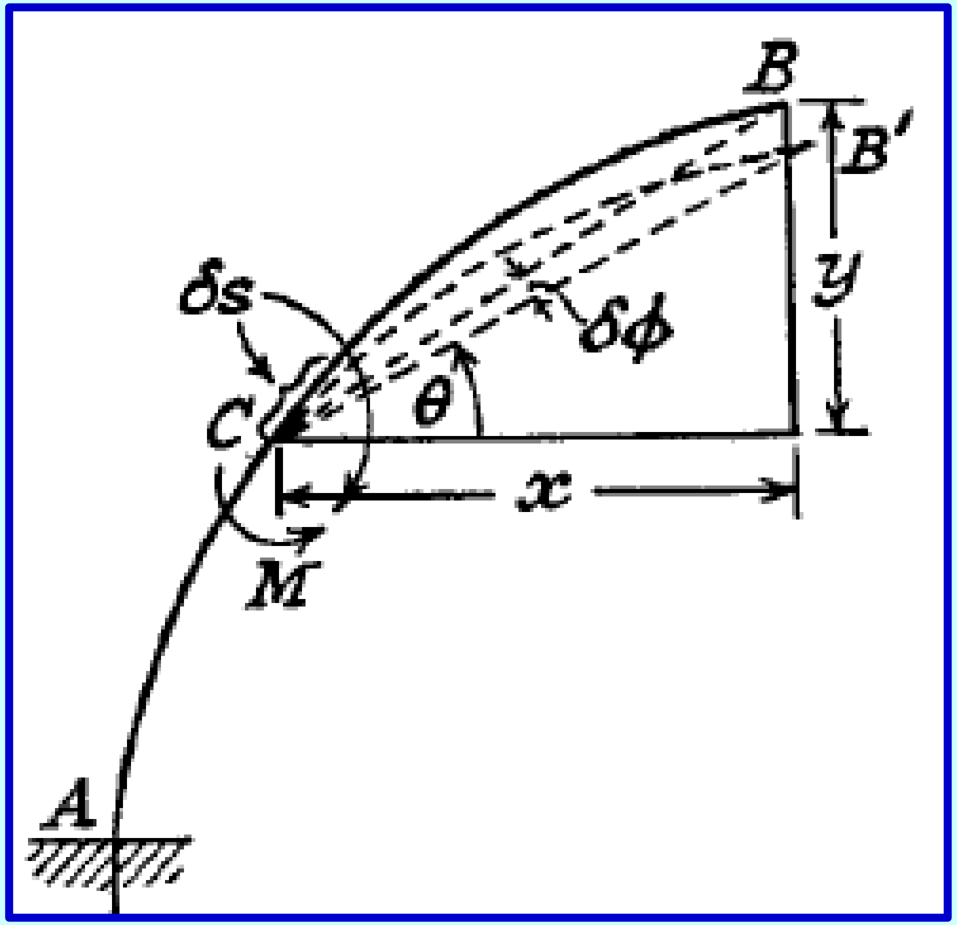


Fig. 7.

- The horizontal deflection of $B = BB' \cdot \sin\theta$

Horizontal Deflection of $B = y \cdot \delta\phi$

- Due to the bending of all the elements along AB,

$$\textbf{Vertical Deflection of } B = x\cdot \delta\phi = \int \frac{Mxds}{EI} \quad \text{from (1)} \qquad (2)$$

and

$$\textbf{Horizontal Deflection of } B = y\cdot \delta\phi = \int \frac{Myds}{EI} \qquad (3)$$

- You can compare this with the moment-area method for deflection of initially straight beams. [Moment-area Method, see Ryder, §9.5, pp. 163-168.]
- The advantage of this method, compared to the next method, is that the deflection can readily be found at any point in any direction, even when there is no load at that point.

Example 3. (Ryder, p. 202).

A steel tube, Fig. 8, having outside diameter 5 cm, bore 3 cm, is bent into a quadrant of 2 m radius. One end is rigidly attached to a horizontal base plate to which a tangent to that end is perpendicular, and the free end supports a load of 100 kg. Determine the vertical and horizontal deflections of the free end under this load. E = 208 GPa.

