EEE 3121 - Signals & Systems

Lecture 2: Signals and waveforms

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References

Our main reference text book in this course is

[1] B. P. Lathi and R. A. Green, Linear Systems and Signals, 3rd Ed., 2018, Oxford University Press, New York. ISBN 978-0-19-020017-6

[2] Kuo Franklin, F., Network Analysis and Synthesis, 3rd Ed., 1986, J. Wiley (SE), ISBN 0-471-51118-8.

[3] Sundararanjan, D., A Practical Approach to Signals and Systems, 2008, John Wiley & Sons (Asia) Pte Ltd, ISBN 978-0-470-82353-8.

However, feel free to use pretty much any additional text which you might find relevant to our course.

Introduction

- In this part of the course we concern ourselves with characterization of signals as functions of time.
- The class of signals encountered in engineering practice is broader than simple AC or DC signals. Thus, characterization of these signals is a daunting task.
- Instead, we will deal only with those signals that can be characterized in simple mathematical terms which serve as building blocks for many other signals.
- Note, that this course will only dwell on deterministic signals which do not exhibit random behavior.

2.1 General Characteristics of signals

- Signals are qualitatively describe as being periodic, symmetrical, and continuous.
- First, signals are either periodic or aperiodic. Periodic signals are described by the equation of the form

$$s(t) = s(t \pm kT) \qquad k = 0, 1, 2, \ldots \tag{2.1}$$

where T is the period of the signal. The sine wave, $\sin t$, is periodic with period $T = 2\pi$. Fig. 2.1 shows yet another example of a periodic signal.

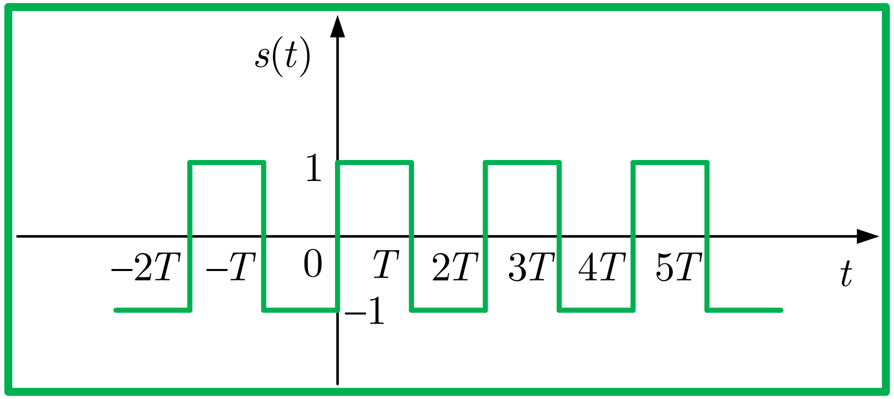

Fig. 2.1: Square wave.

2.1 General Characteristics of signals Cont'd

- Signals exhibited in Fig. 2.2 however, are aperiod since the pulse patterns do not repeat after an interval T. These signals may be considered periodic with an infinite period.

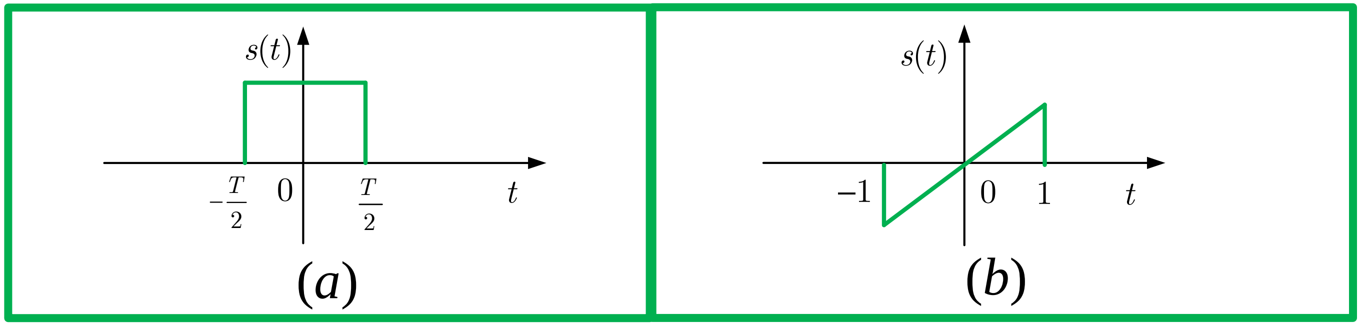

Fig. 2.2: (*a*) Even function. (*b*) Odd function.

