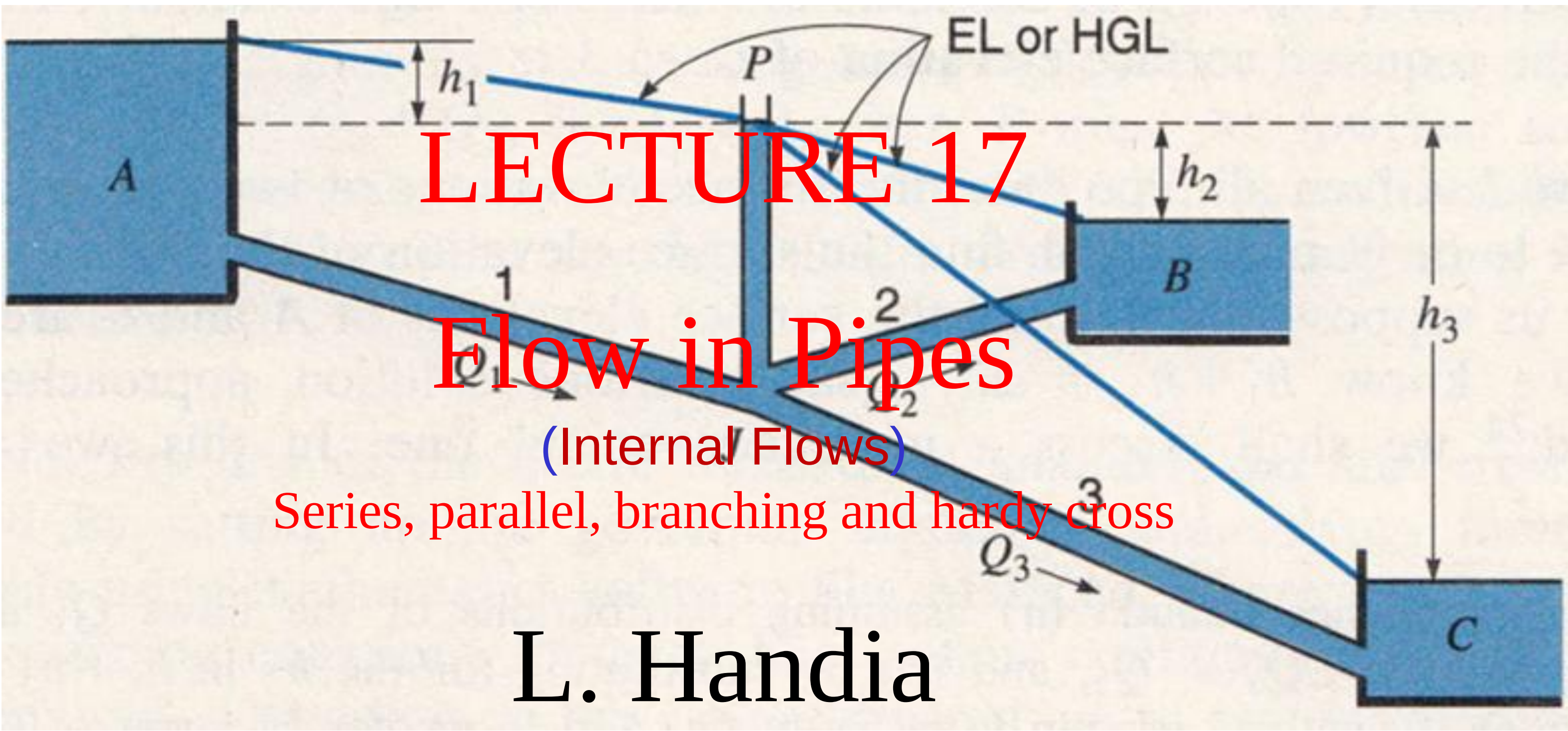
Fluid Mechanics CEE 3311



LECTURE 17

Flow in Pipes

(Internal Flows)

Series, parallel, branching and hardy cross

L. Handia

Branching Pipes

- For convenience, let us consider three pipes connected to three reservoirs as in Fig. 8.27 and connected together or branching at the common junction point J.
- Actually, any of the pipes may be considered to be connected to *some other destination* than a reservoir by simply replacing the reservoir with a *piezometer* tube in which the water level is the same as the reservoir surface.
- We shall suppose that all the pipes are *sufficiently long* (large head loss in the pipes) that *minor losses and velocity heads may be neglected*.

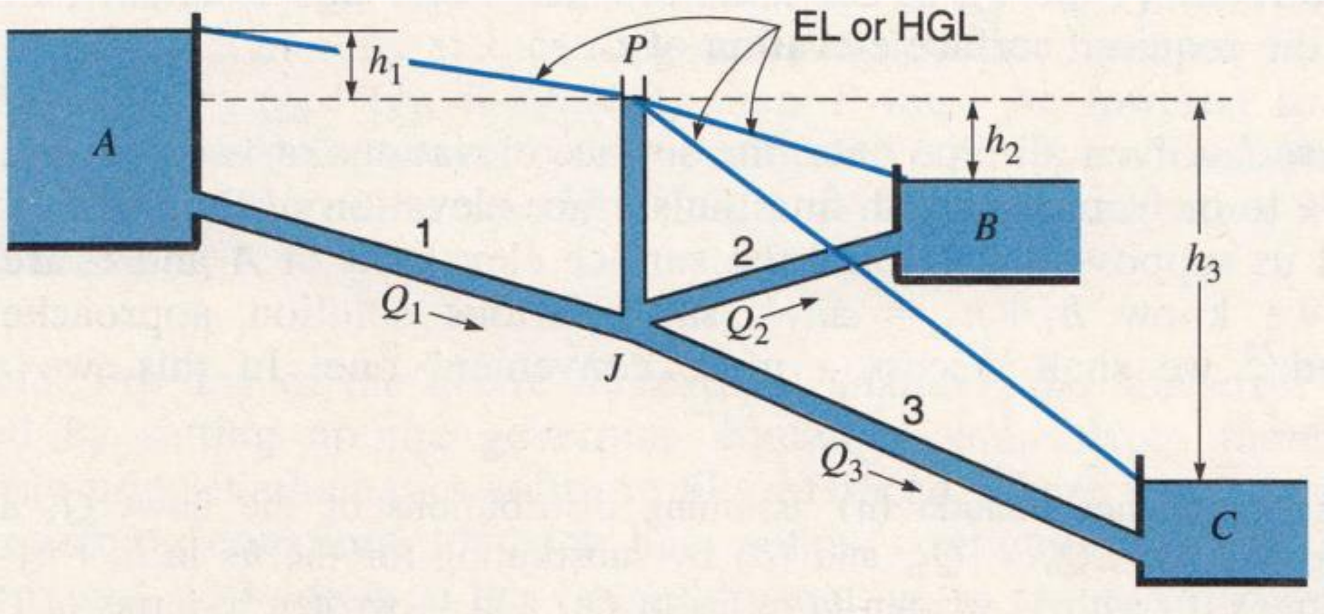


Figure 8.27
Branching pipes.

Branching Pipes

- We name the pipes and flows and corresponding friction losses as shown in the diagram.
- The *continuity and energy equations* require that the flow *entering* the junction (J) *equal* the flow *leaving it* and that the *pressure head at J* (which may be represented schematically by the open piezometer tube shown, with water at elevation P) be *common to all pipes*.

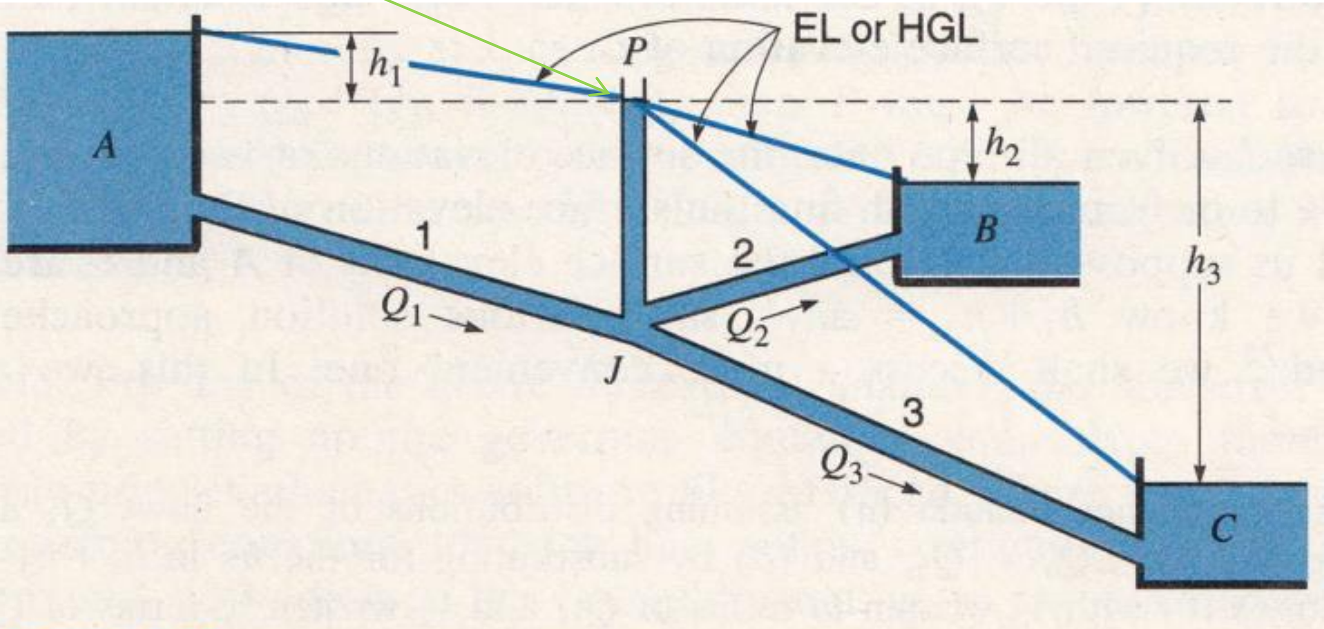


Figure 8.27
Branching pipes.

Branching Pipes

- There being no pumps, the elevation of P must lie between the surfaces of reservoirs A and C.

Since h_L is zero, there is no flow

- If P is level with the surface of reservoir B then h_2 and Q_2 are both zero. If P is above the surface of reservoir B then water must flow into B and $Q_1 = Q_2 + Q_3$.
- If P is below the surface of reservoir B then the flow must be out of B and $Q_1 + Q_2 = Q_3$.
- So for the situation shown in Fig. 8.27 we have the following governing conditions:
  1. $Q_1 = Q_2 + Q_3$
  2. Elevation of P is common to all.
- The diagram suggests several different problems or cases, three of which will be discussed below using different methods of solution.

