Fluid Mechanics CEE 3311

LECTURE 10

Conservation of energy

Simplified forms of the energy equation

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Simplified forms of the energy equation

Energy equation for steady, one dimensional incompressible flow in a pipe

Consider flow through the pipe system shown below.

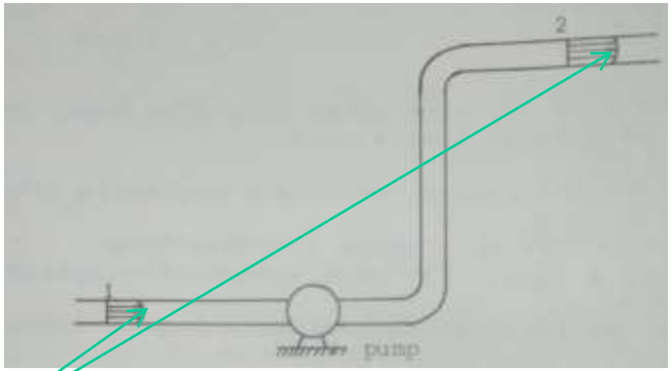

Fig 11.1 Pipe system

$$\frac{dQ}{dt} - \frac{dW_s}{dt} = \frac{d}{dt}\int_{cv} \left(\frac{v^2}{2} + gz + u\right)\rho \, dV + \int_{cs} \left(\frac{v^2}{2} + gz + u + \frac{p}{\rho}\right)\rho \, (\vec{v}\, d\vec{A}) \quad 10.1$$

For a steady flow situation in which there is one entrance and one exit across which **uniform** profiles can be assumed

$$\frac{dQ}{dt} - \frac{dW_s}{dt} = 0 + \int_{cs} \left(\frac{v^2}{2} + gz + u + \frac{p}{\rho}\right)\rho \, (\vec{v}\, d\vec{A}) \quad 10.2$$

$$\frac{dQ}{dt} - \frac{dW_s}{dt} = \int_{A_1} \left(\frac{v_1^2}{2} + gz_1 + u_{1+} + \frac{p_1}{\rho_1}\right)\rho_1(-v_1 \, dA_1) + \int_{A_2} \left(\frac{v_2^2}{2} + gz_2 + u_2 + \frac{p_2}{\rho_2}\right)\rho_2(v_2 \, dA_2) \quad 10.3$$

$$\frac{dQ}{dt} - \frac{dW_s}{dt} = \int_{A_1} \left(\frac{v_1^2}{2} + gz_1 + u_{1+} + \frac{p_1}{\rho_1} \right) \rho_1 (-v_1\, dA_1) + \int_{A_2} \left(\frac{v_2^2}{2} + gz_2 + u_2 + \frac{p_2}{\rho_2} \right) \rho_2 (v_2\, dA_2) \quad 10.3$$

For a flow the term $\left(\frac{v^2}{2} + gz + \frac{p}{\rho} \right)$ in eq is constant across the cross section because v is constant (we assume a uniform velocity profile) and the sum of $\frac{p}{\rho} + gz$ is constant if the streamlines at each section are parallel. Therefore, we take the term outside the integral and separating the velocity term integral

$$\frac{dQ}{dt} - \frac{dW_s}{dt} + \left(\frac{p_1}{\rho_1} + gz_1 + u_1 \right) \int_{A_1} \rho_1\, v_1\, dA_1 + \int_{A_1} \frac{\rho_1 v_1^3}{2} dA_1 = \left(\frac{p_2}{\rho_2} + gz_2 + u_2 \right) \int_{A_2} \rho_2\, v_2\, dA_2 + \int_{A_2} \frac{\rho_2 v_2^3}{2} dA_2 \quad 10.4$$

See Lecture 8, Slide 2

It can be seen that $\int \rho \; v \; dA = \rho \bar{v} A = \frac{dm}{dt}$ = mass rate of flow. Where $\bar{v}$ is the mean or average velocity over a cross section.