- Symmetrical property. A signal function can be even or odd or neither. An even function obeys the relation

$$s(t) = s(-t) \tag{2.2}$$

For an odd function

$$s(t) = -s(-t) \tag{2.3}$$

2.1 General Characteristics of signals Cont'd

- Examples are that, the function $\sin t$ is odd, whereas $\cos t$ is even.
- Notice that a signal need not be even or odd. Examples of signals of this type are shown in Fig. 2.3a and 2.4a.
- It is worth noting that any signal $s(t)$ can be resolved into even and odd components, such that

$$s(t) = s_e(t) + s_o(t) \tag{2.4}$$

- For example, the signals in Figs. 2.3a and 2.4a can be decomposed into odd and even components as depicted in Figs. 2.3b, 2.3c, 2.4b and 2.4c.
- From Eq. 2.4 it follows that

$$\begin{aligned} s(-t) &= s_e(-t) + s_o(-t) \\ &= s_e(t) - s_o(t) \end{aligned} \tag{2.5}$$

2.1 General Characteristics of signals Cont'd

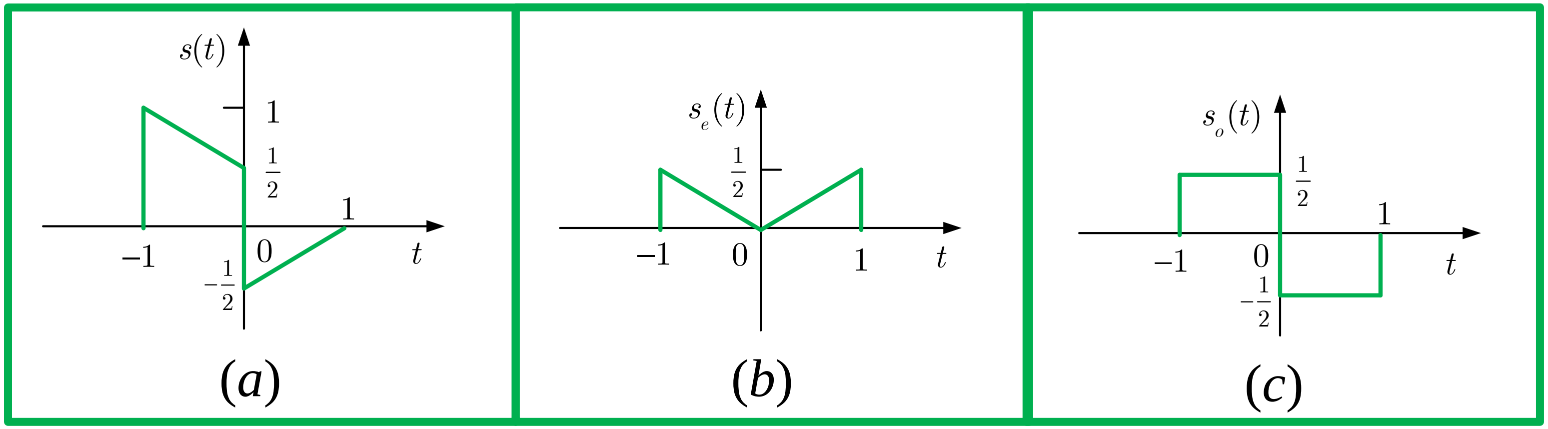

Fig. 2.3: Decomposition into odd and even components (*a*) Original function. (*b*) Even part. (*c*) Odd part.

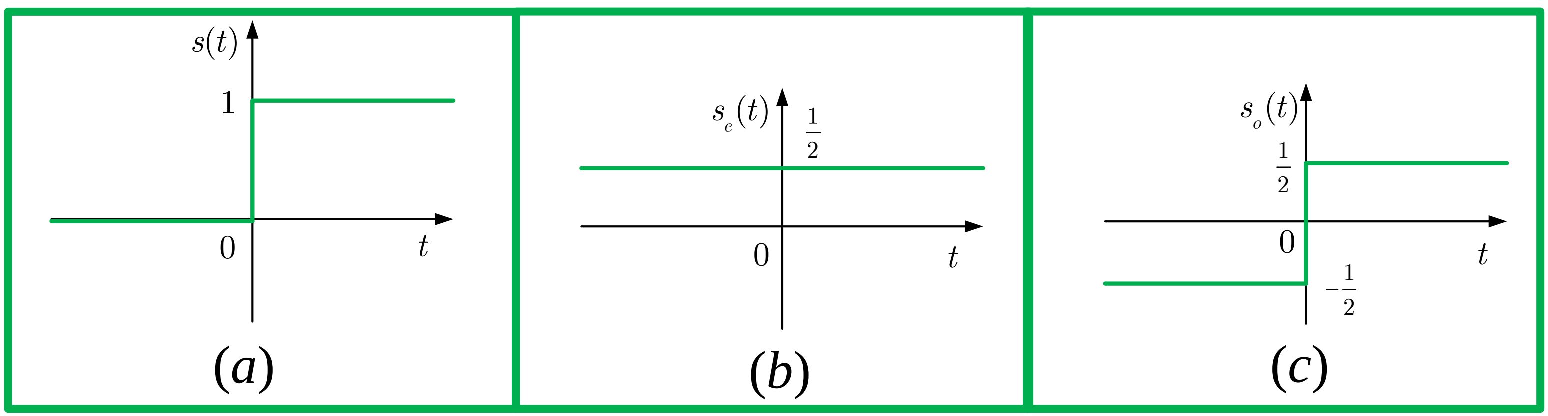

Fig. 2.4: Decomposition into odd and even components (*a*) Unit step function. (*b*) Even part. (*c*) Odd part.