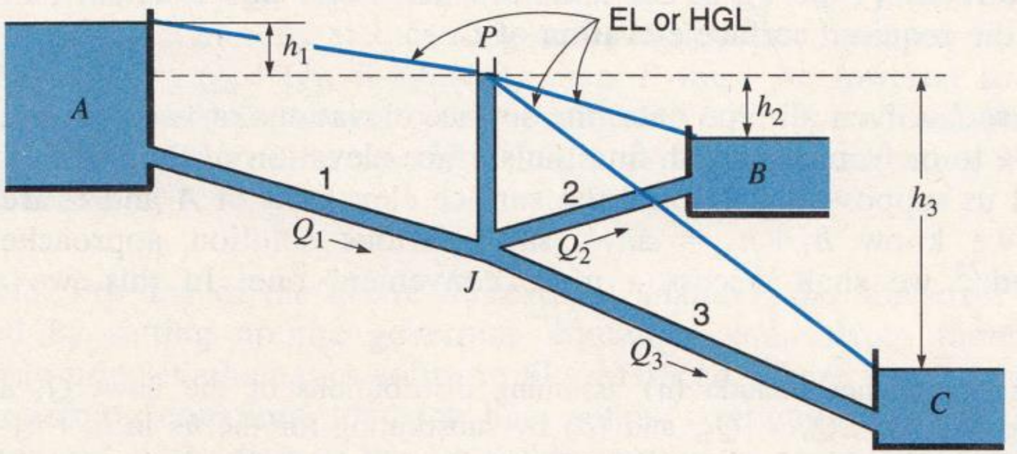


Figure 8.27
Branching pipes.

Rigorous Solutions

- When we know the pipe wall material, we can estimate its e value (from Tables) and we know that the friction factor f varies with the e/D of the pipe and the Reynolds number of the flow.
- Because we are not considering minor losses, we can use the equations in the previous lecture 16.
- In particular, using only a basic scientific calculator, we can solve pipelines for h_L (Type 1 problems), for V or Q (Type 2 problems) and more rarely, we can solve for D (Type 3 problems) using a number of equations (see Lecture 16).
- These equations are preferred because they avoid trial and error, which can become quite confusing when combined with other trial-and-error techniques needed to solve for branching flow.
- These, and the variety of approaches used to "manually" solve the different types of problems that can occur, are illustrated in the following cases.

Rigorous Solutions

Case 1. Given all pipe data (lengths, diameters, and materials for e values), the surface elevations of two reservoirs, and the flow to or from one of these two, ***find the surface elevation of the third reservoir***.

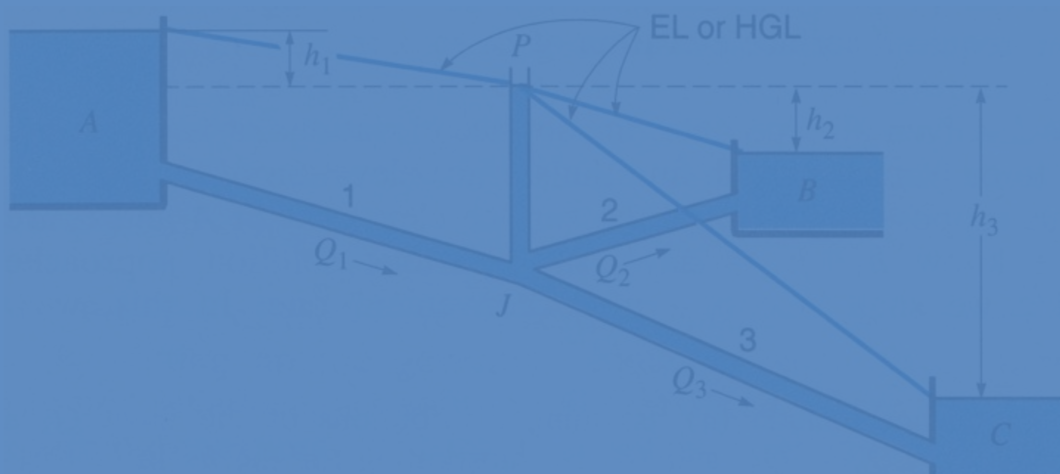
- This problem can be solved directly. Suppose that Q_1 and the elevations of A and B a given.

Turbulent flow, all pipes: $$\frac{1}{\sqrt{f}} = -1.8 \log\left[\left(\frac{e/D}{3.7}\right)^{1.11} + \frac{6.9}{R}\right]$$

- Then the head loss h_L may be determined directly (Type 1), using the moody diagram or Eq. (8.41) to find the proper value of f.

Turbulent flow, Type 2: $$\frac{4Q}{\pi D^2} = -2\sqrt{\frac{2gDh_L}{L}}\log\left(\frac{e/D}{3.7} + \frac{2.51\nu}{D}\sqrt{\frac{L}{2gDh_L}}\right) \quad (8.42b)$$

- Knowing h_L fixes P, so h_3 is now easily obtained. Knowing h_2 enables the flow in pipe 2 to be determined directly using the Type 2 equation (8.42).
- Condition 1 (continuity at junction J) then determines Q_3, which in turn enables h_3 to be found directly (Type 1), in the same manner as for line 1.
- Finally, P and h_3 define the required surface elevation of C.



Rigorous Solutions

Case 2. Given all pipe data, the surface elevations of two reservoirs, and the flow to or from the third, ***find the surface elevation of the third reservoir***.

- Let us suppose that Q_3 and the surface elevations of A and C are given.
- Then we know $h_l + h_3 = \Delta h_{13}$, say.
- Various solution approaches may be used; we shall discuss a more convenient one.
- In this, we ***assume the elevation of P***, which yields values for h_1 and h_3, and so Q_1 and Q_3 via Eq. (8.42).

Turbulent flow, Type 2: $$V = -2\sqrt{\frac{2gDh_L}{L}}\log\left(\frac{e/D}{3.7} + \frac{2.51\nu}{D}\sqrt{\frac{L}{2gDh_L}}\right) \quad (8.42a)$$

- If these do not satisfy the discharge relation at J ($\sum Q=0$)then P must be adjusted until they do.

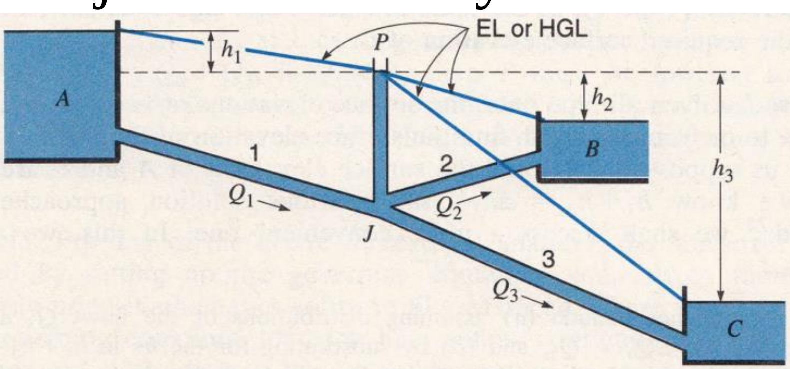


Figure 8.27
Branching pipes.

Rigorous Solutions

- To help us converge on the correct elevation of P, we can plot the results of each assumption on a graph like Fig. 8.28.
- For $\sum Q$ at J, inflows to J are taken as positive and outflows as negative.
- Two or three points, with one fairly close to the vertical axis, determine a curve that ***intersects that axis at the equilibrium level of P, where $\sum Q = 0$, as required***.
- Last, h_2 can be determined from Q_2 and Eqs. (8.41) and (8.10), and the required surface elevation of B found.

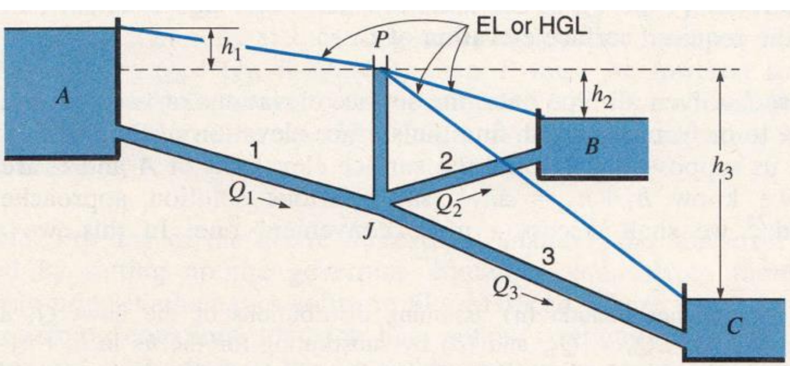


Figure 8.27
Branching pipes.

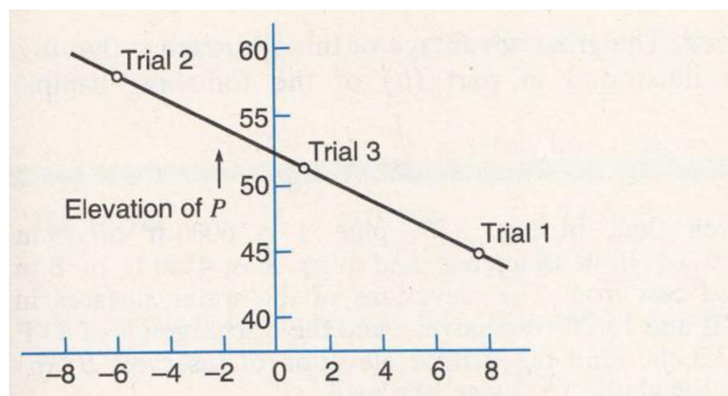


Figure 8.28

Rigorous Solutions

Case 3. Given all pipe lengths and diameters and the elevations of all three reservoirs, ***find the flow in each pipe***.

This is the classic three-reservoir problem, and it differs from the foregoing case that it is not immediately evident whether the flow is into or out of reservoir B. T direction is readily determined by first assuming no flow in pipe 2; that is, the piezometer level P is assumed at the elevation of the surface of B. The head losses h_1 and h_3 then determine the flows Q_1 and Q_3 via Eq. (8.42). If $Q_1 > Q_3$ then P must be raised to satisfy continuity at J, causing water to flow into reservoir B, and we shall have $Q_1 = Q_2 + Q_3$; if $Q_1 < Q_3$ then P must be lowered to satisfy continuity at J, causing water to flow out of reservoir B, and we shall have $Q_1 + Q_2 = Q_3$. From here on the solution proceeds by adjusting P as for Case 2 above.

