MEC 3352

STRENGTH OF MATERIALS II

BEAMS ON ELASTIC FOUNDATIONS

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BEAMS ON ELASTIC FOUNDATIONS

- There are many problems in which a beam is supported on a compressible foundation exerting a distributed reaction on the beam.
- Reaction intensity is proportional to the compressibility.
- Sometimes foundations exerts upward forces only – therefore if a beam is sufficiently long, it might lose contact with foundation,
- In others, pressures may be exerted either way, i.e. upwards or downwards.
- The support may not be truly continuous (e.g. holding down rails on sleepers in railway lines) but can be replaced by an equivalent support.

Remembering that:

If $\boxed{y = Deflection}$

Then: $\boxed{\frac{dy}{dx} = Slope\ (of\ the\ deflection\ curve)}$

$$\boxed{EI\frac{d^2y}{dx^2} = M\ (Bending\ moment)}$$

$$\boxed{EI\frac{d^3y}{dx^3} = \frac{dM}{dx} = F\ (Shear\ force)}$$

$$\boxed{EI\frac{d^4y}{dx^4} = \frac{d^2M}{dx^2} = \frac{dF}{dx} = -w\ (Distributed\ load)}$$

If y is the upward deflection of the foundation at any point, the rate of upward reaction is $-ky$, and

$$\frac{EId^4y}{dx^4} = -ky$$

or

$$\frac{d^4y}{dx^4} = -\frac{1}{EI}ky = -\frac{4k}{4EI}y$$

$$\boxed{\frac{d^4y}{dx^4} = -4\alpha^4 y} \tag{1}$$

where: $$\boxed{\alpha^4 = \frac{k}{4EI}} \tag{2}$$

The general solution to Equation (1) can be written as:

$$\boldsymbol{y = A \sin \alpha x \cdot \sinh \alpha x + B \cos \alpha x \cdot \sinh \alpha x + C \sin \alpha x \cdot cosh\, \alpha x + D \cos \alpha x \cdot cosh\, \alpha x} \tag{3}$$

From this solution, various scenarios and their governing boundary conditions can now be considered.

(a) Long Beam Carrying a Central Load (Fig. 1(a))

Assuming that the foundation can exert upward forces only, let $2l$ be the length of beam in contact with the foundation, and take the origin *O* at the left hand end.