2.1 General Characteristics of signals Cont'd

- The odd and even parts of the signal can thus be expressed as

$$s_e(t) = \frac{1}{2}\left[s(t) + s(-t)\right]$$
$$s_o(t) = \frac{1}{2}\left[s(t) - s(-t)\right] \quad \textbf{(2.6)}$$

- Consider the signal $s(t)$, shown in Fig. 2.5*a*. The function $s(-t)$ is equal to $s(t)$ reflected about the $t = 0$ axis and given in Fig. 2.5*b*. Thus the even and odd functions are respectively shown in Fig. 5*c* and *d*.
- Let us turn our focus on the continuity property of signals. With respect to Fig. 2.6, at $t = T$, the signal is discontinuous with height

$$f(T+) - f(T-) = A \quad \textbf{(2.7)}$$

where

$$f(T+) = \lim_{\epsilon \to 0} f(T + \epsilon), \quad f(T-) = \lim_{\epsilon \to 0} f(T - \epsilon) \quad \textbf{(2.8)}$$

2.1 General Characteristics of signals Cont'd

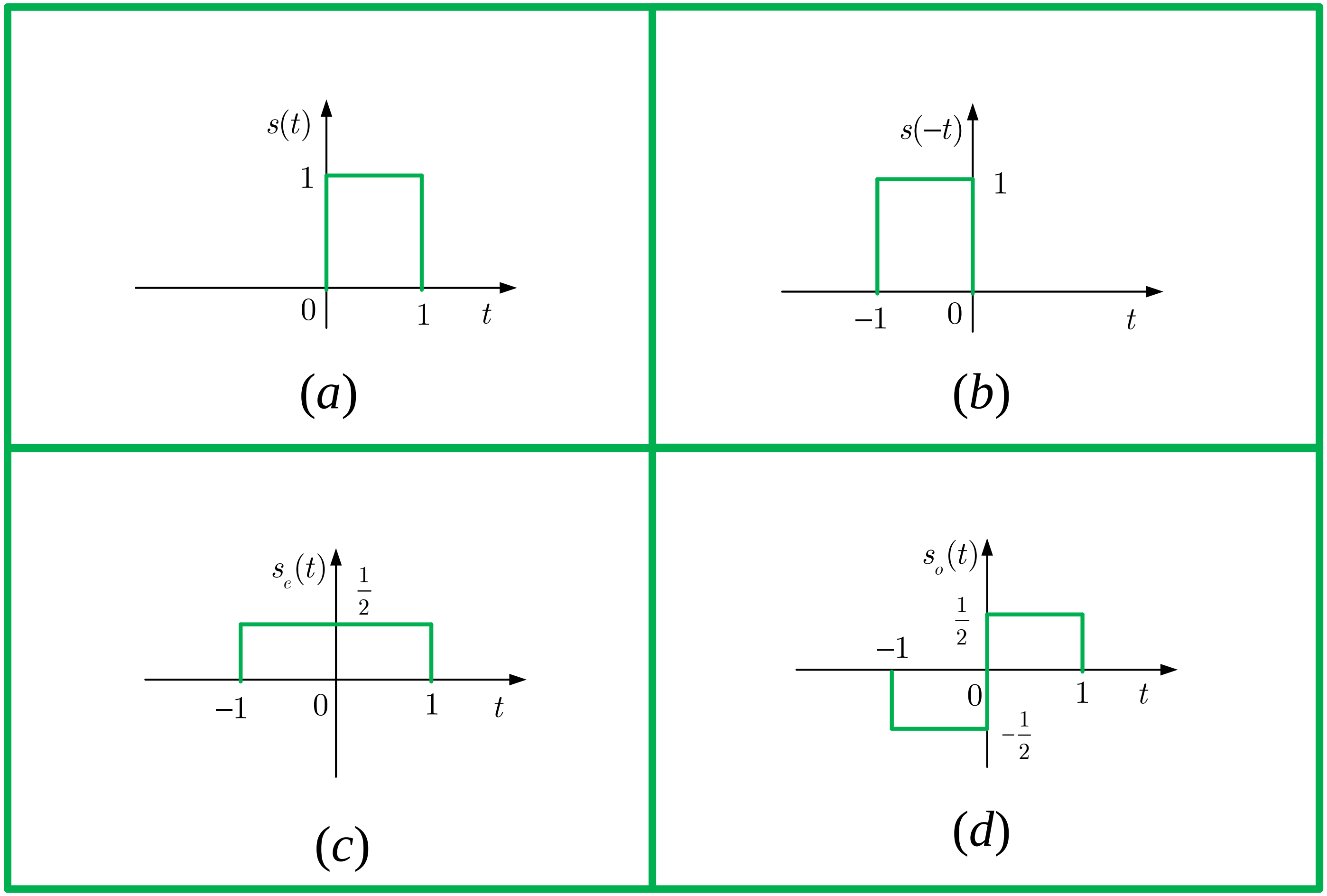

Fig. 2.5: Decomposition into odd and even components from $s(t)$ to $s(-t)$.