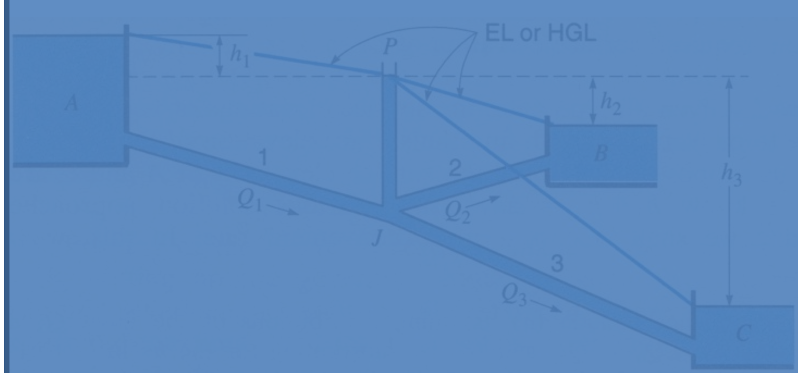


Figure 8.27
Branching pipes.
Branching pipes.

Case 2

SAMPLE PROBLEM 8.12 Given that, in Fig. 8.27, pipe 1 is 6000 ft of 15 in diameter, pipe 2 is 1500 ft of 10 in diameter, and pipe 3 is 4500 ft of 8 in diameter, all asphalt-dipped cast iron. The elevations of the water surfaces in reservoirs A and C are 250 ft and 160 ft respectively, and the discharge Q_2 of 60°F water into reservoir B is 3.3 cfs. Find the surface elevation of reservoir B: (a) using only a basic scientific calculator; (b) using Mathcad.

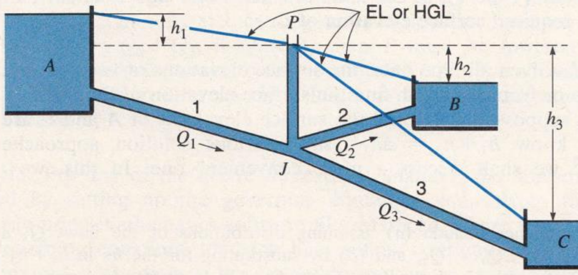


Figure 8.27
Branching pipes.

Solution

This is a Case 2 problem.

Table A.1 for water at 60°F: $\nu = 1.217 \times 10^{-5}\ \text{ft}^2/\text{sec}$.

Line:	1	2	3
L, ft	6000	1500	4500
D, ft	1.25	10/12	8/12
e, ft (Table 8.1)	0.0004	0.0004	0.0004

(a) Find the elevation of P by trial and error.

L/D	4800	1800	6750
A, ft^2	1.227	0.545	0.349
e/D	0.000 32	0.000 48	0.0006

Elevation of P lies between 160 and 250 ft. Calculate V from Eq. (8.42). Trials:

$Q_{2=3.3}$

El. P	h_1	h_3	V_1	V_3	Q_1	Q_3	ΣQ	Move P?
200	50	40	6.444	4.481	7.907	1.564	+3.04	Up
230	20	70	4.013	5.984	4.925	2.088	−0.463	Down

7.907-3.3-1.564

When ∑Q is +ve, there is more flow into junction, producing higher h_L , therefore reduce Q_1, which reduces V_1 since A_1 is constant (Q=AV). Reduced V_1 results in reduced h_L, .Hence P has to move up so that $h_L = h_1$ is reduced as El. A is fixed

h_1 reduces from 50 to 20, V to 4.013, Q to 4.925 while the opposite happens to pipe 3

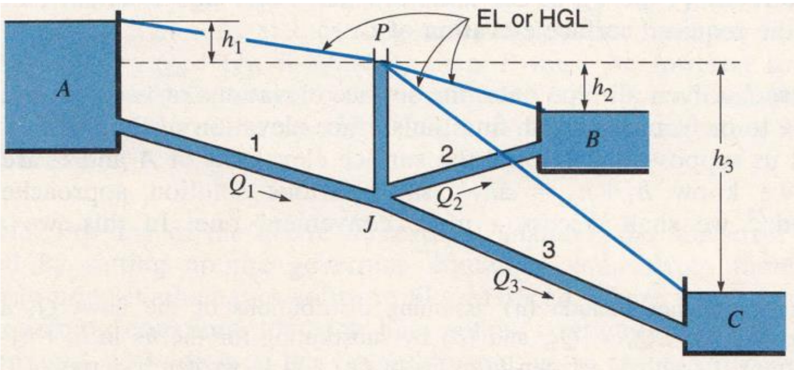


Figure 8.27
Branching pipes.

Sample Problem 8.12 Given that, in Fig. 8.27, pipe 1 is 6000 ft of 15 in diameter, pipe 2 is 1500 ft of 10 in diameter, and pipe 3 is 4500 ft of 8 in diameter, all asphalt-dipped cast iron. The elevations of the water surfaces in reservoirs A and C are 250 ft and 160 ft respectively, and the discharge Q_2 of 60°F water into reservoir B is 3.3 cfs. Find the surface elevation of reservoir B: (a)

Turbulent flow, Type 2:

$$V = -2\sqrt{\frac{2gDh_L}{L}}\log\left(\frac{e/D}{3.7}+\frac{2.51\nu}{D}\sqrt{\frac{L}{2gDh_L}}\right) \qquad (8.42a)$$

Elevation of P lies between 160 and 250 ft. Calculate V from Eq. (8.42). Trials:

$Q_{2=3.3}$

El. P	h_1	h_3	V_1	V_3	Q_1	Q_3	ΣQ	Move P?
200	50	40	6.444	4.481	7.907	1.564	+3.04	Up
230	20	70	4.013	5.984	4.925	2.088	−0.463	Down

Interpolation (Fig. 8.28): (230 − El. P)/(230 − 200) = 0.463/(0.463 + 3.04); El. P = 226.03.

0.463+0

226	24	66	4.412	5.805	5.414	2.026	+0.088	Up

Interpolation (Fig. 8.28): (230 − El. P)/(230 − 226) = 0.463/(0.463 + 0.088); El. P = 226.64.

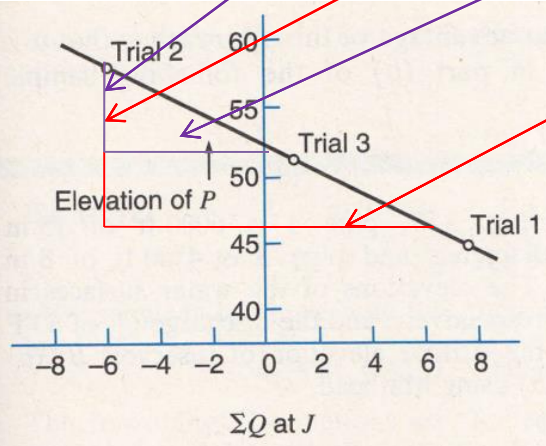


Figure 8.28

Slope between Trial 2 and El P is the same as between Trial 2 and Trial 1 i.e,
Slope 1 = slope 2

$$\frac{230 - El.P}{0.463 - 0} = \frac{230 - 200}{0.463 + 3.04}$$

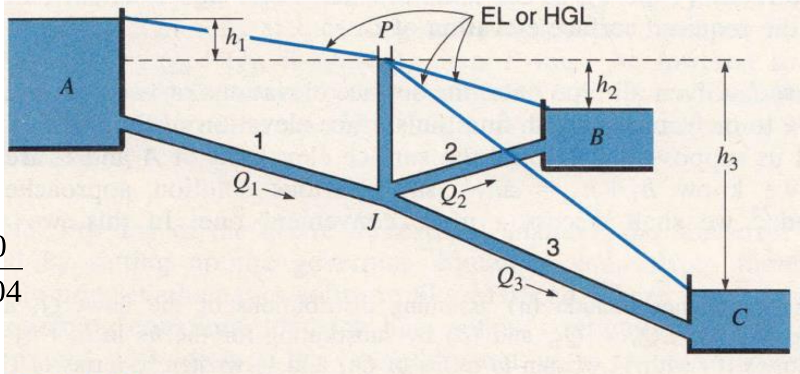


Figure 8.27
Branching pipes.

Close enough! *Note:* these adjustments are very suitable for making on a spreadsheet.

$$V_2 = \frac{Q_2}{A_2} = \frac{3.3}{0.545} = 6.055 \text{ fps}; \qquad \mathbf{R}_2 = \frac{D_2 V_2}{\nu} = 416{,}500$$

Eq. (8.41): $\qquad f_2 = 0.01761; \qquad$ Eq. (8.10): $\quad h_2 = 18.05$ ft

$$\text{El. } B = \text{El. } P - h_2 = 226.64 - 18.05 = 208.59 \text{ ft} \qquad \textbf{\textit{ANS}}$$

$$h_L = f\frac{L}{D}\frac{v^2}{2g}$$

Turbulent flow, all pipes:
$$\frac{1}{\sqrt{f}} = -1.8 \log\left[\left(\frac{e/D}{3.7}\right)^{1.11} + \frac{6.9}{\mathbf{R}}\right] \tag{8.41}$$

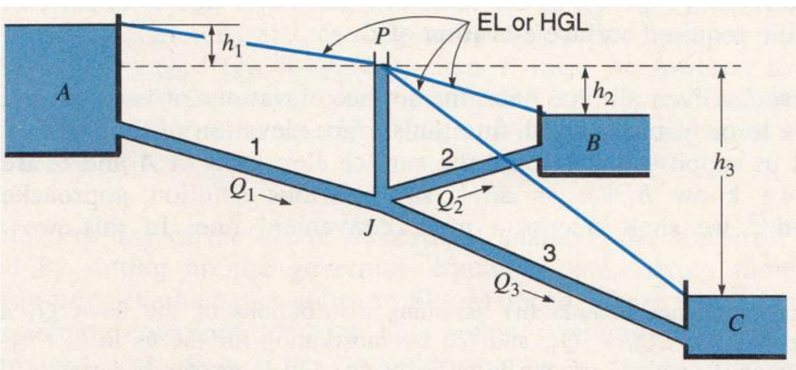


Figure 8.27
Branching pipes.