However, $\frac{dm}{dt}$ does not appear as a factor of $\int \frac{\rho \; v^3}{2} dA$ in

10.4 ..4; so it is common to express $\int \frac{\rho \; v^3}{2} dA$ as

$\alpha\, \rho \left(\frac{\bar{v}^3}{2} \right) A$.equation 11.4 now becomes

$$\int \frac{\rho v^3}{2} dA = \alpha \rho \frac{\bar{v}^3}{2} A = \alpha \frac{\bar{v}^2}{2} \rho \bar{v} A = \alpha \frac{\bar{v}^2}{2} \frac{dm}{dt}$$

$\alpha = \frac{1}{A} \int \left(\frac{v}{\bar{v}} \right)^3 dA$ = kinetic energy correction factor

$\alpha = 1$ for uniform flow ($v = \bar{v}$)

For most cases of turbulent flow, $\alpha \approx 1.05$. because this is very close to unity, it is common practice in engineering applications to let $\alpha = 1$.

$$\frac{dQ}{dt} - \frac{dW_s}{dt} + \left(\frac{p_1}{\rho_1} + gz_1 + u_1 \right) \frac{dm}{dt} + \propto_1 \frac{\bar{v}_1^2}{2} \frac{dm}{dt} = \left(\frac{p_2}{\rho_2} + gz_2 + u_2 \right) \frac{dm}{dt} + \propto_2 \frac{\bar{v}_2^2}{2} \frac{dm}{dt} \quad 10.5$$

$$\frac{dQ}{dt}-\frac{dW_s}{dt}+\left(\frac{p_1}{\rho_1}+gz_1+u_1\right)\frac{dm}{dt}+\propto_1\frac{\bar{v}_1^2}{2}\frac{dm}{dt}=\left(\frac{p_2}{\rho_2}+gz_2+u_2\right)\frac{dm}{dt}+\propto_2\frac{\bar{v}_2^2}{2}\frac{dm}{dt}$$

$$\frac{dQ}{dt}-\frac{dW_s}{dt}+\left(\frac{p_1}{\rho_1}+gz_1+u_1+\propto_1\frac{\bar{v}_1^2}{2}\right)\frac{dm}{dt}=\left(\frac{p_2}{\rho_2}+gz_2+u_2+\propto_2\frac{\bar{v}_2^2}{2}\right)\frac{dm}{dt} \qquad 10.6$$

Dividing by $\frac{dm}{dt}$

$$\frac{1}{\frac{dm}{dt}}\left(\frac{dQ}{dt}-\frac{dW_s}{dt}\right)+\frac{p_1}{\rho_1}+gz_1+u_1+\propto_1\frac{\bar{v}_1^2}{2}=\frac{p_2}{\rho_2}+gz_2+u_2+\propto_2\frac{\bar{v}_2^2}{2} \qquad 10.7$$

The shaft work term is usually the result of a turbine or pump in the flow system. It is therefore convenient to represent the shaft work term as

$$W_s=W_t-W_p \qquad 10.8$$

Lecture 9 slide 11

In the latter case, the fluid system is doing negative work on its surrounding.

where W_t =power delivered by a turbine

W_p = power supplied by a pump

Substituting eq 10.8 into eq 10.7 and divided by g results in

$$\frac{\frac{dW_p}{dt}}{g\frac{dm}{dt}}+\frac{p_1}{\rho g}+z_1+\propto_1\frac{v_1^2}{2g}=\frac{\frac{dW_t}{dt}}{g\frac{dm}{dt}}+\frac{p_2}{\rho g}+z_2+\propto_2\frac{v_2^2}{2g}+\left\{\frac{1}{g}(u_2-u_1)-\frac{\frac{dQ}{dt}}{g\frac{dm}{dt}}\right\} \qquad 10.9$$

$$\frac{\frac{dW_p}{dt}}{g\frac{dm}{dt}}+\frac{p_1}{\rho g}+z_1+\propto_1\frac{v_1^2}{2g}=\frac{\frac{dW_t}{dt}}{g\frac{dm}{dt}}+\frac{p_2}{\rho g}+z_2+\propto_2\frac{v_2^2}{2g}+\left\{\frac{1}{g}(u_2-u_1)-\frac{\frac{dQ}{dt}}{g\frac{dm}{dt}}\right\} \qquad 10.9$$