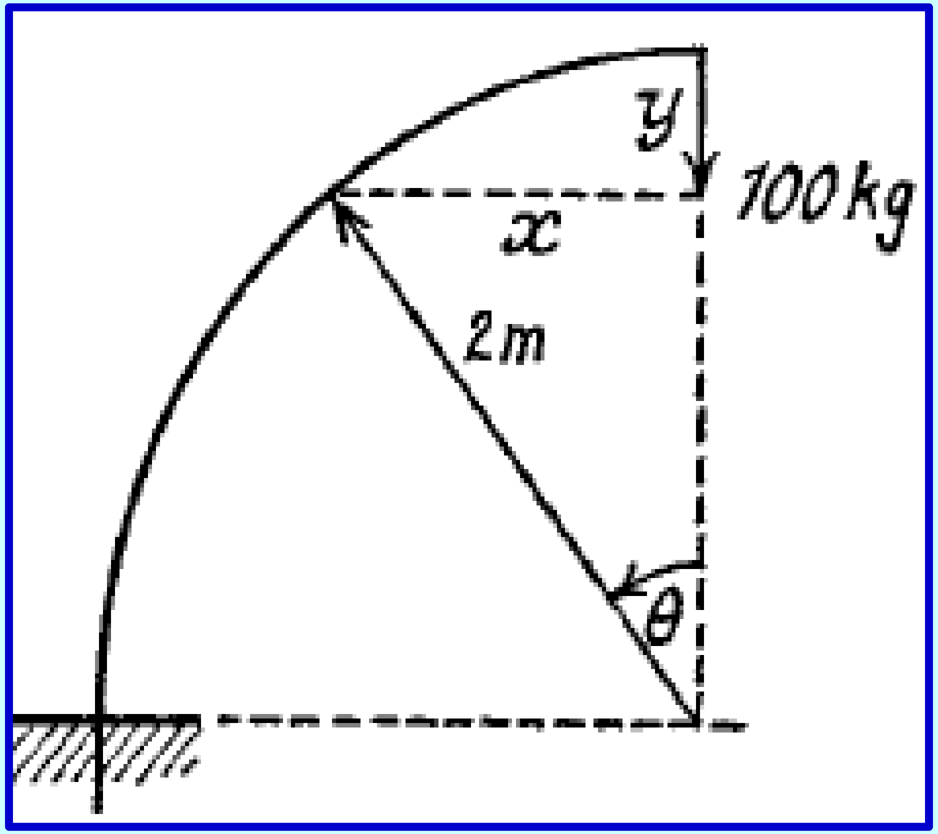


Fig. 8

Solution

$$I = (\pi/64)(50^4 - 30^4) = 267{,}000 \text{ mm}^4$$

$$x = 2000 \sin\theta \text{ mm} \qquad \text{from Fig. 8}$$

$$y = 2000(1 - \cos\theta) \text{ mm}$$

$$M = (100 \times 9.81) \cdot x \text{ Nmm}$$

$$\delta s = 2000 \cdot \delta\theta \text{ mm}$$

$$Vertical\ Deflection = \int \frac{Mxds}{EI} = \int_0^{\pi/2} \frac{100 \times 9.81 \times 2000^3 \sin^2\theta\, d\theta}{208{,}000 \times 267{,}000}$$

$$= 141.31 \int_0^{\pi/2} \frac{(1 - \cos 2\theta)}{2} \cdot d\theta$$

$$= 141.31 \times \frac{\pi}{4}$$

Vertical Deflection = 111 mm

$$Horizontal\ Deflection = \int \frac{Myds}{EI}$$

$$= \frac{100 \times 9.81 \times 2000^3}{208{,}000 \times 267{,}000} \int_0^{\pi/2} \sin\theta (1 - \cos\theta) d\theta$$

$$= 141.31 \left[-\cos\theta + \frac{1}{4}\cos 2\theta \right]_0^{\pi/2}$$

$$= 141.31 \times \frac{1}{2}$$

$$\mathbf{Horizontal\ Deflection = 70.66\ mm}$$

4. Deflection from Strain Energy (Castigliano's Theorem)

- Developed by Italian mathematician and physicist, **Carlo Alberto Castigliano** (1847 - 1884)
- **Theorem:** ***If U is the total strain energy of any structure due to the application of external loads $W_1, W_2, \cdots$, at $O_1, O_2, \cdots$ in the directions O_1X_1, O_2X_2, $\cdots$, and to couples $M_1, M_2, \cdots$, then the deflections at $O_1, O_2, \cdots$ in the directions O_1X_1, $O_2X_2, \cdots$ are $\frac{\partial U}{\partial W_1}, \frac{\partial U}{\partial W_2}, \cdots$, and the angular rotations of the couples are $\frac{\partial U}{\partial M_1}, \frac{\partial U}{\partial M_2}, \cdots$ at their applied points.***

C. A. Castigliano
(1847 - 1884)

Proof for Concentrated Loads.

If the displacements (in the direction of the loads) produced by gradually applied loads $W_1,\ W_2,\ W_3,\cdots$ are $x_1,\ x_2,\ x_3,\cdots$, then

$$\boxed{\boldsymbol{U = \frac{1}{2}W_1x_1 + \frac{1}{2}W_2x_2 + \frac{1}{2}W_3x_3 + \cdots}} \tag{1}$$

Let W_1 alone be increased by δW_1, then

$$\delta U = \text{increase in external work done}$$

$$= \left(W_1 + \frac{\delta W_1}{2}\right)\delta x_1 + W_2\delta x_2 + W_3\delta x_3 + \cdots$$

where $\delta x_1,\ \delta x_2,\ \delta x_3,\cdots$ are the increases in $x_1,\ x_2,\ x_3,\cdots$

$$\boldsymbol{\delta U = W_1 \delta x_1 + W_2 \delta x_2 + W_3 \delta x_3 + \cdots} \tag{2}$$

neglecting the product $\frac{\delta W_1}{2} \delta x_1$.

But if the loads $W_1 + \delta W_1, W_2, \; W_3, \cdots$ were applied gradually from zero, the total strain energy is

$$U + \delta U = \frac{1}{2}(W_1 + \delta W_1)(x_1 + \delta x_1) + \frac{1}{2} W_2 (x_2 + \delta x_2) + \frac{1}{2} W_3 (x_3 + \delta x_3) + \cdots$$

Subtracting (1), and neglecting products of small quantities:

$$\boldsymbol{\delta U = \frac{1}{2} W_1 \delta x_1 + \frac{1}{2} \delta W_1 x_1 + \frac{1}{2} W_2 \delta x_2 + \frac{1}{2} W_3 \delta x_3 + \cdots} \tag{3a}$$

or

$$\boldsymbol{2\delta U = W_1 \delta x_1 + \delta W_1 x_1 + W_2 \delta x_2 + W_3 \delta x_3 + \cdots} \tag{3b}$$

Subtract (2), then $\delta U = \delta W_1 x_1$

and in the limit: $$\boxed{\frac{\partial U}{\partial W_1} = x_1} \qquad (4)$$

Similarly for x_2 and x_3, and so on, and the proof can be extended to incorporate couples.

Important:

- ***U*** is the ***total strain energy***, expressed in terms of loads and not `including statically determinate reactions, and
- the partial derivatives with respect to each load in turn (treating the others as constant) gives the deflection at the load point in the direction of the load.

Principles to be observed in applying this theorem:

1. In finding the deflection of curved beams and similar problems, only strain energy due to bending need normally be taken into account, i.e.:

$$\int \frac{M^2}{2EI}\delta s \qquad \text{from (4) in Section 1.} \tag{5}$$

2. Treat all the loads as “variables” initially, carry out the partial differentiation and integration, putting in numerical values at the final stage.