Sample Problem 8.12 Given that, in Fig. 8.27, pipe 1 is 6000 ft of 15 in diameter, pipe 2 is 1500 ft of 10 in diameter, and pipe 3 is 4500 ft of 8 in diameter, all asphalt-dipped cast iron. The elevations of the water surfaces in reservoirs A and C are 250 ft and 160 ft respectively, and the discharge Q_2 of 60°F water into reservoir B is 3.3 cfs. Find the surface elevation of reservoir B: (a)

Pipes in series

If the pipe is made up of sections of different diameters, as shown in Fig. 8.29, the continuity and energy equations establish the following two simple relations that must be satisfied:

$$Q = Q_1 = Q_2 = Q_3 = \ldots \tag{8.68}$$

$$h_L = h_{L1} + h_{L2} + h_{L3} + \ldots \tag{8.69}$$

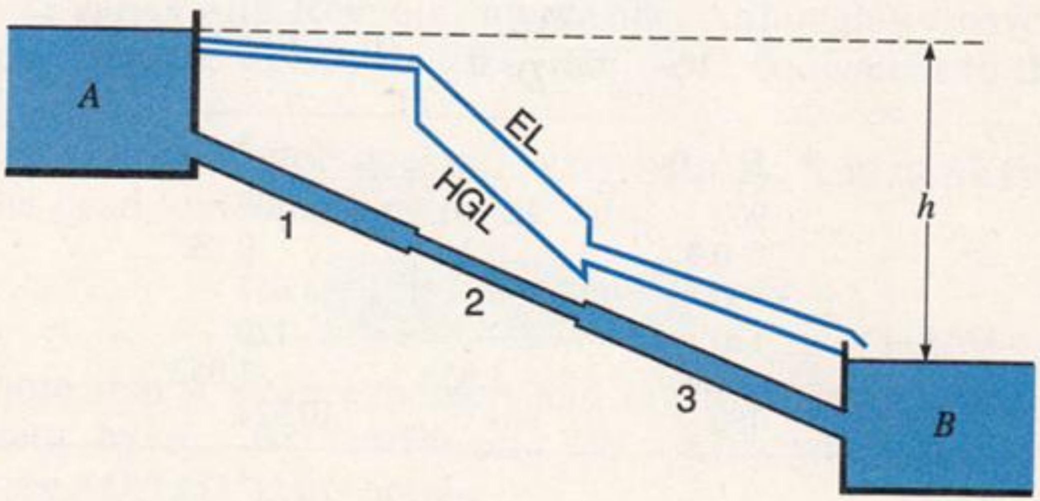


Figure 8.29
Pipes in series.

Type 1: h_L

If the rate of discharge Q is given, the problem is straightforward. The head loss may be found directly by adding the contributions from the various sections, as in Eq. (8.69). If empirical coefficients or constant f values are given, we can do this using Eq. (8.66) and the appropriate values of K and n. If, however, the pipe material or e is given, this is preferred, because the Darcy–Weisbach approach is more accurate. Then we use Eq. (8.10) to find the individual head loss contributions after finding e/D, V, **R**, and f for each pipe.

$$h_L = h_{L1} + h_{L2} + h_{L3}$$

$$h_L = KQ^n$$

$$h_L = f\frac{L}{D}\frac{v^2}{2g}$$

Type 2: Q

If the total head loss h_L is given and the flow is required, the problem is a little more involved. Using the empirical equations, we again substitute Eq. (8.66) into Eq. (8.69), to get

$h_L = KQ^n$ $h_L = h_{L1} + h_{L2} + h_{L3}$

$$h_L = K_1 Q_1^n + K_2 Q_2^n + K_3 Q_3^n + \ldots$$

But since all the Qs are equal from Eq. (8.68), this becomes

$Q = Q_1 = Q_2 = Q_3 = \ldots$

$$h_L = (K_1 + K_2 + K_3 + \ldots)Q^n = \left(\sum K\right)Q^n \qquad (8.70)$$

So, knowing the pipe information and the empirical equation to be used, we can solve for Q.

If we wish to use the more accurate Darcy–Weisbach approach to find Q, we must note that in Eq. (8.70) each K has now become a function of a different f. The preferred manual method of solution is similar to the above, and is known as the ***equivalent-velocity-head method***. Substituting from Eq. (8.10) into Eq. (8.69) and including minor losses if needed (usually if $L/D < 1000$),

$h_L = f\frac{L}{D}\frac{v^2}{2g}$ $\quad h_L = h_{L1} + h_{L2} + h_{L3}$ $\quad h_L = K\frac{v^2}{2g}$ $\quad h_L = (K_1 + K_2 + K_3 + \dots)Q^n = \left(\sum K\right)Q^n$

$$h_L = \left(f_1\frac{L_1}{D_1} + \sum k_1\right)\frac{V_1^2}{2g} + \left(f_2\frac{L_2}{D_2} + \sum k_2\right)\frac{V_2^2}{2g} + \dots$$

$Q_1 = Q_2 = Q_3$

Using the equation of continuity, we know $D_1^2V_1 = D_2^2V_2 = D_3^2V_3$, etc., from which all the velocities may be expressed in terms of one chosen velocity. So, by assuming reasonable values for each f (e.g., from Eq. (8.39) or Fig. 8.11), for any pipeline, however complex, the total head loss may be written as

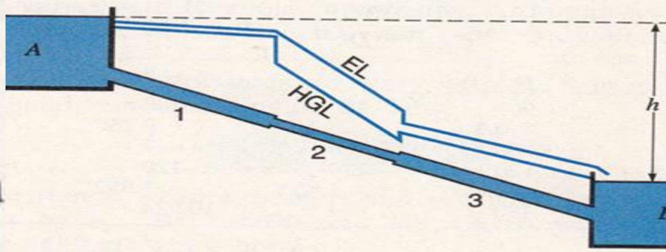
$\frac{1}{\sqrt{f}} = 2\log\left(\frac{3.7}{e/D}\right)$ or $\quad \frac{1}{\sqrt{f}} = -0.86\ln\frac{e}{3.7D} \quad (7.6.27)$

$$h_L = K\frac{V^2}{2g} \tag{8.71}$$

where V is the chosen velocity. This equation may be solved for the chosen V, and so the V and **R** and f values obtained for each pipe. For better accuracy, the assumed values of f should be replaced by the values just obtained, and an improved solution obtained. When the f values have converged V is correct, and Q may be calculated.

i.e., either V_1, V_2 or V_3

SAMPLE PROBLEM 8.15 Suppose in Fig. 8.29 the pipes 1, 2, and 3 are 300 m of 30 cm diameter, 150 m of 20 cm diameter, and 250 m of 25 cm diameter, respectively, of new cast iron and are conveying 15°C water. If $h = 10$ m, find the rate of flow from A to B.



Solution

Table 8.1 for cast iron pipe: $e = 0.25$ mm $= 0.000\ 25$ m

Pipe:	1	2	3
L, m	300	150	250
D, m	0.3	0.2	0.25
e/D	0.000 833	0.001 25	0.001 00
f_{min} (Fig. 8.11)	0.019	0.021	0.020

Using e/D and at Re = 10^8 the max Re value in Moody diagram, meaning this is the minimum f value for a given e/D. As Re decreases into transition zone, the f values go up

Assuming these friction factor values, and neglecting minor losses

$$h_L = \left(f_1 \frac{L_1}{D_1} + \sum k_1\right)\frac{V_1^2}{2g} + \left(f_2 \frac{L_2}{D_2} + \sum k_2\right)\frac{V_2^2}{2g} + \ldots$$

$$h = 10 = 0.019\left(\frac{300}{0.3}\right)\frac{V_1^2}{2g} + 0.021\left(\frac{150}{0.2}\right)\frac{V_2^2}{2g} + 0.020\left(\frac{250}{0.25}\right)\frac{V_3^2}{2g}$$

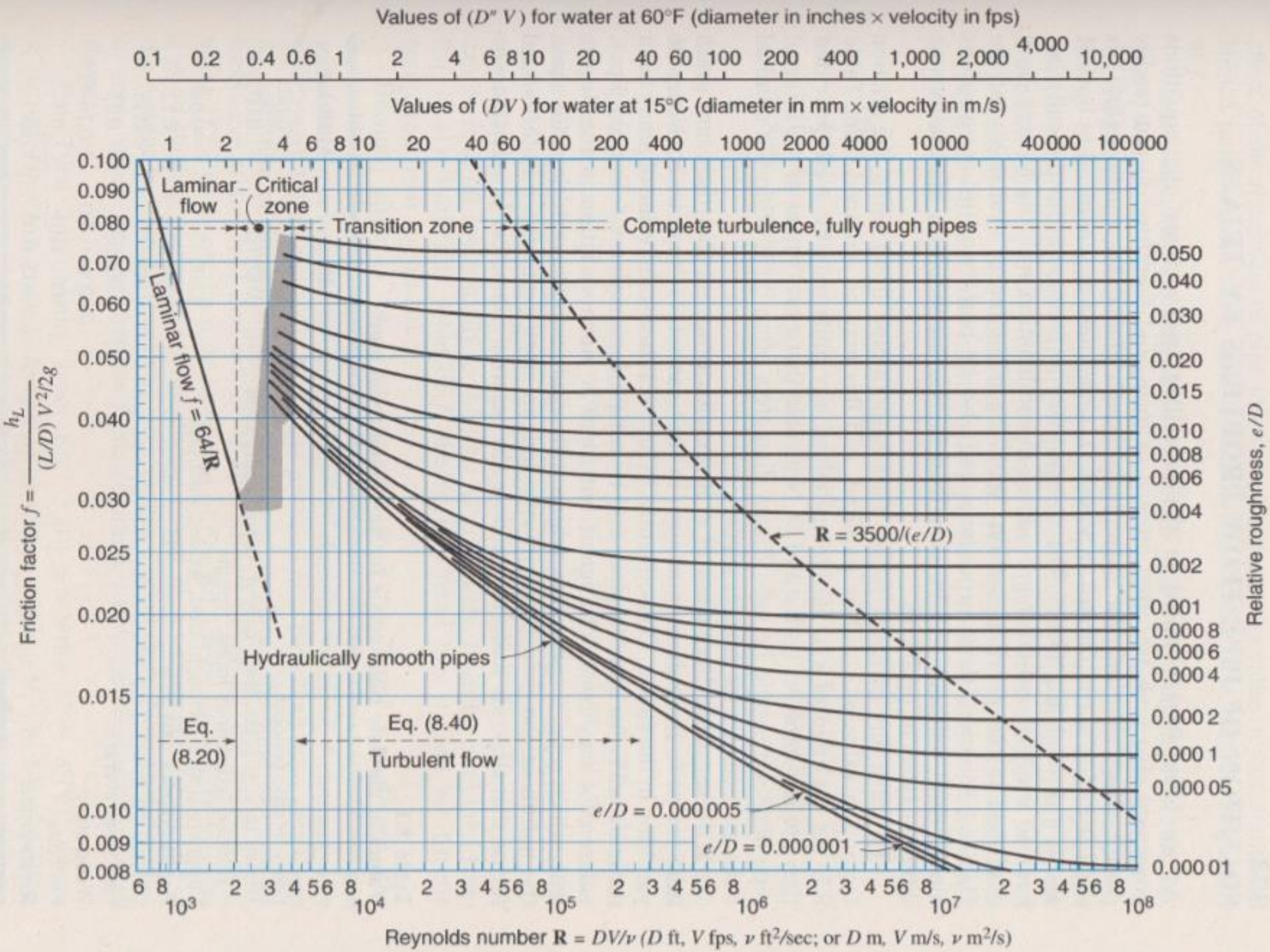


Figure 8.11 Moody chart for pipe friction factor (Stanton diagram).