2.1 General Characteristics of signals Cont'd

and ϵ is a real positive quantity.

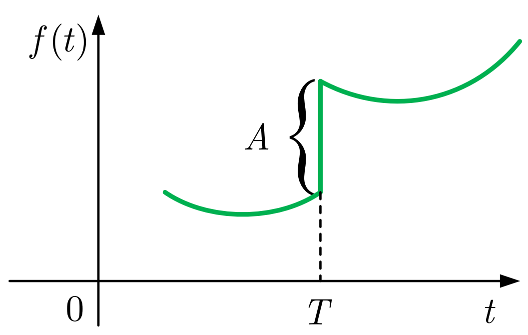

Fig. 2.6: Signal with discontinuity.

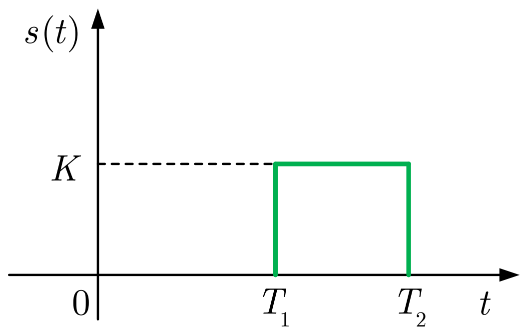

Fig. 2.7: Signal two with discontinuity.

- Since of particular concern are discontinuities in the neighborhood of $t = 0$, by Eq. 2.8, the points $f(0+)$ and $f(0-)$ are

$$f(0+) = \lim_{\epsilon \to 0} f(\epsilon)$$
$$f(0-) = \lim_{\epsilon \to 0} f(-\epsilon) \qquad \textbf{(2.9)}$$

2.1 General Characteristics of signals Cont'd

- The square pulse in Fig. 2.7 has two discontinuities, at T_1 and T_2. The height of discontinuity at T_1 is

$$s(T_1+) - s(T_1-) = K \tag{2.10}$$

- Similarly, the height of the discontinuity at T_2 is $-K$.

2.2 General Descriptions of Signals

- Time constant. In many physical problems, it is important to know how quickly a waveform decays. Thus, a useful measure of decay of an exponential is the time constant τ. Let the exponential waveform be

$$r(t) = Ke^{-t/\tau}u(t) \tag{2.11}$$

- From a plot of $r(t)$ in Fig. 2.8, we see that when $t = \tau$,

$$r(\tau) = 0.37r(0) \tag{2.12}$$

Also

$$r(4\tau) = 0.02r(0) \tag{2.13}$$

- Clearly, the larger the time constant, the longer it requires for the waveform to reach 37% of its peak value. The common time constants in circuit analysis are RC and R/L.

2.2 General Descriptions of Signals Cont'd

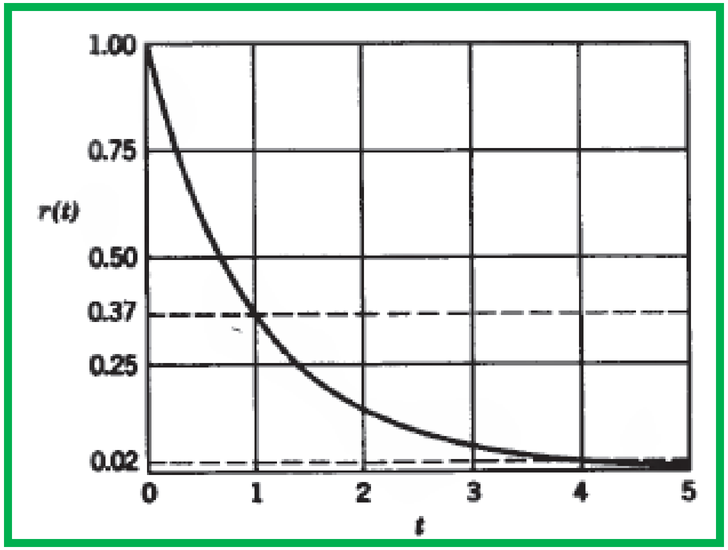

Fig. 2.8: Normalized curve for time constant $\tau = 1$.

- RMS Value. The rms (root mean square) value of a periodic waveform $e(t)$ is defined as

$$e_{rms} = \left[\frac{1}{T} \int_{t_0}^{t_0+T} e^2(t)\,dt \right]^{1/2} \qquad \textbf{(2.14)}$$

- here T is the period. For non periodic waveforms, the term rms does not apply.

[Example 2.1] RMS Value

Calculate the rms voltage for the waveform in Fig. 2.9.