3. If the deflection is to be found at a point where, or in a direction in which, there is no load, a load may be put in where required and given a value zero in the final reckoning ($x = (\partial U/\partial W_1)_{W_1=0}$).

Generally, this method:

- Requires less thought in application than the direct method.
- It is only necessary to obtain an expression for the bending moment.
- Strain energy is bound to be positive, and deflection is positive in the direction of the load, so no problem with signs.

Disadvantage of this method:

- The only disadvantage occurs when a case such as principle (3) above has to be dealt with, when the direct method of Section 3 will probably be shorter.

Example 4. (Ryder, p. 204).

Obtain an expression for the vertical displacement at A of the beam shown in Fig. 9.

Solution:

The bending moments in the various sections can be written as follows:

AB: $M = Wx$ (at x from A going left)

BC: $M = Wa$ (constant)

CD: $M = Wx'$ (at x' from D)

DE: $M = Wx''$ (at x'' from D)

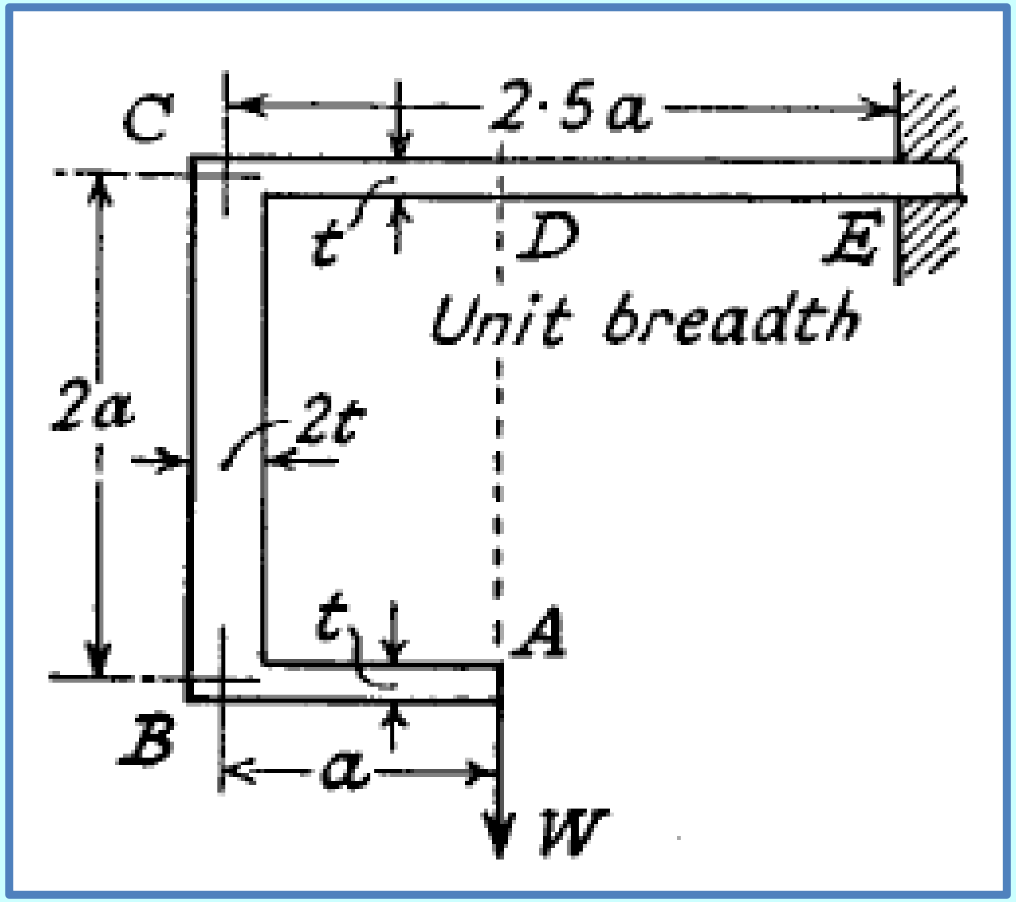


Fig. 9.

$$U = \int M^2 \cdot ds/2EI$$

$$= \int_0^a \frac{W^2x^2 \cdot dx}{2E \times t^3/12} + \int_0^{2a} \frac{W^2a^2 \cdot ds}{2E \times t^3/12} + \int_0^a \frac{W^2x'^2 \cdot dx'}{2E \times t^3/12} + \int_0^{1.5a} \frac{W^2x''^2 \cdot dx''}{2E \times t^3/12}$$

$$= \left(\frac{6W^2}{Et^3}\right)\left[\frac{a^3}{3} + 2\frac{a^3}{8} + \frac{a^3}{3} + 1.5^3\frac{a^3}{3}\right]$$

$$\boldsymbol{U = \frac{24.5W^2a^3}{2Et^3} = \frac{12.25W^2a^3}{Et^3}}$$

An allowance could be made for the linear extension of the portion BC

$$\delta_{BC} = (W \cdot 2a)/(2t \cdot E)$$

which is clearly negligible compared with the deflection due to bending.

Example 5. (Ryder, p. 204-205).

Fig. 10 shows a steel rod of 12 mm diameter with one end fixed into a horizontal table. The remainder of the rod is bent into the form of three-quarters of a circle and the free end is constrained to move vertically. Determine the vertical deflection for a load of 10 kg. E = 208 GPa.