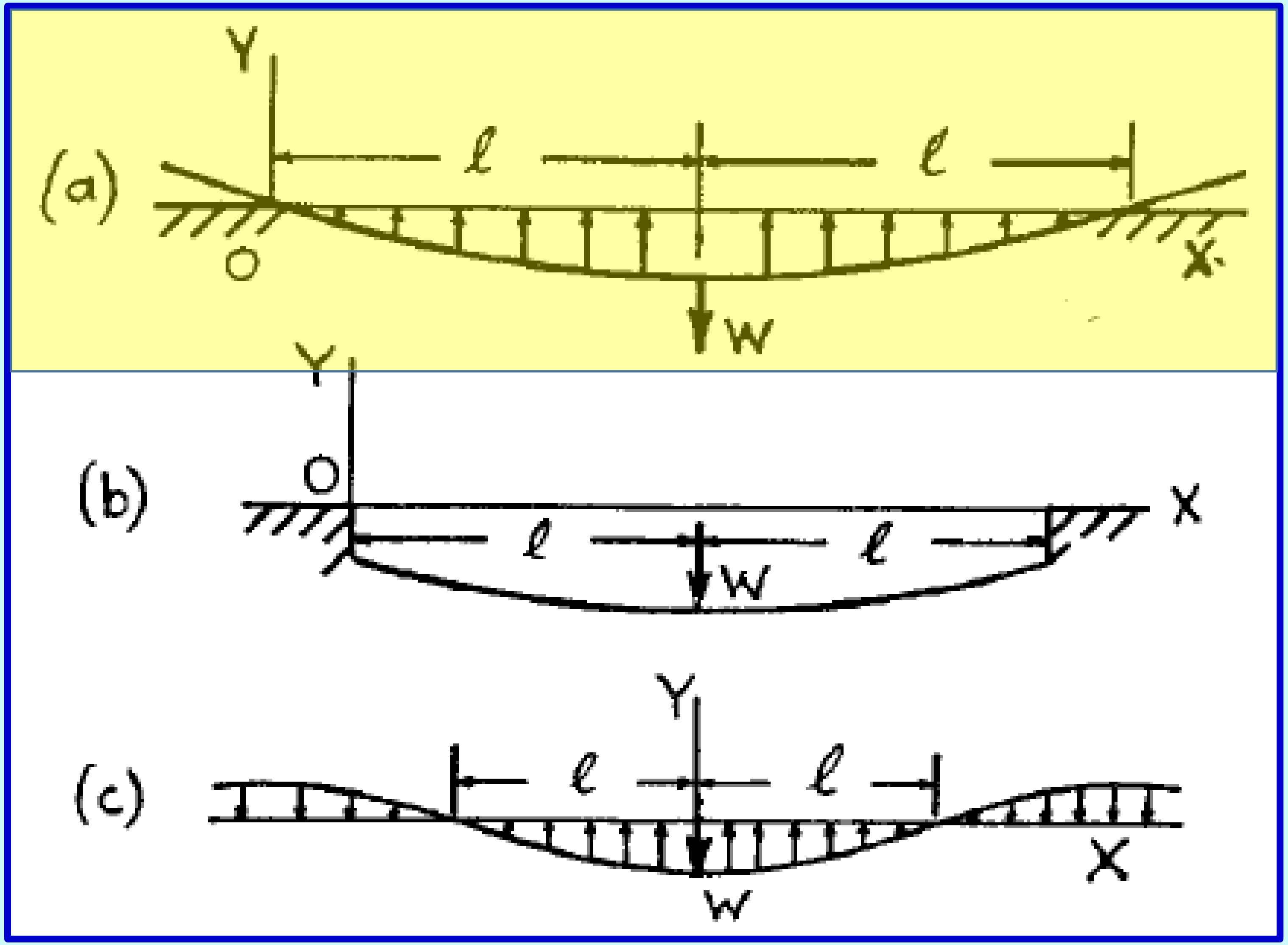
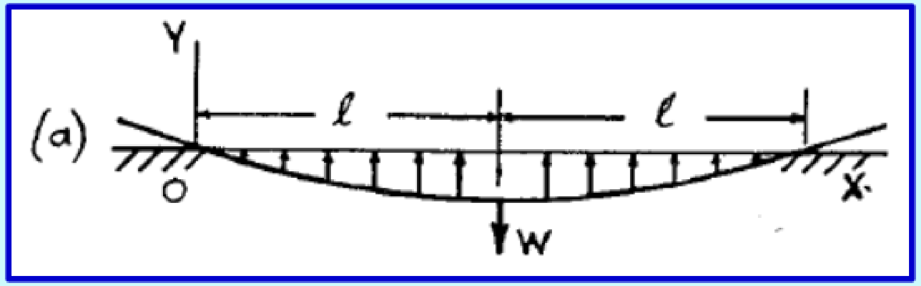


Figure 1

The solution to Equation (1) can be written as:

At $x = 0, \quad y = 0;$



$$0 = A \sin 0 \cdot \sinh 0 + B \cos 0 \cdot \sinh 0$$
$$+ C \sin 0 \cdot \cosh 0 + D \cos 0 \cdot \cosh 0$$

$$0 = A\,(0) \cdot (0) + B\,(1) \cdot (0) + C\,(0) \cdot (1) + D(1) \cdot (1) = D$$