From continuity, $$\frac{V_2^2}{2g} = \frac{V_1^2}{2g}\left(\frac{D_1}{D_2}\right)^4 = \frac{V_1^2}{2g}\left(\frac{30}{20}\right)^4 = 5.06\,\frac{V_1^2}{2g}$$

$$Q_1 = Q_2$$

$$V_1 D_1^2 = V_2 D_2^2$$

$$V_2 = \frac{V_1 D_1^2}{D_2^2}$$

$$V_2^2 = V_1^2\left(\frac{D_1}{D_2}\right)^4$$

Similarly, $$\frac{V_3^2}{2g} = 2.07\,\frac{V_1^2}{2g}$$

$$10 = 0.019\left(\frac{300}{0.3}\right)\frac{V_1^2}{2g} + 0.021\left(\frac{150}{0.2}\right)\frac{V_2^2}{2g} + 0.020\left(\frac{250}{0.25}\right)\frac{V_3^2}{2g}$$

and thus $$10 = \frac{V_1^2}{2g}\left(0.019\,\frac{1000}{1} + 0.021\,\frac{750}{1}\,5.06 + 0.020\,\frac{1000}{1}\,2.07\right)$$

if V_1 is chosen

from which $$\frac{V_1^2}{2g} = 0.071 \text{ m}$$

Hence $$V_1 = \sqrt{2(9.81 \text{ m/s}^2)(0.072 \text{ m})} = 1.18 \text{ m/s}$$

The corresponding values of **R** are 0.31×10^6, 0.47×10^6, and 0.37×10^6; the corresponding friction factors are only slightly different from those originally assumed, since the flow is at Reynolds numbers very close to those at which the pipes behave as rough pipes. Hence

$$Q = A_1 V_1 = \tfrac{1}{4}\pi(0.30)^2 1.18 = 0.083 \text{ m}^3/\text{s} \qquad \textit{ANS}$$

Greater accuracy would have been obtained if the friction factors had been adjusted to match the pipe-friction chart more closely or were calculated by Eq. (8.41), and if minor losses had been included. In that case $Q = 0.081 \text{ m}^3/\text{s}$.

Pipes in parallel

In the case of flow through two or more parallel pipes, as in Fig. 8.30, the continuity and energy equations establish the following relations that must be satisfied:

$$Q = Q_1 + Q_2 + Q_3 \tag{8.72}$$

$$h_L = h_{L1} = h_{L2} = h_{L3} \tag{8.73}$$

as the pressures at A and B are common to all pipes. Problems may be posed in various ways.

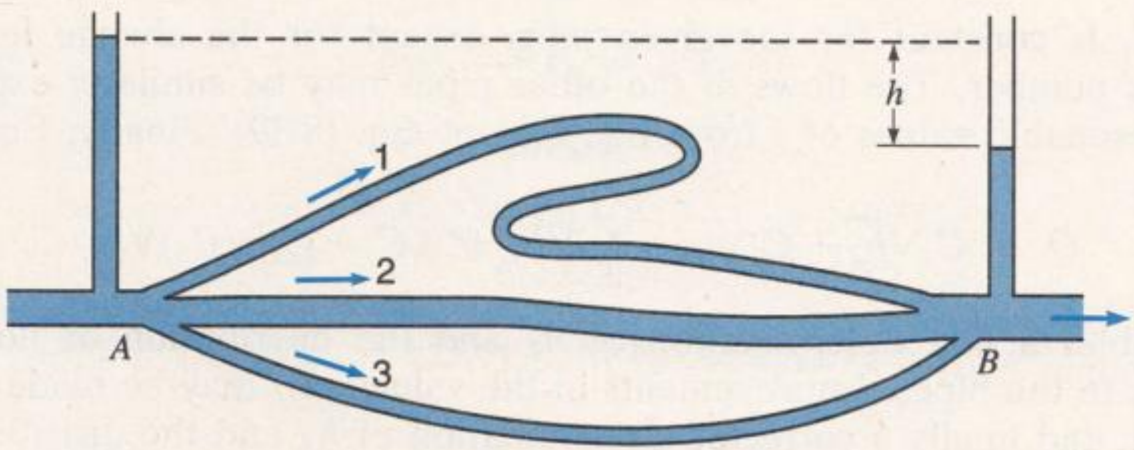


Figure 8.30
Pipes in parallel.

Pipes in parallel

Solutions

1. If the head loss is given, the problem is straight forward. The head loss may be found directly by adding the contributions from the various pipes, as in Eq. 8.72.
2. If empirical coefficients or constant f values are given, we can do this using Eq. 8.67 and the appropriate values of K and n.

$$h_L = KQ^n \Rightarrow Q = \left(\frac{h_L}{K}\right)^{1/n}$$

3. If, however, the pipe material or e is given, this is preferred because the Darcy-Weisbach approach is more accurate. Then we have an independent Type 2 problem for each pipe, see Lecture 16, which can be solved directly by Eq. 8.42, for example.

Turbulent flow, Type 2:

$$\frac{4Q}{\pi D^2} = -2\sqrt{\frac{2gDh_L}{L}}\log\left(\frac{e/D}{3.7} + \frac{2.51\nu}{D}\sqrt{\frac{L}{2gDh_L}}\right) \quad (8.42b)$$

4. If the total flow Q is given and the head loss and individual flows are required, the problem is a little more involved. Using the empirical equations, we again substitute Eq. (8.67) into Eq. (8.72), to get

$$h_L = KQ^n \Rightarrow Q = \left(\frac{h_L}{K}\right)^{1/n}$$

$$Q = Q_1 + Q_2 + Q_3 +$$

$$Q = \left(\frac{h_{L1}}{K_1}\right)^{1/n} + \left(\frac{h_{L2}}{K_2}\right)^{1/n} + \left(\frac{h_{L3}}{K_3}\right)^{1/n} + \ldots$$

Pipes in parallel

Solutions

$$Q = \left(\frac{h_{L1}}{K_1}\right)^{1/n} + \left(\frac{h_{L2}}{K_2}\right)^{1/n} + \left(\frac{h_{L3}}{K_3}\right)^{1/n} + \ldots$$

But since all the h_Ls are equal from Eq. (8.73), this becomes

$$Q = (h_L)^{1/n}\left[\left(\frac{1}{K_1}\right)^{1/n} + \left(\frac{1}{K_2}\right)^{1/n} + \left(\frac{1}{K_3}\right)^{1/n} + \ldots\right] = (h_L)^{1/n} \sum \left(\frac{1}{K}\right)^{1/n} \quad (8.74)$$

So, knowing the pipe information and the empirical equation to be used, we can solve for h_L. We can then find the individual flows using Eq. (8.67).

$$\boxed{\begin{gathered} h_L = KQ^n \\ Q = \left(\frac{h_L}{K}\right)^{1/n} \end{gathered}}$$

Solutions

5. If we wish to use the more accurate Darcy-Weisbach approach to find h_L

and the individual Qs, we must note that in Eq. (8.74) now each K has become a function of a different f. The preferred manual method of solution is similar to the above. Writing Eq. (8.10) for each line, including minor losses if needed,

$$h_L = f\frac{L}{D}\frac{v^2}{2g}$$

$$Q = (h_L)^{1/n}\left[\left(\frac{1}{K_1}\right)^{1/n} + \left(\frac{1}{K_2}\right)^{1/n} + \left(\frac{1}{K_3}\right)^{1/n} + \ldots\right] = (h_L)^{1/n}\sum\left(\frac{1}{K}\right)^{1/n} \quad (8.74)$$

$$h_L = \left(f\frac{L}{D} + \sum k\right)\frac{V^2}{2g}$$

where $\sum k$ is the sum of the minor-loss coefficients, which may usually be neglected if the pipe is longer than 1000 diameters. Solving for V and then Q, the following is obtained for pipe 1:

$$Q_1 = A_1V_1 = A_1\sqrt{\frac{2gh_L}{f_1(L_1/D_1) + \sum k_1}} = C_1\sqrt{h_L} \quad (8.75)$$

where C_1 is constant for the given pipe, except for the change in f with Reynolds number. The flows in the other pipes may be similarly expressed, using reasonable values of f from Fig. 8.11 or Eq. (8.39). Finally, Eq. (8.72) becomes

$$Q = Q_1 + Q_2 + Q_3 +$$

$$Q = C_1\sqrt{h_L} + C_2\sqrt{h_L} + C_3\sqrt{h_L} = (C_1 + C_2 + C_3)\sqrt{h_L} \quad (8.76)$$

Since h_L is the same for all pipes

This enables a first determination of h_L and the distribution of flows and velocities in the pipes. Improvements in the values of f may be made next, if indicated, and finally a corrected determination of h_L and the distribution of flows.

If the turbulent flow equation (8.40) or (8.41) is used to obtain f, it is important to remember to confirm that the Reynolds number is in the turbulent range. We can pre-check the likelihood of laminar flow occurring in any of the pipes by calculating an "average" flow velocity from the total flow divided by the total area of all the pipes, and using this to obtain an indicator **R** for each pipe.