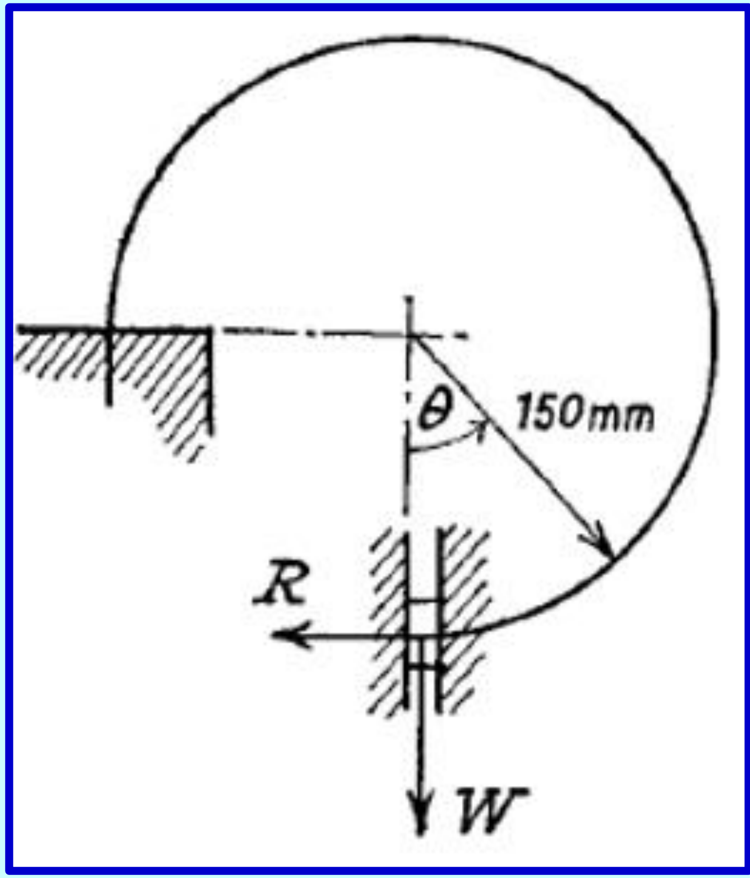


Fig. 10.

Solution:

Let the vertical load be W, and the normal reaction due to the constraint be R. Then:

$$M = R \times 150(1 - \cos\theta) - W \times 150 \sin\theta$$

$$\delta s = \int 150\delta\theta \qquad\qquad U = \int M^2 \cdot \delta s/2EI$$

$$U = \frac{150^3}{2EI} \int_0^{3\pi/2} [R(1-\cos\theta) - W\sin\theta]^2 \cdot d\theta$$

Since there is no horizontal displacement, $\partial U/\partial R = 0$, i.e.

$$\frac{\partial U}{\partial R} = \int_0^{\frac{3\pi}{2}} 2[R(1-\cos\theta) - W\sin\theta](1-\cos\theta)d\theta = 0$$

$$= \int_0^{\frac{3\pi}{2}} [2R - 4R\cos\theta + R(1-\cos 2\theta) - 2W\sin\theta + W\sin 2\theta]d\theta = 0$$

i.e. $$3R\left(\frac{3\pi}{2}\right) - 4\mathrm{R}(-1) + \frac{R}{2}(0) + 2W(0-1) - \left(\frac{W}{2}\right)(-1-1) = 0$$

giving

$$\boldsymbol{R} = \frac{W}{\frac{9\pi}{2} + 4} = \mathbf{0.55\ kg} = \mathbf{5.4\ N}$$

$$= \mathbf{0.55\ kg}$$

$$\boxed{\boldsymbol{R} = \mathbf{5.4\ N}}$$

Vertical Displacement = $\partial U/\partial W$

$$= (150^3/2EI)\int_0^{\frac{3\pi}{2}} 2[R(1-\cos\theta) - W\sin\theta](1-\cos\theta)d\theta$$

$$= (150^3/2EI)\int_0^{\frac{3\pi}{2}} [-2R\sin\theta + R\sin 2\theta + W(1-\cos 2\theta)]d\theta$$

$$= (150^3/2EI)\left[2R(-1) - \frac{R}{2}(-1-1) + W\frac{3\pi}{2} - \frac{W}{2}(0)\right]$$

$$= \left(\frac{3.375\times 10^6\times 64}{2\times 208{,}000\times \pi\times 12^4}\right)\left[-5.4 + \frac{98.1\times 3\pi}{2}\right]$$

Vertical Displacement = 3.65 mm

Example 6. (Ryder, p. 205-206).

If a ring of mean radius R is acted upon by equal and opposite pulls P along a diameter, find the expressions for the maximum bending moment and deflection along the line of P.

Solution

- The bending moment cannot immediately be obtained in terms of P and R.
- But, making use of the symmetry, let M_0 be the bending moment on cross-sections perpendicular to P (Fig. 11).
- There will also be a normal pull of $P/2$ on these cross-sections.

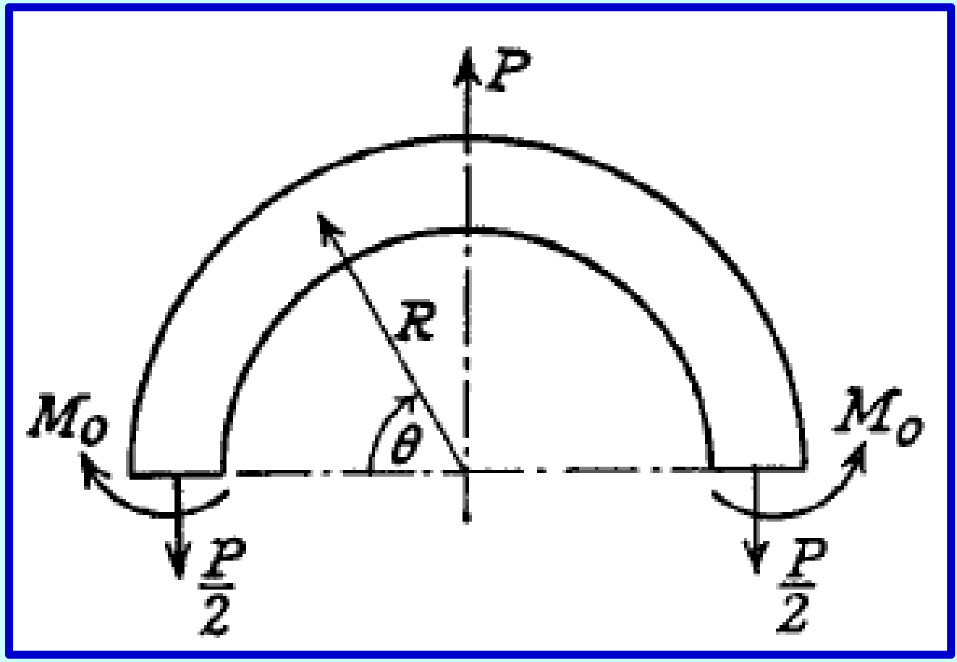


Fig. 11.