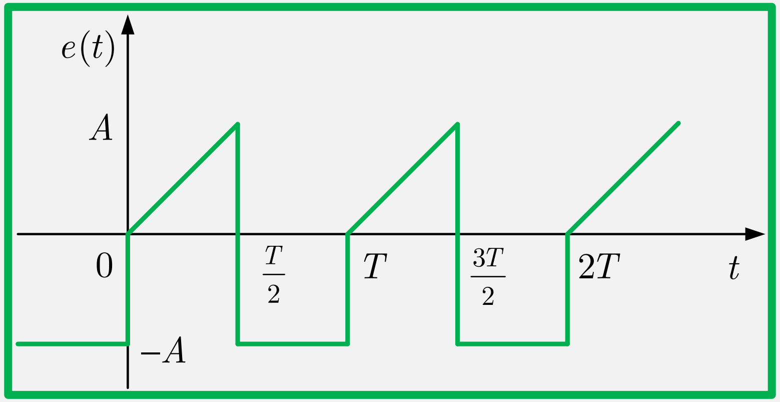

Fig. 2.9: Periodic waveform.

[Solution]

$$e_{rms} = \left\{\frac{1}{T}\left[\int_0^{T/2}\left(\frac{2A}{T}t\right)^2 dt + \int_{T/2}^{T}(-A)^2 dt\right]\right\}^{1/2} = \left\{\frac{1}{T}\left[\frac{4A^2}{T^2}\frac{t^3}{3}\Bigg|_0^{T/2} + A^2 t\Big|_{T/2}^{T}\right]\right\}^{1/2}$$

$$= \left(\frac{A^2}{6} + \frac{A^2}{2}\right)^{1/2} = \sqrt{2/3}A\,[\mathrm{V}] \qquad \textbf{(2.15)}$$

2.2 General Descriptions of Signals Cont'd

- DC value. The dc value of a waveform has meaning only when the waveform is periodic. It is the average value of the waveform over one period.

$$e_{dc} = \frac{1}{T}\int_{t_0}^{t_0+T} e(t)\,dt \qquad \textbf{(2.16)}$$

- Vividly, the squarewave in Fig. 2.1 has zero dc value, whereas the waveform in Fig. 2.9 has a dc value of

$$e_{dc} = \frac{1}{T}\left[\frac{AT}{4} - \frac{AT}{2}\right] = -\frac{A}{4}\left[\mathrm{V}\right] \qquad \textbf{(2.17)}$$

2.2 General Descriptions of Signals Cont'd

- Duty cycle. The term duty cycle, D, is defined as the ratio of the time duration of the positive cycle t_0 of a periodic waveform to the period, T, that is,

$$D = \frac{t_0}{T} \qquad (2.18)$$

- For the pulse train shown in Fig. 2.10 below, where most of the energy is concentrated in a narrow pulse of width t_0, the rms voltage of the waveform in is

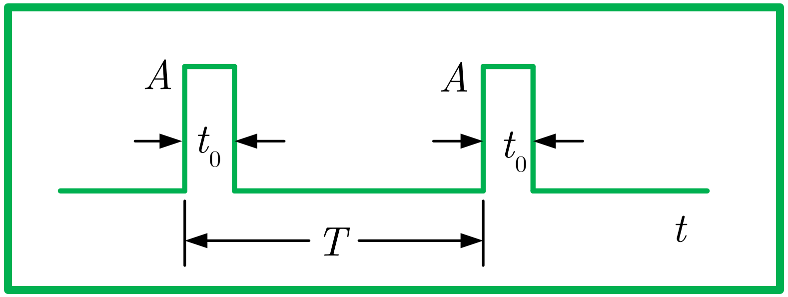

Fig. 2.10: Periodic waveform with small duty cycle.

$$e_{rms} = \left(\frac{1}{T} \int_0^{t_0} A^2 dt \right)^{1/2} = A\sqrt{t_0/T} = A\sqrt{D} \qquad (2.19)$$

2.2 General Descriptions of Signals Cont'd

- Crest factor is defined as the ratio of the peak voltage of a periodic waveform to the rms value (with the dc component removed). Explicitly, for any waveform with zero dc such as the one in Fig. 2.11, crest factor, *CF*, is