Sample Problem 8.16 Three pipes A, B, and C are interconnected as shown. The pipe characteristics are given as follows:

Pipe	D (in)	L (ft)	f
A	6	2000	0.020
B	4	1600	0.032
C	8	4000	0.024

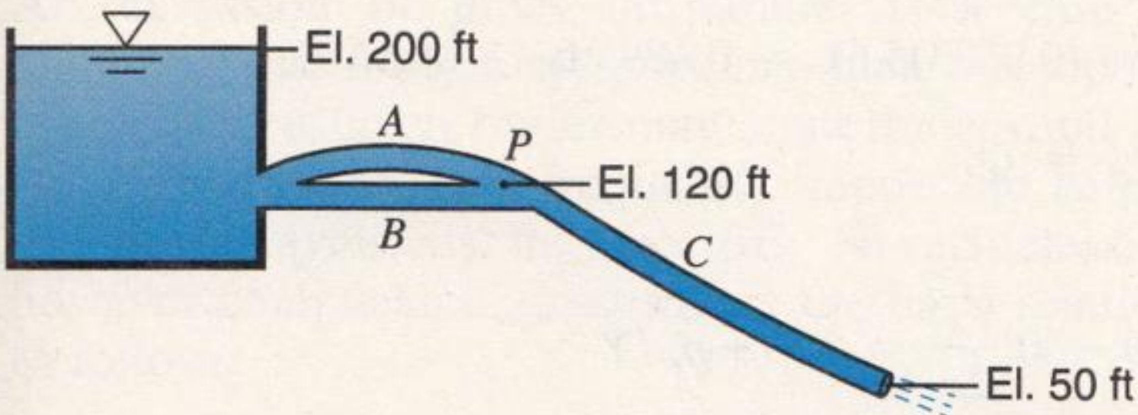
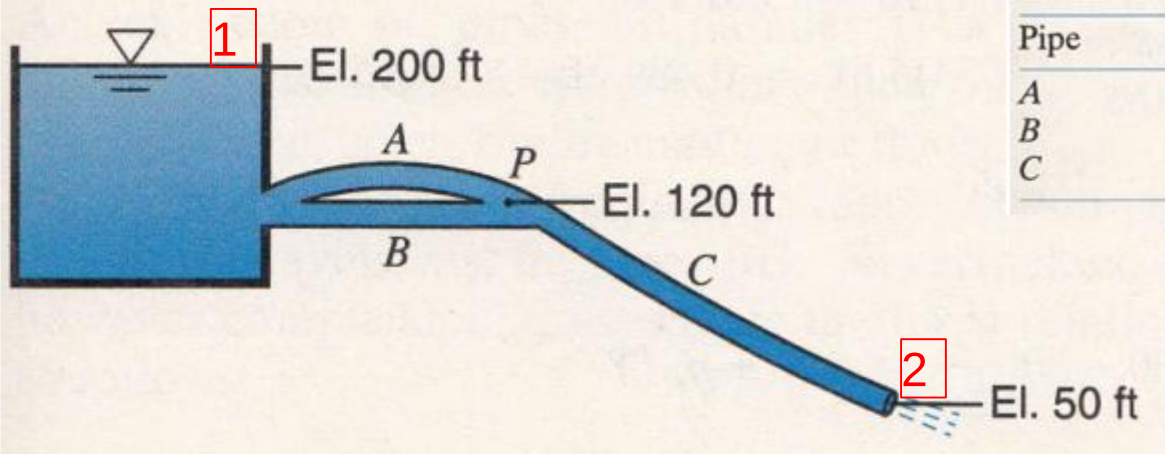


Figure S8.16

Find the rate at which water will flow in each pipe. Find also the pressure at point P. All pipe lengths are much greater than 1000 diameters, therefore minor losses may be neglected.



Pipe	D (in)	L (ft)	f
A	6	2000	0.020
B	4	1600	0.032
C	8	4000	0.024

$$h_L = f\frac{L}{D}\frac{v^2}{2g}$$

$$\alpha_1 \cancelto{0}{\frac{v_1^2}{2g}} + \cancelto{0}{\frac{p_1}{\rho g}} + z_1 + \cancelto{0}{h_p} = \alpha_2 \frac{v_2^2}{2g} + \cancelto{0}{\frac{p_2}{\rho g}} + z_2 + \cancelto{0}{h_t} + h_l$$

Solution

Energy Eq.: $$200 - 0.020\frac{2000}{6/12}\frac{V_A^2}{2g} - 0.024\frac{4000}{8/12}\frac{V_C^2}{2g} = 50 + \frac{V_C^2}{2g}$$

Between points 1 & 2

i.e., $$150 = 80\frac{V_A^2}{2g} + 145\frac{V_C^2}{2g}$$

Continuity: $Q_A + Q_B = Q_C$ $$6^2V_A + 4^2V_B = 8^2V_C$$

i.e. $$36V_A + 16V_B = 64V_C$$

Also, $$h_{L_A} = h_{L_B} = 0.020\frac{2000}{6/12}\frac{V_A^2}{2g} = 0.032\frac{1600}{4/12}\frac{V_B^2}{2g}$$

i.e. $$80V_A^2 = 153.6V_B^2, \quad V_B = 0.722V_A$$

Note that h_L is the same in pipe A and B and so only one pipe is used, in this case pipe A. If pipe B is used, then pipe A will not be used

Substituting into the continuity equation,

$$36V_A + 16(0.722V_A) = 64V_C$$

$$47.5V_A = 64V_C, \quad V_A = 1.346V_C$$

Substituting into the energy equation,

$$150 = 80\frac{V_A^2}{2g} + 145\frac{V_C^2}{2g}$$

$$150 = 80\frac{(1.346V_C)^2}{2g} + 145\frac{V_C^2}{2g} = 289.9\frac{V_C^2}{2g}$$

$$V_C^2 = 2(32.2)150/289.9 = 33.3$$

$$V_C = 5.77 \text{ fps}, \quad Q_C = A_C V_C = (0.349)5.77 = 2.01 \text{ cfs} \quad \textit{ANS}$$

$$V_A = 1.346V_C = 7.77 \text{ fps} \quad Q_A = (0.1963)7.77 = 1.526 \text{ cfs} \quad \textit{ANS}$$

Continuity:

$$36V_A + 16V_B = 64V_C$$

$$36(7.77) + 16V_B = 64(5.77)$$

$$279.7 + 16V_B = 369.4$$

$$V_B = 89.7/16 = 5.61 \text{ fps}$$

$$Q_B = A_B V_B = (0.0873)5.61 = 0.489 \text{ cfs} \quad \textit{ANS}$$

As a check, note that $Q_A + Q_B = Q_C$.

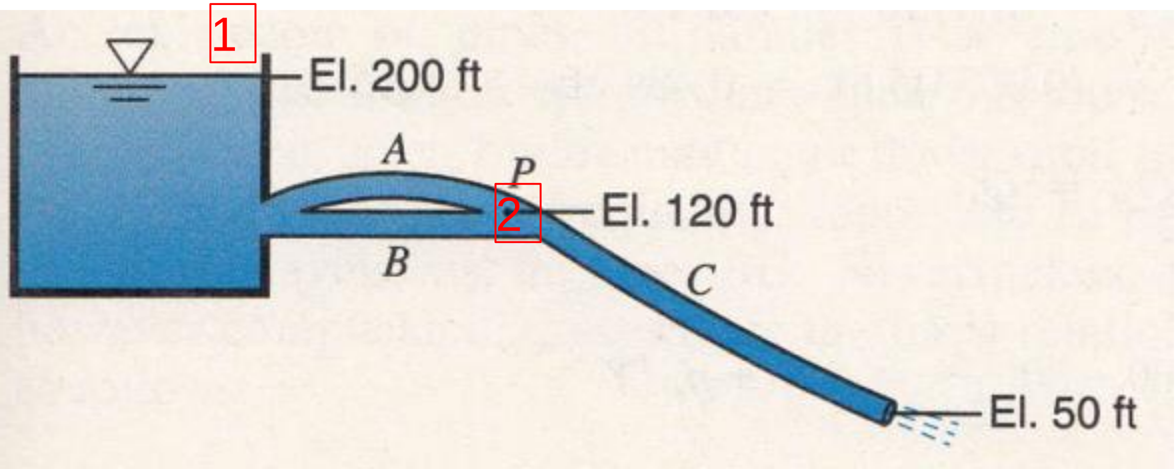
To find the pressure at P

$$\propto_1 \frac{v_1^2}{2g} + \frac{p_1}{\rho g} + z_1 + h_p = \propto_2 \frac{v_2^2}{2g} + \frac{p_2}{\rho g} + z_2 + h_t + h_l$$

$$h_L = f \frac{L}{D} \frac{v^2}{2g}$$

$$200 - 80 \frac{V_A^2}{2g} = 120 + p_P/\gamma$$

$$p_P/\gamma = 80 - 80 \frac{(7.77)^2}{2g} = 5.01 \text{ ft}$$



Pipe	D (in)	L (ft)	f
A	6	2000	0.020
B	4	1600	0.032
C	8	4000	0.024

$$\propto_1 \frac{v_1^2}{2g} + \frac{p_1}{\rho g} + z_1 + h_p = \propto_2 \frac{v_2^2}{2g} + \frac{p_2}{\rho g} + z_2 + h_t + h_l$$

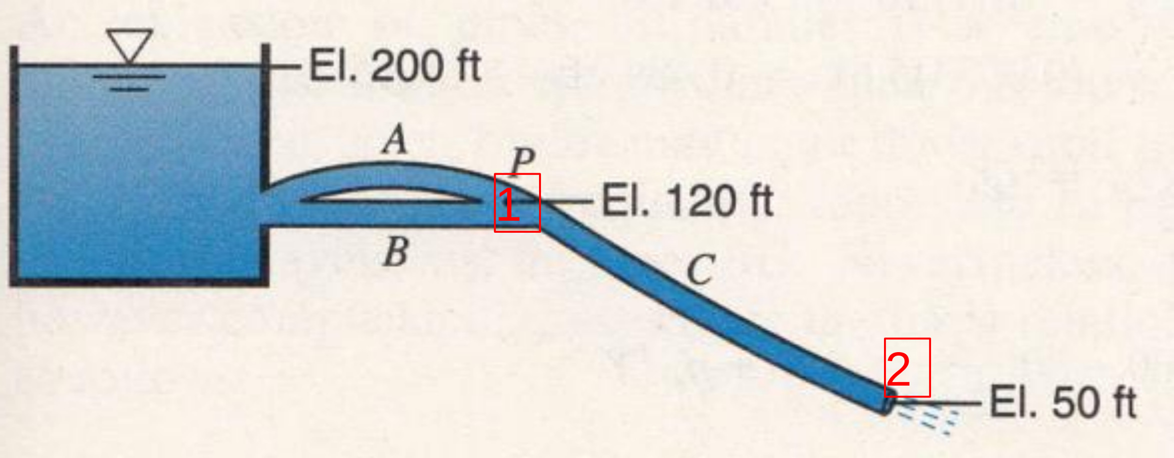
$$h_L = f\,\frac{L}{D}\frac{v^2}{2g}$$

Check: $$120 + p_P/\gamma - 144\,\frac{V_C^{\,2}}{2g} = 50 + \frac{V_C^{\,2}}{2g}$$

$$p_P/\gamma = 145\,\frac{(5.77)^2}{2g} - 70 = 5.01\text{ ft}$$

So $p_P/\gamma = 5.01$ ft and $p_P = (62.4/144)5.01 = 2.17$ psi. ***ANS***

In this example it was assumed that the values of f for each pipe were known. Actually f depends on **R** [Fig. 8.11 or Eq. (8.41)]. Usually the absolute roughness e of each pipe is known or assumed, and an accurate solution is achieved through trial and error until the fs and **R**s for each pipe have converged.