At an angle θ

$$M = (PR/2)(1 - \cos\theta) - M_0$$

and

$$U = 4\int_0^{\pi/2} \frac{[PR(1 - \cos\theta) - 2M_0]^2}{4 \times 2EI} R d\theta$$

$$\frac{\partial U}{\partial M_0} = \frac{R}{2EI}\int_0^{\pi/2} 2[PR(1 - \cos\theta) - 2M_0](-2) d\theta$$

$$= \text{rotation of } M_0$$

$$= 0 \text{ by symmetry}$$

$$\therefore \quad \int_0^{\pi/2} [PR - PR\cos\theta - 2M_0] d\theta = 0$$

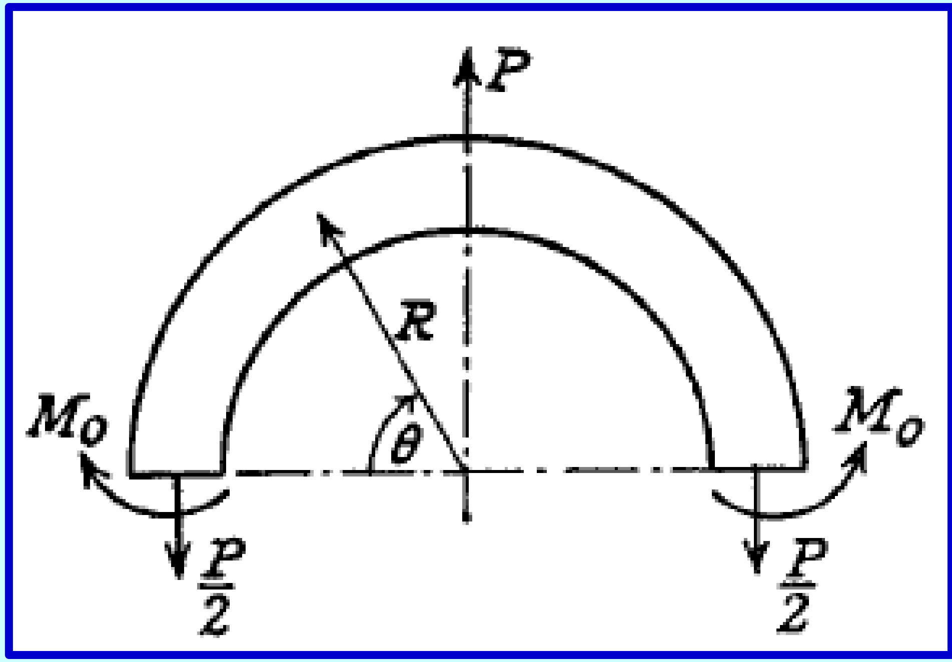


Fig. 11.

i.e. $$PR \cdot \frac{\pi}{2} - \mathrm{PR} - 2M_0 \cdot \frac{\pi}{2} = 0$$

giving $$\boxed{\boldsymbol{M_0 = PR\left(\frac{1}{2} - \frac{1}{\pi}\right)}}$$

The maximum bending moment occurs when $\theta = \pi/2$, and

$$\boxed{\boldsymbol{\widehat{M} = \frac{1}{2}PR - M_0 = \frac{PR}{\pi}}}$$

The deflection of $\boldsymbol{P}$ $= \partial U/\partial P$

$$= (R/2EI)\int_0^{\pi/2} 2[PR(1-\cos\theta) - 2M_0](1-\cos\theta)Rd\theta$$

$$= (R^2/2EI)\int_0^{\pi/2}[2PR - 4PR\cos\theta + PR(1+\cos\theta) - 4M_0 + 4M_0\cos\theta]d\theta$$

$$= (R^2/2EI)\left[PR\pi - 4PR + PR\left(\frac{\pi}{2}\right) + \frac{PR}{2}(0) - 2M_0 + 4M_0\right]$$

$$= (PR^3/2EI)\left[\pi - 4 + \left(\frac{\pi}{2}\right) - \pi + 2 + 2 - \frac{4}{\pi}\right]$$

$$\textbf{Deflection of } \boldsymbol{P = \frac{PR^3}{4EI}\cdot\frac{\pi^2 - 8}{\pi}}$$

5. Portal Frame by Strain Energy

- A frame with stiff joints could be analysed by a “direct” method based on the moment-area equations.

- It is frequently simpler to make use of Castigliano’s theorem to solve this type of problem, as the following example will illustrate.