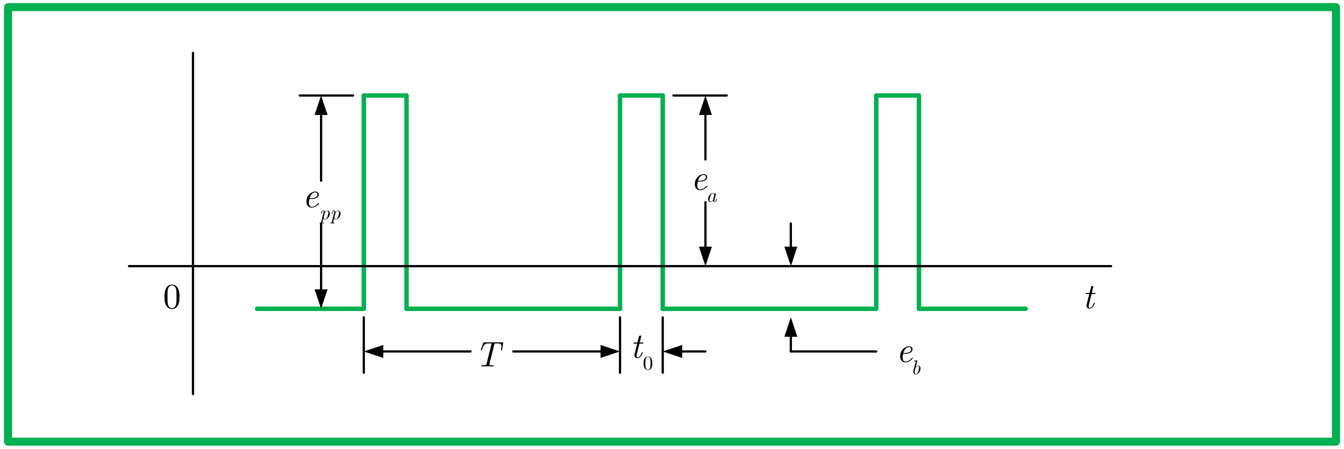

Fig. 2.11: Periodic waveform with zero dc and small duty cycle.

$$CF = \frac{e_a}{e_{rms}} \text{ or } \frac{e_b}{e_{rms}} \quad \textbf{(2.20)}$$

whichever is greater. Thus for the waveform in Fig. 2.11, the peak-to-peak voltage is

2.2 General Descriptions of Signals Cont'd

- Thus for the waveform in Fig. 2.11, the peak-to-peak voltage is

$$e_{pp} = e_a + e_b \qquad (2.21)$$

- Since the waveform has zero dc value

$$e_a t_0 = e_b (T - t_0) \qquad (2.22)$$

Also,

$$e_b = e_{pp} D \qquad (2.23)$$

and

$$e_a = e_{pp} (1 - D) \qquad (2.24)$$

2.2 General Descriptions of Signals Cont'd

- The rms value of the waveform in Fig. 2.11 is

$$e_{rms} = \left(\frac{e_{pp}{}^2 (1-D)^2 t_0 + e_{pp}{}^2 D^2 (T - t_0)}{T} \right)^{1/2} = e_{pp} \sqrt{D(1-D)} \qquad \textbf{(2.25)}$$

- Since crest factor $CF = e_a / e_{rms}$, we have

$$CF = \frac{e_{pp}(1-D)}{e_{pp}\sqrt{D(1-D)}} = \sqrt{1/D - 1} \qquad \textbf{(2.26)}$$

- For example, if $D = 1/100$,

$$CF = \sqrt{\frac{1}{1/100} - 1} = \sqrt{100 - 1} \cong 10 \qquad \textbf{(2.27)}$$

2.2 General Descriptions of Signals Cont'd

- If $D = 1/10,000$,

$$CF = \sqrt{10,000 - 1} \cong 100 \qquad \textbf{(2.28)}$$

- A voltmeter with high crest factor is able to read accurately rms values of signals whose waveforms differ from sinusoids, that is, signals with low duty factor. It is worth noting that the smallest value of crest factor occurs for the maximum value of D , that is, $D_{\max} = 0.5$,

$$CF_{\min} = \sqrt{1/D_{\max} - 1} = 1 \qquad \textbf{(2.29)}$$

2.3 The Step Function and Associated Waveforms

- The unit step function $u(t)$ shown in Fig. 2.12 is defined as

$$u(t) = \begin{cases} 0 & \forall\, t < 0 \\ 1 & \forall\, t \geq 0 \end{cases} \qquad \textbf{(2.30)}$$

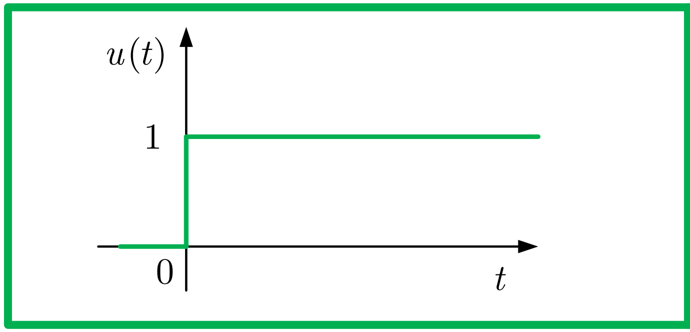

Fig. 2.12: Unit step function.

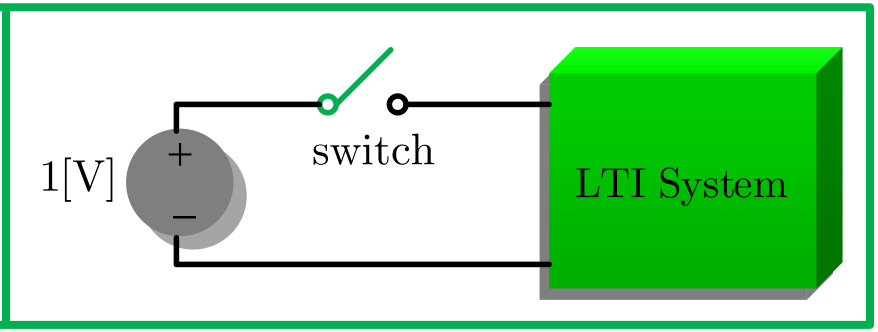

Fig. 2.13: System analogy of unit step.