Therefore, $D = 0$;

and $\boldsymbol{M = \dfrac{EId^2y}{dx^2} = 0;}$ Therefore, $\boldsymbol{A = 0}$ (4)

also $\boldsymbol{F = \dfrac{EId^3y}{dx^3} = 0}$ (5)

giving

$$EI \cdot 2\alpha^3 [B(-\cos 0 \cdot \cosh 0 - \sin 0 \cdot \sinh 0) + C(-\sin 0 \cdot \sinh 0 + \cos 0 \cdot \cosh 0)] = 0$$

$$EI \cdot 2\alpha^3 [B(-1 \cdot 1 - 0 \cdot 0) + C(-0 \cdot 0 + 1 \cdot 1)] = 0$$

$$-B + C = 0$$

i.e.: $C = B$

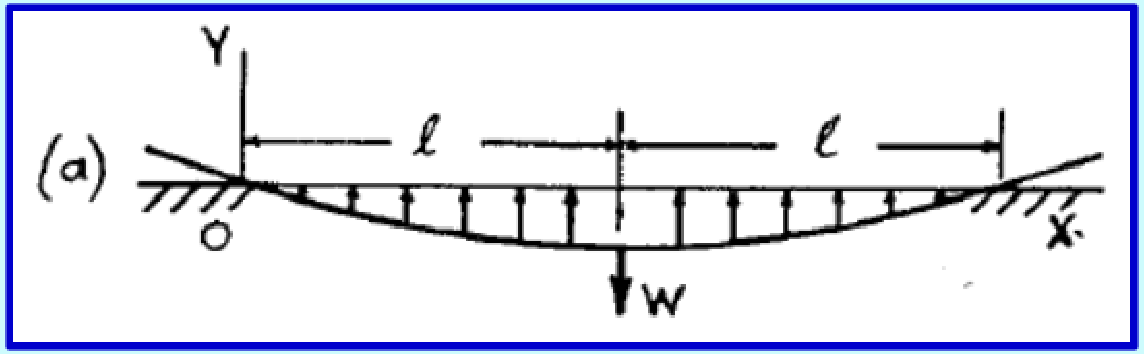
The solution to Equation (1) is now reduced to:

$$\boldsymbol{y = B(\cos \alpha x \cdot \sinh \alpha x + \sin \alpha x \cdot \cosh \alpha x)} \qquad (6)$$

At $x = l$, $\dfrac{dy}{dx} = 0$

Therefore:

$$B\alpha \cos \alpha l \cdot \cosh \alpha l = 0$$



The least solution of this is $\alpha l = \pi/2$ which determines the length in contact with the ground.

The value of the constant B is obtained from the condition that the shear force at the centre is $W/2$, since by symmetry it must be numerically the same on either side of the load and it must change by an amount W on passing through the load. Hence:

$$\frac{W}{2} = EI\frac{d^3y}{dx^3} = -EI \cdot 4\alpha^3 B \sin \alpha l \ \cdot \sinh \alpha l$$

$$B = -\frac{W\alpha}{2k}\sinh\frac{\pi}{2}$$

The maximum deflection and bending moment at the centres, $\alpha x = \pi/2$,

$$\boldsymbol{\hat{y} = -\left(\frac{W\alpha}{2k}\coth\frac{1}{2}\pi\right)} \tag{7}$$

$$\hat{M} = EI\left(\frac{W\alpha^3}{k}\right)\coth\frac{1}{2}\pi$$

From (2), $k = 4EI\alpha^4$ and substituting for it above:

We have:

$$\boldsymbol{\hat{M} = \left(\frac{W}{4\alpha}\right) coth\frac{1}{2}\pi} \tag{8}$$

(b) Short Beam Carrying Central Load W (Fig. 1(b))

If $\alpha l < \pi/2$ in case (a), the beam will sink below the unstressed level of the foundation at all points.

Again taking the origin at the left-hand end and the overall length of beam as $2l$, the following are obtained for the constants of integration of the general solution of the previous paragraph.