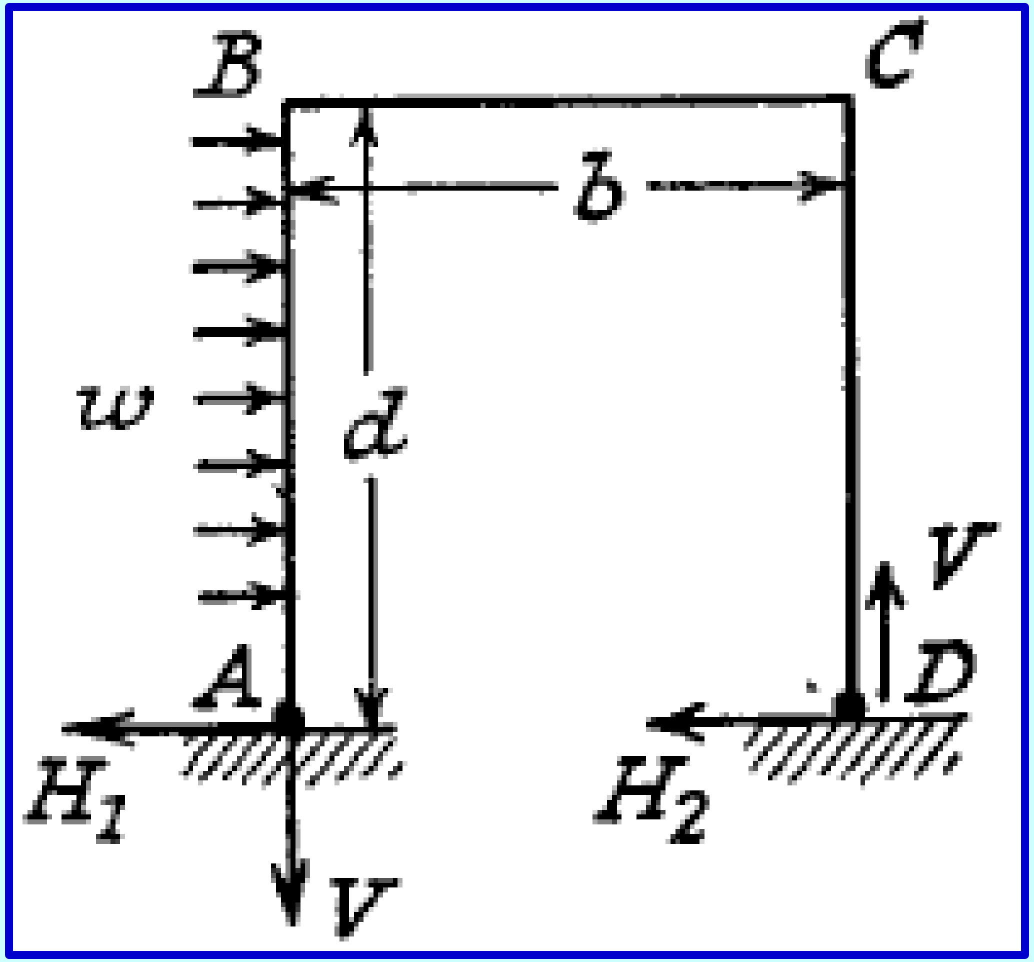


Fig. 12

Example 7. (Ryder, pp. 206-207)

The framework shown in Fig. 12 is pin-jointed to the ground at A and D and is loaded along AB with a distributed load w. If the flexural rigidity EI is constant throughout, obtain expressions for the reactions at A and D.

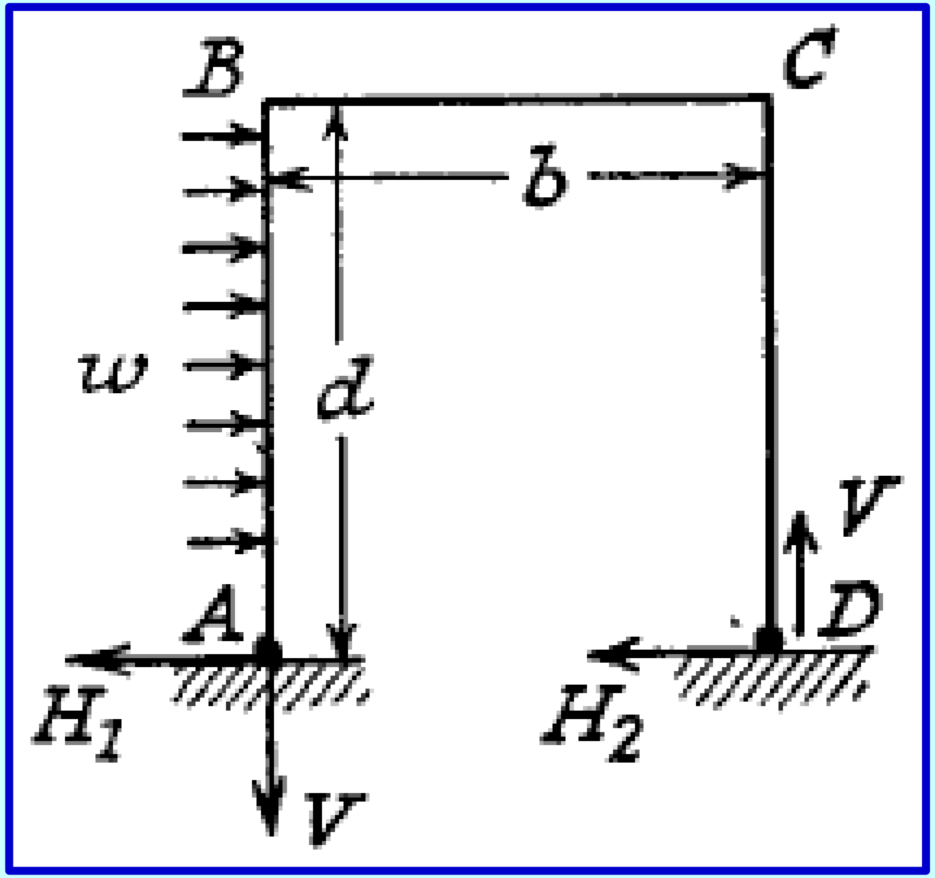


Fig. 12

Solution:

Resolving vertically, the vertical components of reaction, V, must be equal and opposite at A and D, and by moments about A:

$$V = \frac{wd^2}{2b} \qquad \text{(i)}$$

Resolving horizontally:

$$H_1 + H_2 = wd \qquad \text{(ii)}$$

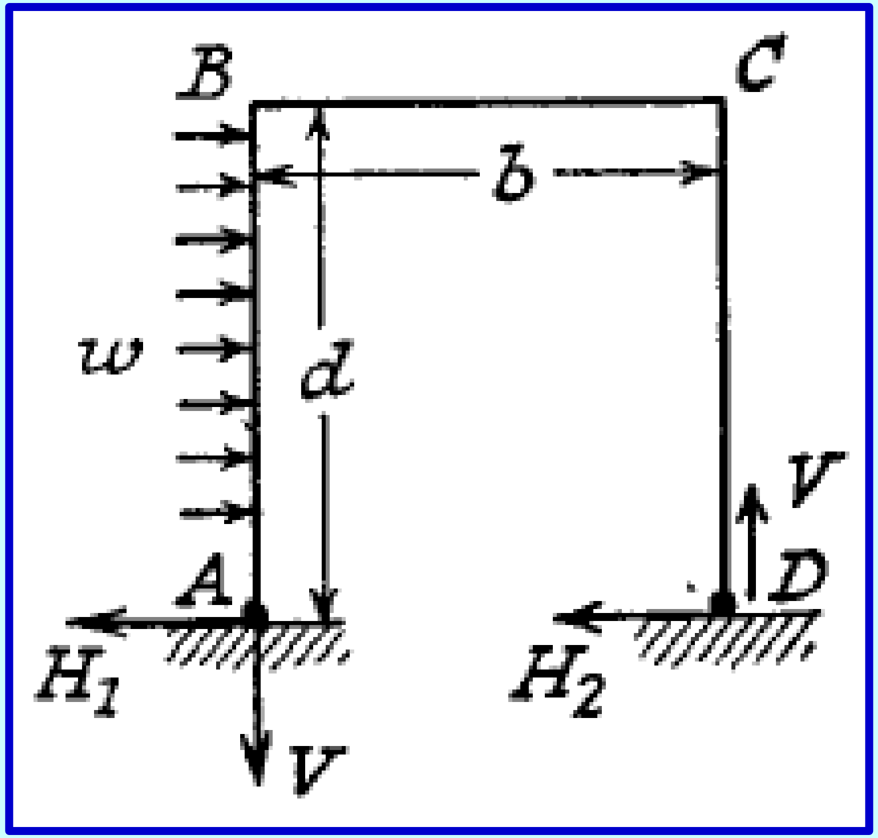


Fig. 12

Bending moment along AB, at a distance x from A, is

$$M_1 = H_1 x - wx^2/2 \qquad \text{from (i)}$$

Bending moment along BC, at a distance x' from B, is

$$M_2 = H_1 d - V x' - w d^2/2$$

$$M_2 = H_1 d - w d^2 x'/2b - w d^2/2 \quad \text{from (i)}$$

$$M_3 = H_2 x''$$

$$M_3 = (wd - H_1) x'' \quad \text{from (ii)}$$

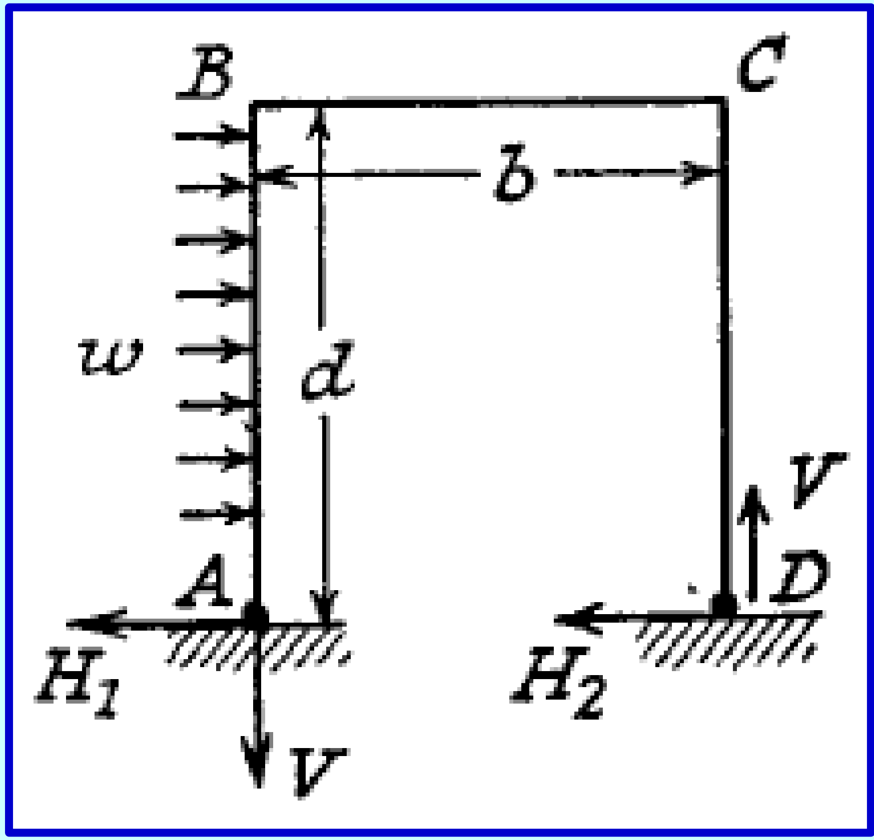


Fig. 12

Total strain energy due to bending

$$U = \int_0^d \frac{M_1^2 dx}{2EI} + \int_0^b \frac{M_2^2 dx'}{2EI} + \int_0^d \frac{M_3^2 dx''}{2EI}$$

But since the supports are fixed in position

$$\partial U/\partial H_1 = 0$$

$$\therefore \quad \int_0^d M_1(\partial M_1/\partial H_1)dx + \int_0^b M_2(\partial M_2/\partial H_1)dx' + \int_0^d M_3(\partial M_3/\partial H_1)dx'' = 0$$

i.e.