- The physical analogy of a unit step excitation corresponds to a switch S, which closes at $t = 0$ and connects a dc battery of 1 volt to a given circuit, as shown in Fig. 2.13. The unit step is zero whenever the argument within the parentheses is negative and unit otherwise.

2.3 The Step Function and Associated Waveforms Cont'd

- Thus the function $u(t-a)$, where $a>0$, is defined by

$$u(t-a)=\begin{cases}0 & \forall\, t<a\\ 1 & \forall\, t\geq a\end{cases} \qquad \textbf{(2.31)}$$

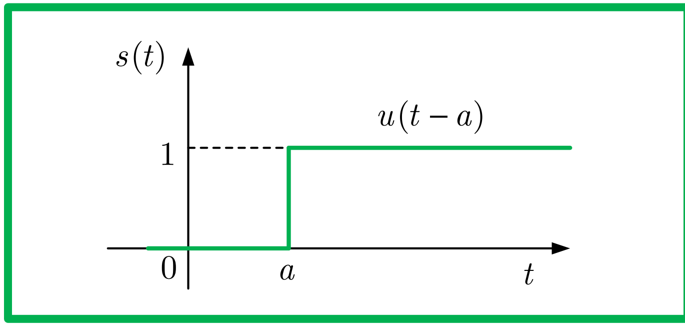

Fig. 2.14: Shifted step function.

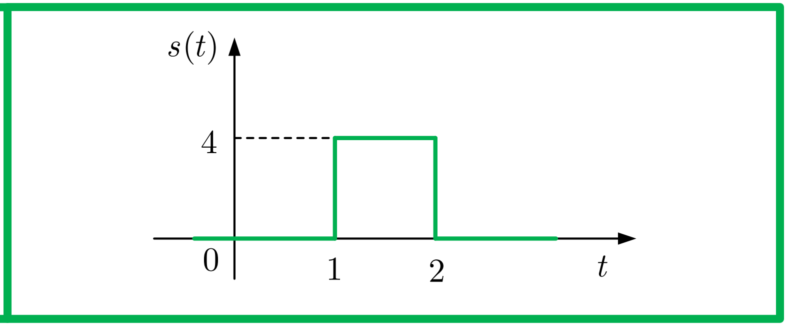

Fig. 2.15: Square pulse.

- Consider the change of amplitude and shifting properties of the step function. It follows that the square pulse in Fig. 2.15 can be constructed by the sum of two step functions.

$$s(t)=4u(t-1)+(-4)u(t-2) \qquad \textbf{(2.32)}$$

2.3 The Step Function and Associated Waveforms

- Eq. 2.32 is depicted by Fig. 2.16.

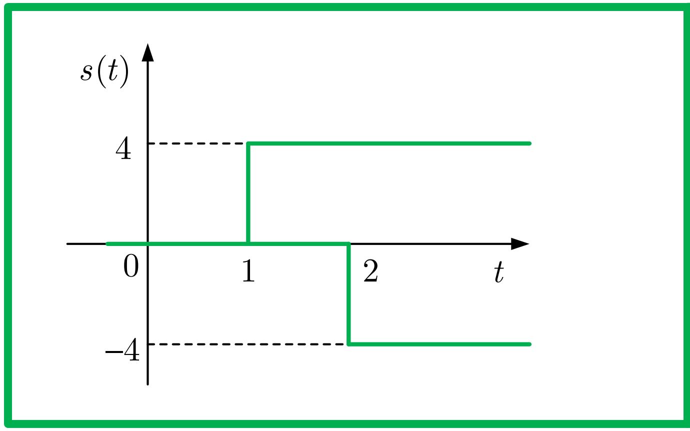

Fig. 2.16: Construction of square pulse by step function

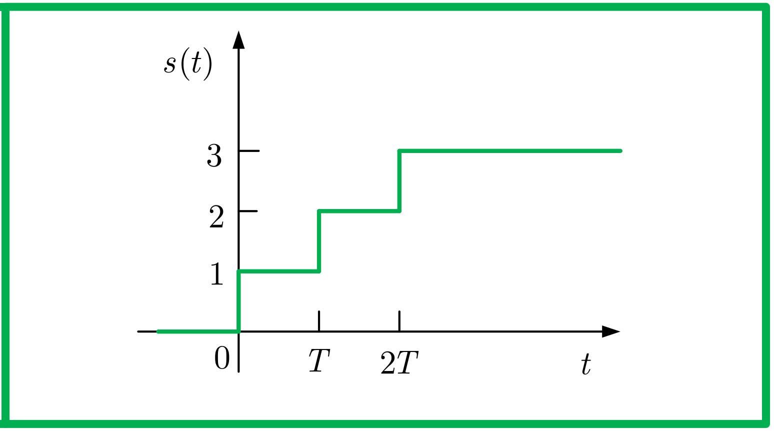

Fig. 2.17: Staircase function

- The equation of the staircase function in Fig. 2.17, is given by

$$s(t) = \sum_{k=0}^{2} u(t - kT) \qquad \textbf{(2.33)}$$

2.3 The Step Function and Associated Waveforms Cont'd

- Similarly, by the shifting property, the square wave in Fig. 2.1, for $t \geq 0$, is given by the equation of the form