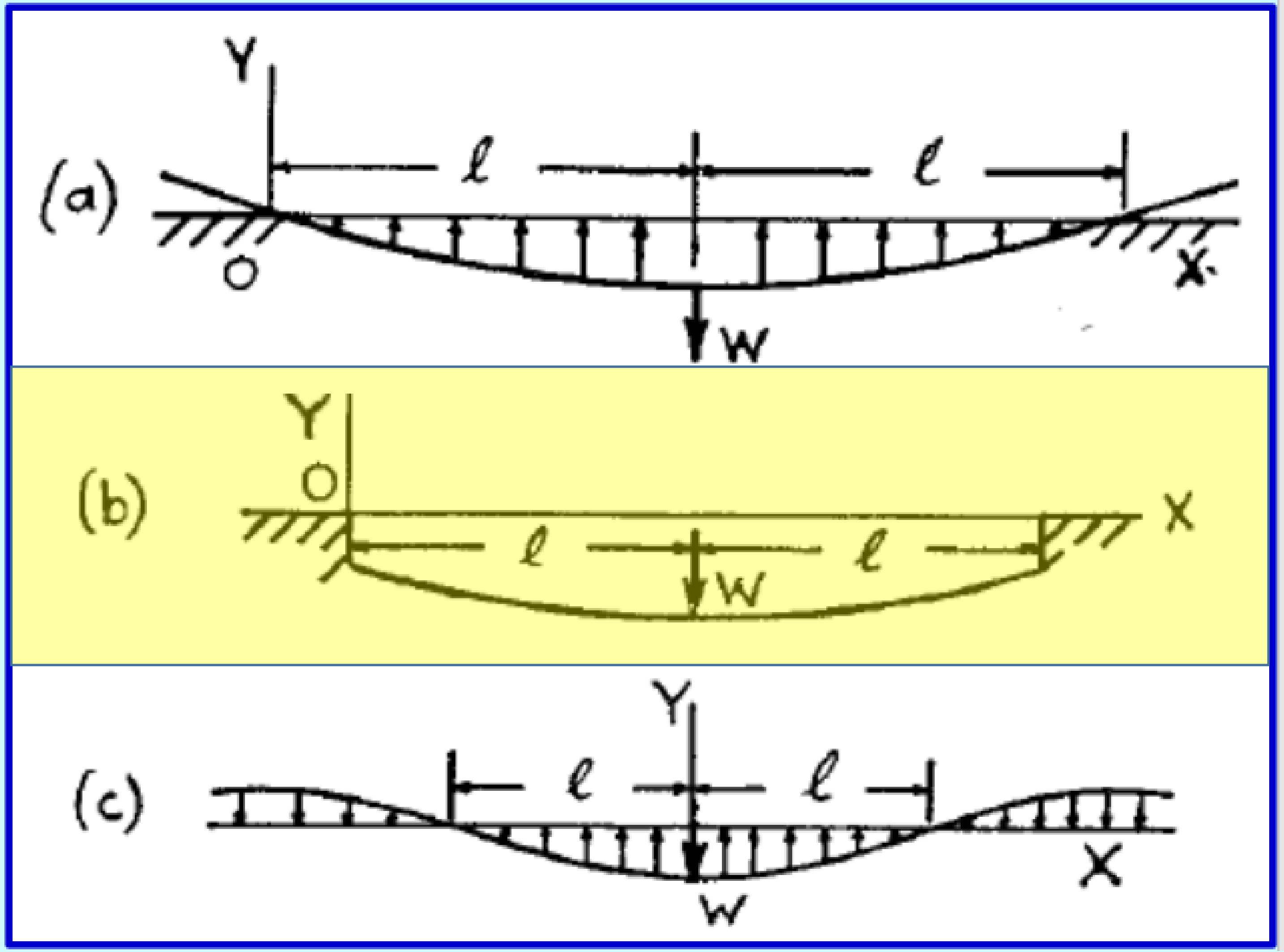
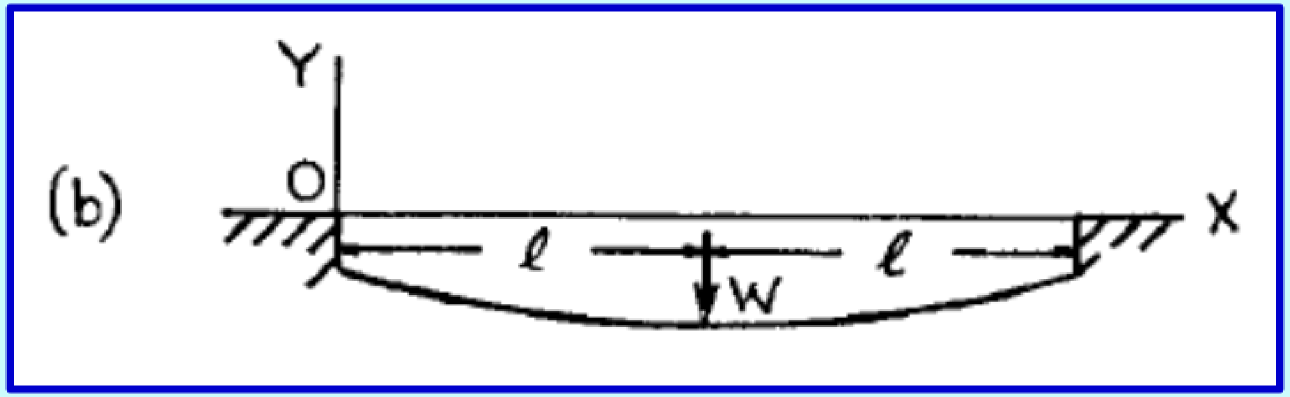