$$\int_0^d \left(H_1 x - \frac{wx^2}{2}\right)dx + \int_0^b \left[H_1 d - \left(\frac{wd^2}{2b}\right)x' - \left(\frac{wd^2}{2}\right)\right]dx'$$

$$+ \int_0^d (wd - H_1)x''(-x'')dx'' = 0$$

$$\frac{H_1 d^3}{3} - \frac{wd^4}{8} + H_1 d^2 b - \frac{wd^3 b}{4} - \frac{wd^3 b}{2} - \frac{wd^3}{3} + \frac{H_1 d^3}{3} = 0$$

giving:

$$\boldsymbol{H_1 = \left(\frac{wd}{8}\right)\left[\frac{11d + 18b}{2d + 3b}\right]}$$

and from (ii),

$$\boldsymbol{H_2 = \left(\frac{wd}{8}\right)\left[\frac{5d + 6b}{2d + 3b}\right]}$$

ooooooooooOOOOOoooooooooo

The End

But wait...

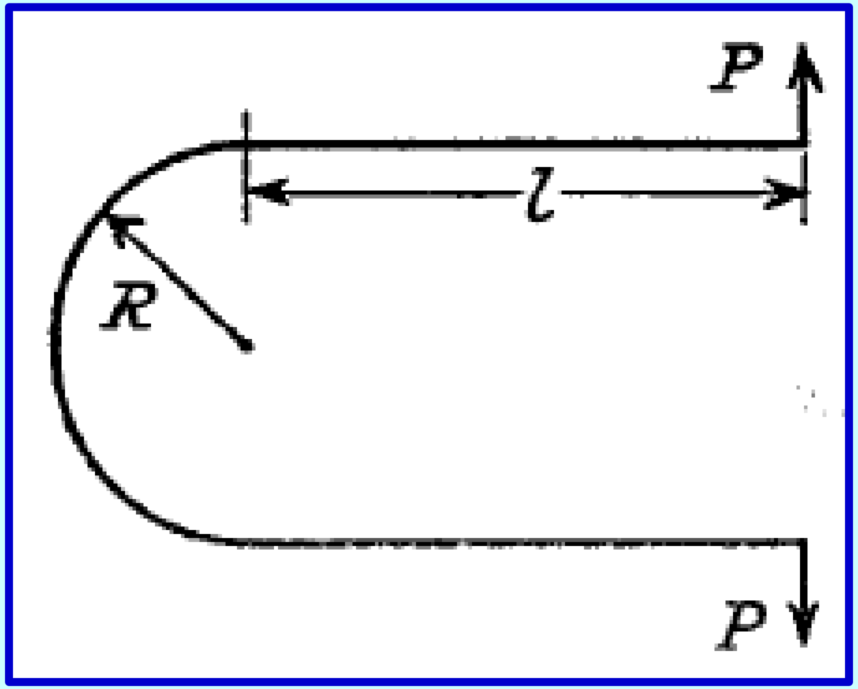
For your practice, solve the problems (1 to 7) at the end of Chapter 11 in Ryder, pp. 207 & 208. Take note that Problem 7 is numbered 6 also.

Hand in as your assignment <u>**Problems 3 and 6**</u>, reproduced in the next two slides.

Assignment — Due in One Week

Assignment Q1 (Ryder, Chapter 11, Problem 3, p.208):

(a) A bar of diameter *d* is bent as shown. Prove that the stiffness *s* is given by the expression:

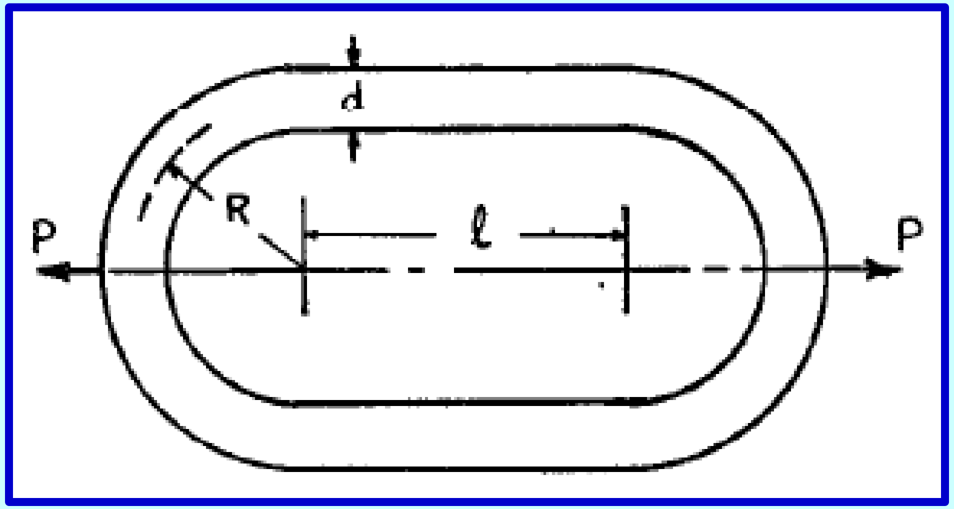


$$s = \frac{P}{\delta} = \frac{\frac{3\pi E d^4}{32}}{4l^3 + 6\pi R l^2 + 24R^2 l + 3\pi R^3}$$

(b) If *s* = 165 n/m, *d* = 6 mm, *R* = 36 mm, find *l*. *E* = 206,00N/mm^2.

Assignment Q2 (Ryder, Chapter 11, Problem 6, p.208):

(a) A chain link made of circular section has the dimensions shown. Prove that if d, the diameter of the section, is assumed small compared with R, then the maximum bending moment occurs at the point of application of the load and is equal to:



$$\frac{PR}{2}\left(\frac{l+2R}{l+\pi R}\right)$$

(b) If $R = 24$ mm, $d = 6$ mm, and $l = 42$ mm, calculate the ratio of the maximum tensile stress at the section where the load is applied to that at a section half way along the straight portion.

The End

Now for real!!!