Pipe	D (in)	L (ft)	f
A	6	2000	0.020
B	4	1600	0.032
C	8	4000	0.024

Manual iteration for f may be avoided by solving simultaneous eq using mathematics software like Mathcad. There are the usual four eq for each pipe, plus Eq. (8.72); if necessary, minor losses may easily be accounted for by using head loss equations with the form of Eq. (8.60). For the three pipes of Fig. 8.30, for example, there are therefore 13 simultaneous equations, which may be solved in the usual manner (see Sample Prob. 8.12*b*) for 13 unknowns. These unknowns are either the head loss or the total flow rate, and three values each of Q, V, $\mathbf{R}$, and f.

It is instructive to compare the solution methods for pipes in parallel with those for pipes in series. The role of the head loss in one case becomes that of the discharge rate in the other, and vice versa. The student should be already familiar with this situation from the elementary theory of dc circuits. The flow corresponds to the electrical current, the head loss to the voltage drop, and the frictional resistance to the ohmic resistance. The outstanding deficiency in this analogy occurs in the variation of potential drop with flow, which is with the first power in the electrical case ($E = IR$) and with the second power in the hydraulic case ($h_L \propto V^2 \propto Q^2$) for fully developed turbulent flow.

Pipe networks: Hardy Cross method

An extension of pipes in parallel is a case frequently encountered in municipal distribution systems, in which the pipes are interconnected so that the flow to a given outlet may come by several different paths, as shown in Fig. 8.31. Indeed, it is frequently impossible to tell by inspection which way the flow travels, as in pipe *BE*. Nevertheless, the flow in any network, however complicated, must satisfy the basic relations of continuity and energy as follows:

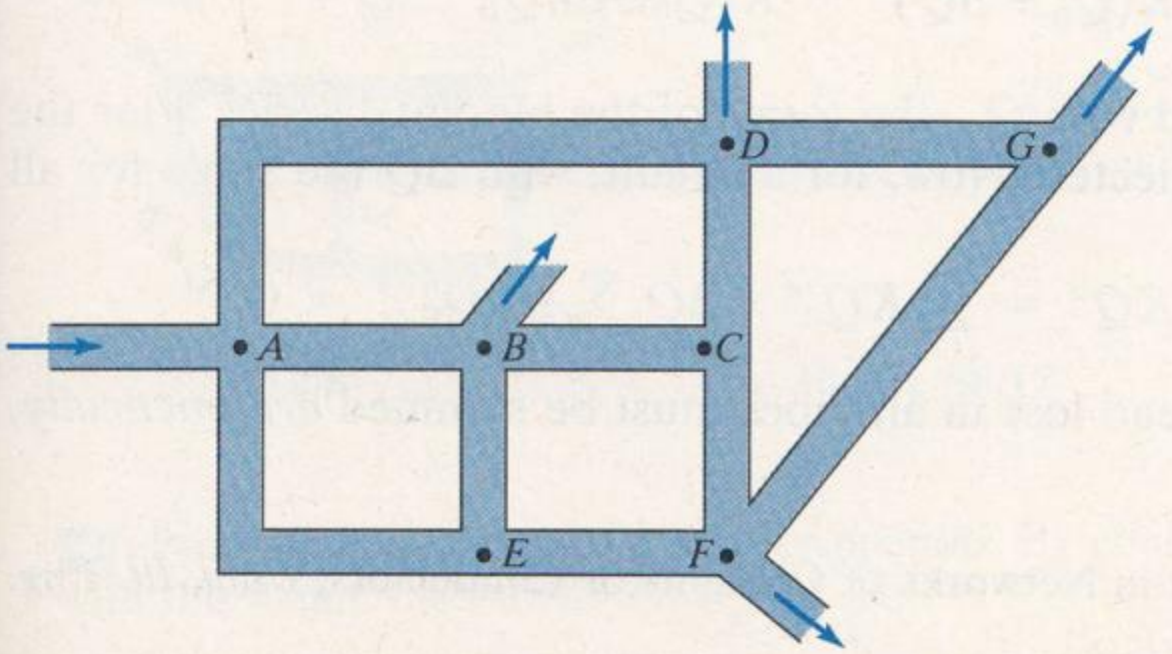


Figure 8.31
Pipe network.

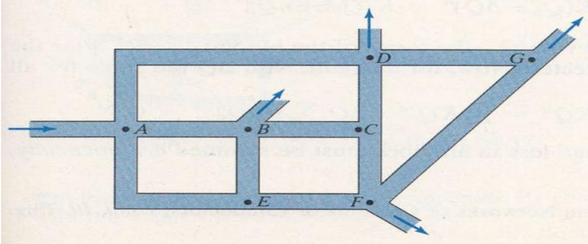


Figure 8.31
Pipe network.

1. The flow into any junction must equal the flow out of it

2. The flow in each pipe must satisfy the pipe-friction laws for flow in a single pipe

3. The algebraic sum of the head losses around any closed loop must be zero
   (As it is the same as considering that there is no flow from the start point, therefore no head loss)

Pipe networks are generally too complicated to solve analytically, as was possible in the simpler cases of parallel pipes (Sec. 8.28). Furthermore, it is seldom possible to predict the capacity requirements of water distribution systems with high precision, and flows in them vary considerably throughout the day, so high accuracy in calculating their flows is not important. As a result, the use of empirical equations (Sec. 8.15) and constant values of f are very acceptable for this purpose. A practical procedure is the method of successive approximations, introduced by Cross.[25] It consists of the following elements, in order:

1. By careful inspection assume the most reasonable distribution of flows that satisfies condition 1.

1. The flow into any junction must equal the flow out of it.

2. Write condition 2 for each pipe in the form

2. The flow in each pipe must satisfy the pipe-friction laws for flow in a single pipe

$$h_L = KQ^n \tag{8.77}$$

where K and n are constants for each pipe as described in Sec. 8.26 under Approximate Solutions [Eq. (8.66), etc.]. If minor losses are important they may be included as in Eq. (8.75), which yields $K = 1/C^2$ and $n = 2$ for constant f. Minor losses may be included within any pipe or loop, but they are neglected at the junction points.

$$Q_1 = A_1V_1 = A_1\sqrt{\frac{2gh_L}{f_1(L_1/D_1)+\sum k_1}} = C_1\sqrt{h_L}$$

$$Q = C\sqrt{h_L} \Rightarrow h_L = \frac{Q^2}{C^2} = KQ^n \therefore K = \frac{1}{C^2}, n = 2$$

3. To investigate condition 3, compute the algebraic sum of the head losses around each elementary loop, $\sum h_L = \sum KQ^n$. Consider losses from clockwise flows as positive, counterclockwise negative. Only by good luck will these add to zero on the first trial.

3. The algebraic sum of the head losses around any closed loop must be zero.

4. Adjust the flow in each loop by a correction ΔQ to balance the head in that loop and give $\sum KQ^n = 0$. The heart of this method lies in the determination of ΔQ. For any pipe, we may write

$$Q = Q_0 + \Delta Q$$

where Q is the correct discharge and Q_0 is the assumed discharge. Then, for each pipe,

$$h_L = KQ^n = K(Q_0 + \Delta Q)^n = K(Q_0{}^n + nQ_0^{n-1}\Delta Q + \ldots)$$

If ΔQ is small compared with Q_0, the terms of the binomial series after the second one may be neglected. Now, for a circuit, with ΔQ the same for all pipes,

$$\sum h_L = \sum KQ^n = \sum KQ_0{}^n + \Delta Q \sum KnQ_0^{n-1} = 0$$

[redacted]

[25] H. Cross, Analysis of Flow in Networks of Conduits or Conductors, *Univ. Ill. Eng. Expt. Sta. Bull.* 286, 1936.

$$\sum h_L = \sum KQ^n = \sum KQ_0^n + \Delta Q \sum KnQ_0^{n-1} = 0$$

As the corrections of head loss in all pipes must be summed *arithmetically*,

we may solve this equation for ΔQ by making ΔQ the subject of the formula in the above equation

$$\sum KQ_0^{\ n} + \Delta Q \sum KnQ_0^{\ n-1} = 0$$

$$\Delta Q \sum KnQ_0^{\ n-1} = -\sum KQ_0^{\ n}$$

$$\Delta Q = \frac{-\sum KQ_0^{\ n}}{\sum \left| KnQ_0^{\ n-1} \right|}$$

Since $$h_L = KQ^n$$;and

$$h_L = KQ^n = KQ^{n+(1-1)} = KQ^{n-1}Q$$

$$KQ^{n-1} = \frac{h_L}{Q}$$

Therefore

$$\Delta Q = \frac{-\sum KQ_0{}^n}{\sum \left|KnQ_0{}^{n-1}\right|} = \frac{-\sum KQ_0{}^n}{n\sum \left|KQ_0{}^{n-1}\right|} = \frac{-\sum h_L}{n\sum \left|\frac{h_L}{Q_0}\right|} \qquad 8.78$$

It must be emphasized again that the numerator of Eq. (8.78) is to be summed algebraically, with due account of sign, while the denominator is summed arithmetically. The negative sign in Eq. (8.78) indicates that when there is an excess of head loss around a loop in the clockwise direction, the ΔQ must be subtracted from clockwise Q_0s and added to counterclockwise ones. The reverse is true if there is a deficiency of head loss around a loop in the clockwise direction.

5. After each circuit is given a first correction, the losses will still not balance, because of the interaction of one circuit upon another (pipes which are common to two circuits receive two independent corrections, one for each circuit). The procedure is repeated, arriving at a second correction, and so on, until the corrections become negligible.

Either form of Eq. (8.78) may be used to find ΔQ. As values of K appear in both numerator and denominator of the first form, values proportional to the actual K may be used to find the distribution. The second form will be found most convenient for use with pipe-friction diagrams for water pipes.