Figure 1

At $x = 0$, $\frac{d^2y}{dx^2} = 0.$



therefore: $A = 0$

and $\frac{d^3y}{dx^3} = 0.$

therefore: $B = C$

and

$$y = B\,(\cos\alpha x\cdot \sinh\alpha x + \sin\alpha x\cdot \cosh\alpha x) + D\cos\alpha x\cdot \cosh\alpha x \qquad (9)$$

At $x = l$, $\frac{dy}{dx} = 0$,

giving

$$B \cdot 2\cos\alpha l \cdot \cosh\alpha l + D(-\sin\alpha l \cdot \cosh\alpha l + \cos\alpha l \cdot \sinh\alpha l) = 0$$

and $EI\dfrac{d^3y}{dx^3} = \dfrac{W}{2}$

giving

$$-B \cdot 2\sin\alpha l \cdot \sinh\alpha l - D(\sin\alpha l \cdot \cosh\alpha l + \cos\alpha l \cdot \sinh\alpha l)$$

$$= \frac{W}{4EI\alpha^3} = \frac{W\alpha}{k}$$

Solving for B and D gives:

$$\boldsymbol{B = -\frac{W\alpha}{k} \cdot \frac{\sin \alpha l \cdot \cosh \alpha l + \cos \alpha l \cdot \sinh \alpha l}{\sin 2\alpha l + \sinh 2\alpha l}} \tag{10}$$

$$\boldsymbol{D = -\frac{2W\alpha}{k} \cdot \frac{\cos \alpha l \cdot \cosh \alpha l}{\sin 2\alpha l + \sinh 2\alpha l}} \tag{11}$$

The complete solution for y is now known, the maximum deflection and bending moment being under the load.

(c) Infinite Beam Carrying Load W (Fig. 1(c))

- Assuming that the support can exert pressure either upwards or downwards;
- taking the Y axis through the load and the X axis at the un-deformed level;
- a solution of Eq. (1) can be written in the form:

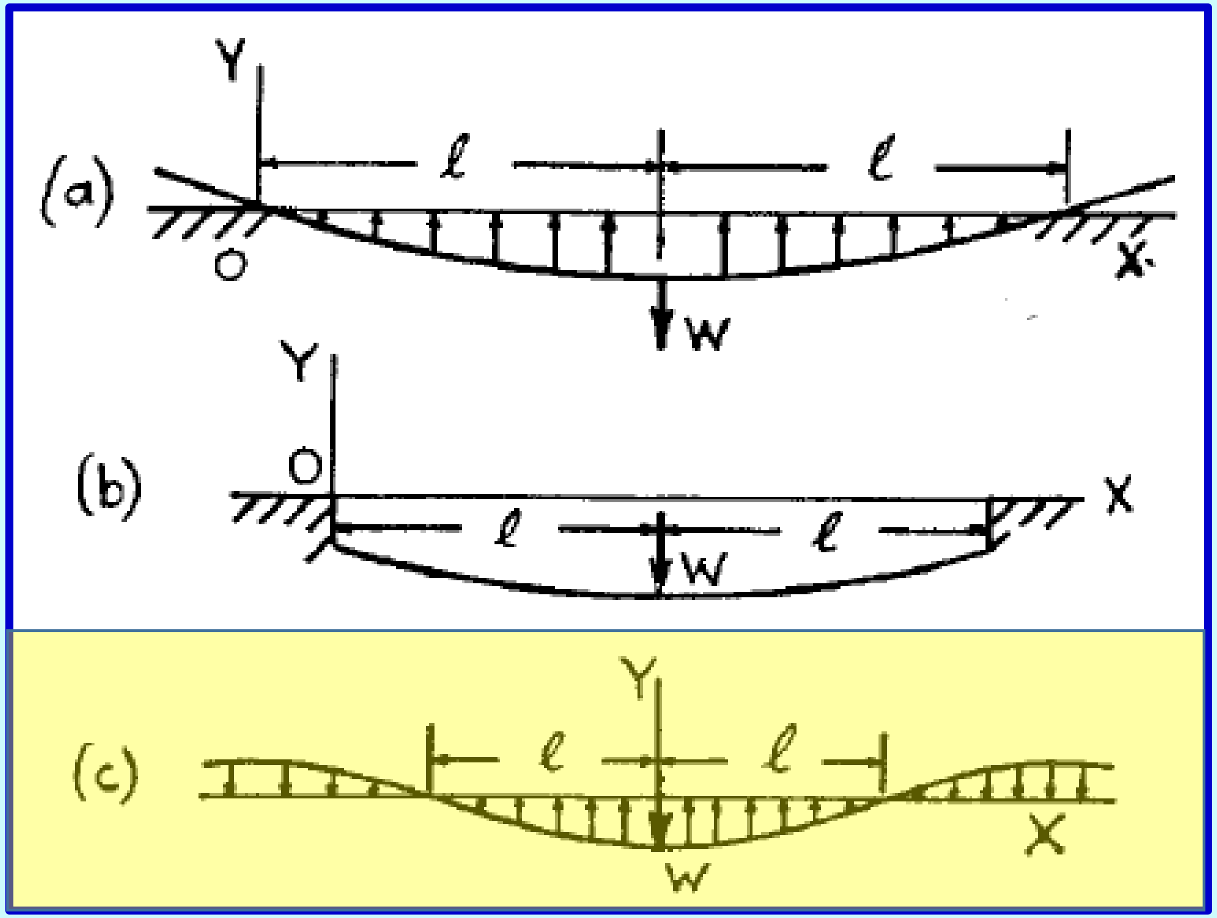
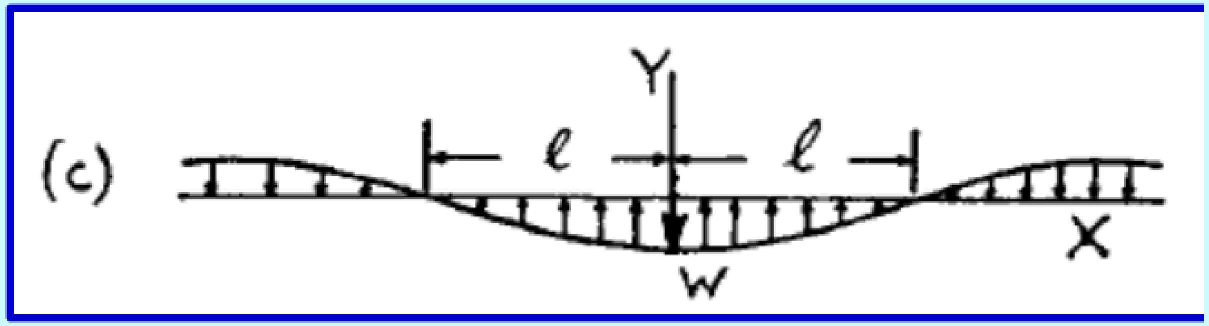