Replacing g(dm/dt) by dw/dt

$$\frac{\frac{dW_p}{dt}}{\frac{dw}{dt}}+\frac{p_1}{\rho g}+z_1+\propto_1\frac{v_1^2}{2g}=\frac{\frac{dW_t}{dt}}{\frac{dw}{dt}}+\frac{p_2}{\rho g}+z_2+\propto_2\frac{v_2^2}{2g}+\left\{\frac{1}{g}(u_2-u_1)-\frac{\frac{dQ}{dt}}{\frac{dw}{dt}}\right\} \qquad 10.10$$

Where $\frac{dw}{dt}=g\frac{dm}{dt}$ weight flow rate (from w = gm)

- $\frac{\frac{dW_p}{dt}}{\frac{dw}{dt}} + \frac{p_1}{\rho g} + z_1 + \propto_1 \frac{v_1^2}{2g} = \frac{\frac{dW_t}{dt}}{\frac{dw}{dt}} + \frac{p_2}{\rho g} + z_2 + \propto_2 \frac{v_2^2}{2g} + \left\{\frac{1}{g}(u_2 - u_1) - \frac{\frac{dQ}{dt}}{\frac{dw}{dt}}\right\}$ 10.10
- All of the terms of eq11.10 have one dimension:
- $\frac{\text{energy}}{\text{unit weight of fluid}} = \frac{[\text{FORCE}][\text{LENGTH}]}{[\text{FORCE}]} = [\text{LENGTH}]$

Hence $\frac{\frac{dW_p}{dt}}{\frac{dw}{dt}}$ may be designated as h_p (**pump head**) and similarly $\frac{\frac{dW_t}{dt}}{\frac{dw}{dt}}$ as h_t (**turbine head**).

The term $\left\{\frac{1}{g}(u_2 - u_1) - \frac{\frac{dQ}{dt}}{\frac{dw}{dt}}\right\}$ represents a loss of mechanical energy due to viscous stresses, which is usually lumped together in a single term called head loss and symbolised by h_l. Thus eq 10.10 becomes

$$\frac{p_1}{\rho g} + z_1 + \propto_1 \frac{v_1^2}{2g} + h_p = \frac{p_2}{\rho g} + z_2 + \propto_2 \frac{v_2^2}{2g} + h_t + h_l \qquad 10.11$$

This is the steady flow energy equation.

$$\propto_1 \frac{v_1^2}{2g} + \frac{p_1}{\rho g} + z_1 + h_p = \propto_2 \frac{v_2^2}{2g} + \frac{p_2}{\rho g} + z_2 + h_t + h_l$$

- The sum of the terms on the left hand side of eq represents the total energy, stated in energy per unit weight of flowing liquid, plus the *energy supplied* by a pump.
- The sum of the terms on the right hand side represents the total energy per unit weight at the downstream section plus the *energy given up* to a turbine and energy lost to friction between the two sections

Water flows from a reservoir through a 0.8-m-diameter pipeline to a turbine-generator unit and exits to a river that is 30 m below the reservoir surface (Fig. E4.7). If the flow rate is