An attractive feature of the approximation method is that errors in computation have the same effect as errors in judgment and will eventually be corrected by the process.

$$\Delta Q = \frac{-\sum K Q_0{}^{n}}{\sum \left| K n Q_0{}^{n-1} \right|} = \frac{-\sum h_L}{n \sum \left| \frac{h_L}{Q_0} \right|}$$

The **Hardy Cross method** is an iterative method for determining the flow in pipe network systems where the inputs and outputs are known, but the flow inside the network is unknown.[1] The method was first published in November 1936 by its namesake, Hardy Cross, a structural engineering professor at the University of Illinois at Urbana–Champaign.[2] The Hardy Cross method is an adaptation of the Moment distribution method, which was also developed by Hardy Cross as a way to determine the forces in statically indeterminate structures.

The introduction of the Hardy Cross method for analyzing pipe flow networks revolutionized municipal water supply design. Before the method was introduced, solving complex pipe systems for distribution was extremely difficult due to the nonlinear relationship between head loss and flow. The method was later made obsolete by computer solving algorithms employing the Newton-Raphson method or other numerical methods that eliminate the need to solve nonlinear systems of equations by hand.

In 1930, Hardy Cross published a paper called "Analysis of Continuous Frames by Distributing Fixed-End Moments" in which he described the moment distribution method, which would change the way engineers in the field performed structural analysis.[3] The moment distribution method was used to determine the forces in statically indeterminate structures and allowed for engineers to safely design structures from the 1930s through the 1960s, till the computer oriented methods.[3] In November 1936, Cross applied the same geometric method to solving pipe network flow distribution problems, and published a paper called "Analysis of flow in networks of conduits or conductors."[1]

Hardy Cross	
Born	1885 Nansemond County, Virginia, United States
Died	1959
Nationality	United States
Education	Massachusetts Institute of Technology, Cambridge, Massachusetts, United States Norfolk Academy Harvard University
Occupation	Engineer
	Engineering career
Discipline	Structural engineer
Institutions	Institution of Structural Engineers Brown University University of Illinois at Urbana-Champaign Yale University
Significant advance	moment distribution method for reinforced concrete
Awards	Frank P. Brown Medal (1959) American Society for Engineering Education Lamme Medal (1944), ACI (1935) Wason Medal for Most Meritorious Paper, IStructE Gold Medal

Sample Problem 8.17 If the flow into and out of a two-loop pipe system are as shown in Fig. S8.17, determine the flow in each pipe. The K values for each pipe were calculated from the pipe and minor loss characteristics and from an assumed value of f.

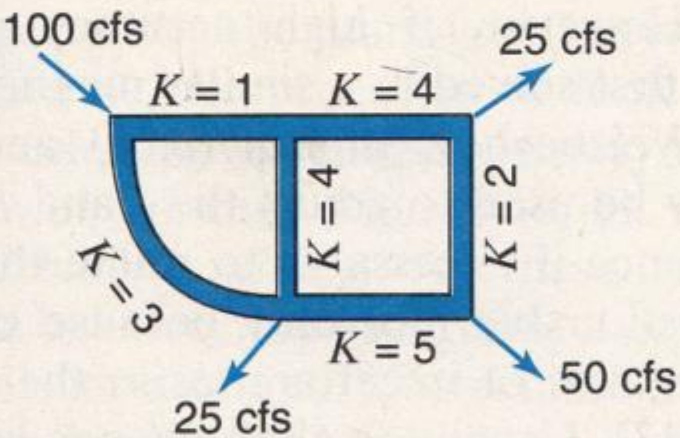


Figure S8.17

Solution

As a first step, assume flow in each pipe such that continuity is satisfied at all junctions. Calculate ΔQ for each loop, make corrections to the assumed Qs, and repeat several times until the ΔQs are quite small.

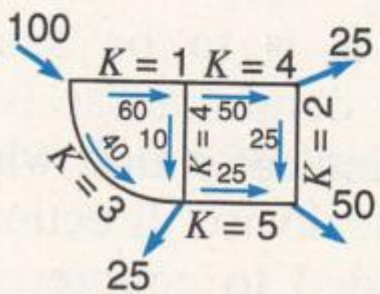
n=2

$$\Delta Q = \frac{-\sum KQ_0^{\,n}}{\sum \left|KnQ_0^{\,n-1}\right|} = \frac{-\sum h_L}{n\sum \left|\frac{h_L}{Q_0}\right|}$$

Left loop

KQ_0^n	$\lvert KnQ_0^{n-1}\rvert$
$1 \times 60^2 = 3{,}600$ ↻	$1 \times 2 \times 60 = 120$
$4 \times 10^2 = 400$ ↻	$4 \times 2 \times 10 = 80$
$3 \times 40^2 = 4{,}800$ ↺	$3 \times 2 \times 40 = 240$
800 ↺	440

$$\Delta Q_1 = \frac{-800}{440} \approx 2 \circlearrowright$$

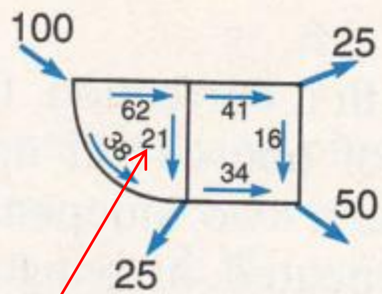


First approximation

Right loop

KQ_0^n	$\lvert KnQ_0^{n-1}\rvert$
$4 \times 50^2 = 10{,}000$ ↻	$4 \times 2 \times 50 = 400$
$2 \times 25^2 = 1{,}250$ ↻	$2 \times 2 \times 25 = 100$
$4 \times 10^2 = 400$ ↺	$4 \times 2 \times 10 = 80$
$5 \times 25^2 = 3{,}125$ ↺	$5 \times 2 \times 25 = 250$
$7{,}725$ ↻	830

$$\Delta Q_1 = \frac{-7{,}725}{830} \approx 9 \circlearrowleft$$



After first correction

$1 \times 62^2 = 3{,}844$ ↻	$1 \times 2 \times 62 = 124$
$4 \times 21^2 = 1{,}764$ ↻	$4 \times 2 \times 21 = 168$

$4 \times 41^2 = 6{,}724$ ↻	$4 \times 2 \times 41 = 328$
$2 \times 16^2 = 512$ ↻	$2 \times 2 \times 16 = 64$

$Q = Q_0 + \Delta Q$ =10 + (2 + 9)= 21

Q =50 – 9 = 41

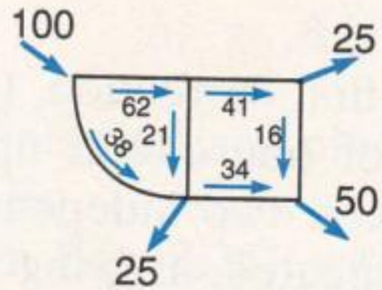
First approximation

$$3 \times 40^2 = 4{,}800 \qquad 3 \times 2 \times 40 = 240$$
$$\overline{\;800\;} \qquad \overline{\;440\;}$$

$$\Delta Q_1 = \frac{-800}{440} \approx 2$$

$$4 \times 10^2 = 400 \qquad 4 \times 2 \times 10 = 80$$
$$5 \times 25^2 = 3{,}125 \qquad 5 \times 2 \times 25 = 250$$
$$\overline{\;7{,}725\;} \qquad \overline{\;830\;}$$

$$\Delta Q_1 = \frac{-7{,}725}{830} \approx 9$$



After first correction

$$1 \times 62^2 = 3{,}844 \qquad 1 \times 2 \times 62 = 124$$
$$4 \times 21^2 = 1{,}764 \qquad 4 \times 2 \times 21 = 168$$

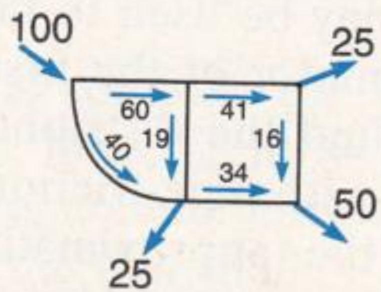
$$4 \times 41^2 = 6{,}724 \qquad 4 \times 2 \times 41 = 328$$
$$2 \times 16^2 = 512 \qquad 2 \times 2 \times 16 = 64$$

$$3 \times 38^2 = 4{,}332 \qquad 3 \times 2 \times 38 = 228$$
$$\overline{\;1{,}276\;} \qquad \overline{\;520\;}$$

$$\Delta Q_2 = \frac{-1{,}276}{520} \approx 2$$

$$4 \times 21^2 = 1{,}764 \qquad 4 \times 2 \times 21 = 168$$
$$5 \times 34^2 = 5{,}780 \qquad 5 \times 2 \times 34 = 340$$
$$\overline{\;308\;} \qquad \overline{\;900\;}$$

$$\Delta Q_2 = \frac{-308}{900} \approx 0$$



After second correction

Further corrections can be made if greater accuracy is desired.

As noted earlier, varying demand rates usually make high accuracy unnecessary with pipe networks. However, if high accuracy is required for some reason, the problem can be first solved in a similar manner to the preceding example using the Darcy–Weisbach K in Eq. (8.66) and constant f values. Then the resulting flows may be used to adjust the f and K values, and the process repeated (more than once if necessary) to refine the answers. The value of such refinement is questionable, not only because of uncertainties in the demand flows, but also because of uncertainties in the e values (pipe roughness) to be used (see Sec. 8.13). Usually f values change by only a few percent when they are adjusted, but we can see in Fig. 8.11 that for smoother pipe it is possible for them to change by as much as a factor of five.

Simple networks can be solved without approximation and manual iteration by solving simultaneous equations using mathematics software like

Mathcad. For networks containing i pipes, $5i$ equations are required i the Darcy–Weisbach equation with variable f, and $2i$ equations are required if using the simplified equation (8.77) with constant friction factors. These required equations include (a) the usual (condition 2) flow equations for each pipe (four or one per pipe, depending on the equations used); (b) flow continuity equations (condition 1) at all but one of the j nodes (as these imply continuity at the last node); (c) equations for the sum of the head losses around $i - j + 1$ loops (condition 3). The unknowns to be determined for each pipe are h_L, Q, V, $\mathbf{R}$, and f using the Darcy–Weisbach equation, or only h_L and Q using Eq. (8.77).

The pipe-network problem lends itself well to solution by use of a digital computer. Programming takes time and care, but once set up, there is great flexibility and many hours of repetitive labor can be saved. Many software packages are now available to simulate water distribution networks; see Appendix B.