Figure 1

$$\boldsymbol{y = e^{\alpha x}(A\sin\alpha x + B\cos\alpha x) + e^{-\alpha x}(C\sin\alpha x + D\cos\alpha x)} \qquad (12)$$

For the length to the right of W, since as $x \to \infty$, $y \to 0$, then $A = B = 0$

At $x = 0$, $\quad \frac{dy}{dx} = 0$,



Therefore $\quad C = D$

and $$\boldsymbol{EI\frac{d^3y}{dx^3} = -\frac{W}{2}} \tag{13}$$

Giving $$C = D = -\frac{W}{8\alpha^3 EI} = -\frac{W\alpha}{2k}$$

and $$\boldsymbol{y = -\left(\frac{W\alpha}{2k}\right)e^{-\alpha x}(\sin\alpha x + \cos\alpha x)} \tag{14}$$

The distance from the load at which $y = 0$ is given by

$$\sin \alpha l + \cos \alpha l = 0$$

The least solution being $\boldsymbol{\alpha l = \frac{3\pi}{4}}$ (15)

The maximum deflection and bending moment at the centres, $x = 0$,

$$\boldsymbol{\hat{y} = -\frac{W\alpha}{2k}} \tag{16}$$

$$\boldsymbol{\hat{M} = EI\frac{W\alpha^3}{k} = \frac{W}{4\alpha}} \tag{17}$$

Example:

A steel railway track is supported on timber sleepers which exert an equivalent load of 2 800 N/m length of rail per mm deflection from its unloaded position. For each rail $I = 12 \times 10^6$ mm^4, section modulus $Z = 16 \times 10^4$ mm^3 and $E =$ 205 000 MPa. if a point load of 100 kN acts on each rail, find the length of rail over which the sleepers are depressed and the maximum bending stress in the rail.

Solution:

Given:

Equivalent load, k = 2 800 N/m per mm deflection

$I = 12 \times 10^6$ mm^4

$Z = 16 \times 10^4$ mm^3

$E = 205\ 000$ N/mm^2

$W = 100$ kN

Required:

- Length of rail over which the sleepers are depressed
- Maximum bending stress in the rail

Solving the problem:

$$\alpha^4 = \frac{k}{4EI} = \frac{2\,800}{4 \times 10^3 \times 205\,000 \times 12 \times 10^6}$$

giving

$$\boldsymbol{\alpha = 0.731 \times 10^{-3}/\text{mm.}}$$

Each rail can be treated as an infinitely long beam, for which the length over which downward deflection occurs is given by scenario (c).

$$2l = \frac{3\pi}{2\alpha} = \frac{3\pi \times 10^3}{2 \times 0.731}$$

$$\mathbf{Length\ 2\boldsymbol{l} = 6\,440\ mm = 6.44\ m}$$

and

$$\widehat{M} = \frac{W}{4\alpha} = \frac{100 \times 10^3}{4 \times 0.731} = 34\,200\ \mathrm{Nm}$$

$$\sigma = \frac{\widehat{M}}{Z} = \frac{34\,200}{16 \times 10^4 \times 10^{-9}} = 213.75 \times 10^6\ \ \mathrm{N/m^2}$$

$$\boldsymbol{\sigma} = \mathbf{214\ N/mm^2}$$

The End

But wait!!!!

Assignment

1. A beam rests on three supports A, B and C. A and C are rigid, but B compresses 0.000 5 mm per kg of load carried. If $AB = BC = 4.5$ m, what is the deflection at B when the beam is loaded with 16 kN/m run? What is the maximum bending moment and where does it occur? $E = 204$ GPa; $I = 9\ 350$ cm^4.

   (Ans.: 4.2 mm; 28.5 kNm, 1.85 m.)

2. A timber beam 15 cm wide and 10 cm deep, rests on compressible ground which exerts an upward pressure of 7 000 N/m^2 per mm compression. It supports a load of 1 000 kg at its mid-point. Compute the maximum bending stresses when the beam is (a) 1.8 m long, (b) 3 m long. $E = 10$ GPa.

   (Ans.: $\alpha = 0.0012\ mm^{-1}$, (a) 12.6 MPa; (b) 9.3 MPa.)

The End