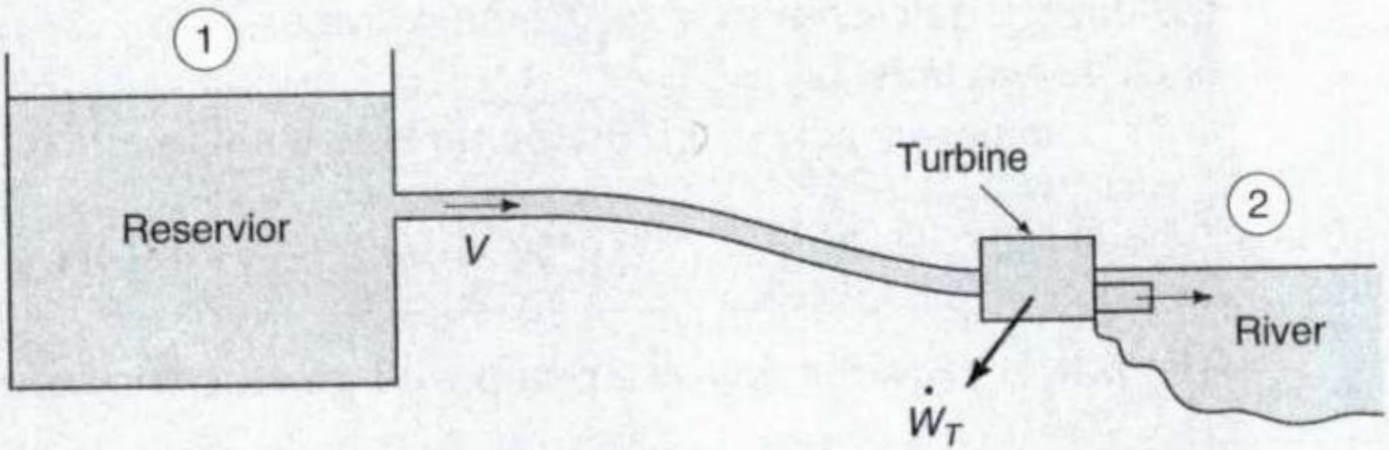

Figure E4.7

3 m^3/s and the turbine-generator efficiency is 80%, calculate the power output. Assume the loss coefficient in the pipeline (including the exit) to be $K = 2$.

Solution: The control volume to be used extends from section 1 to section 2; we consider the water surface of the left reservoir to be the entrance and the water surface of the river to be the exit. Because we assume the water surfaces to be large, the velocities at the surfaces are negligible. The velocity in the pipe is

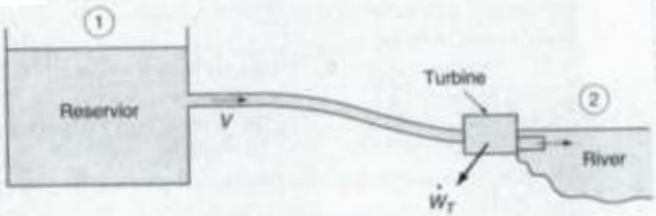

Figure E4.7

$$V = \frac{Q}{A}$$

$$= \frac{3}{\pi \times 0.8^2/4} = 5.968 \text{ m/s}$$

Now, consider the energy equation. We will use gage pressures so that $p_1 = p_2 = 0$; the datum is placed through the lower section 2 so that $z_2 = 0$; the velocities V_1 and V_2 are negligibly small; K is assumed to be based on the 0.8-m-diameter pipe velocity. The energy equation (4.4.24) then becomes

$$\cancelto{0}{H_P} + \cancelto{0}{\frac{V_1^2}{2g}} + \cancelto{0}{\frac{p_1}{\gamma}} + z_1 = H_T + \cancelto{0}{\frac{V_2^2}{2g}} + \cancelto{0}{\frac{p_2}{\gamma}} + \cancelto{0}{z_2} + K\frac{V^2}{2g}$$

$$30 = H_T + 2\frac{5.968^2}{2 \times 9.81}$$

$$\therefore H_T = 26.4 \text{ m}$$

From this the power output is found, using Eq. 4.4.25, to be

$$\dot{W}_T = Q\gamma H_T \eta_T$$

$$= 3 \times 9810 \times 26.4 \times 0.8 = 622\,000 \text{ W} \quad \text{or} \quad 622 \text{ kW}$$

In this example we have used gage pressure; the potential-energy datum was assumed to be placed through section 2, V_1 and V_2 were assumed to be insignificantly small, and K was assumed to be based on the 0.8-m-diameter pipe velocity.

Energy equation for non-viscous steady, one dimensional incompressible flow in a pipe

- If :
- the losses are negligible
- there is no shaft work
- the flow is incompressible

The energy equation becomes $\propto_1 \frac{v_1^2}{2g} + \frac{p_1}{\rho g} + z_1 + h_p = \propto_2 \frac{v_2^2}{2g} + \frac{p_2}{\rho g} + z_2 + h_t + h_l$

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho g} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho g} + z_2 \qquad 10.12$$

The energy equation has been reduced to a form identical with the Bernoulli equation. Assumptions are similar in both equations.