$$s(t) = u(t) - 2u(t-T) + 2u(t-2T) - 2u(t-3T) + \cdots \quad \textbf{(2.34)}$$

- A simpler way to represent the square wave is by using the property that the step function is zero if and only if the argument is negative. By confining us to the interval $t \geq 0$, the function

$$s(t) = u\left(\sin\frac{\pi t}{T}\right) \quad \textbf{(2.35)}$$

is zero whenever $\sin(\pi t/T)$ is negative, see Fig. 2.18. Thus, the square wave in Fig. 2.1 can also be given by the equation of the form

$$s(t) = u\left(\sin\frac{\pi t}{T}\right) - u\left(-\sin\frac{\pi t}{T}\right) \quad \textbf{(2.36)}$$

2.3 The Step Function and Associated Waveforms

- Consider a generalized step function called the sgn function defined as

$$\operatorname{sgn}\left[f(t)\right]=\begin{cases}1 & \forall f(t)>0\\ 0 & \forall f(t)=0\\ -1 & \forall f(t)<0\end{cases} \tag{2.37}$$

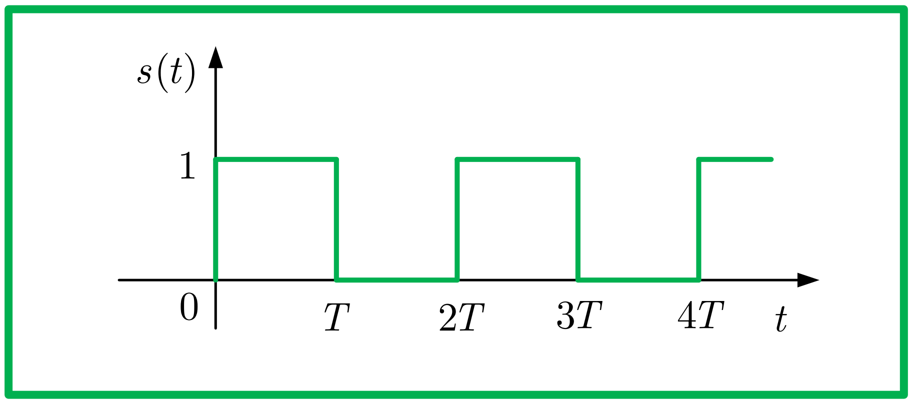

Fig. 2.18: The signal $u(\sin \pi t/T)$.

- The square wave in Fig. 2.1 is thus simply expressed as

2.3 The Step Function and Associated Waveforms Cont'd

$$s(t) = \operatorname{sgn}\left(\sin\frac{\pi t}{T}\right) \quad (2.38)$$

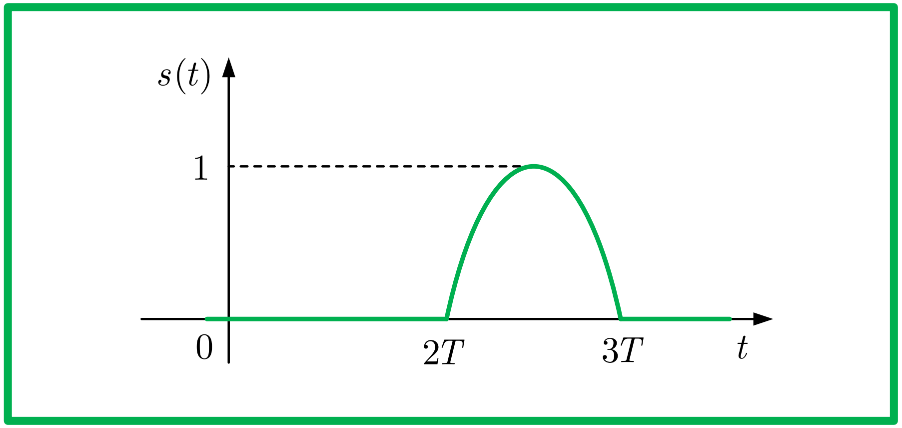

Fig. 2.19: Sine pulse.

- By the shifting property of the step function, the sine pulse in Fig. 2.19 can be represented as

$$s(t) = \sin\frac{\pi t}{T}\left[u(t-2T) - u(t-3T)\right] \quad (2.39)$$

2.3 The Step Function and Associated Waveforms

- The step function is also extremely useful in representing the shifted or delayed version of any signal. For instance, consider the unit ramp function

$$\rho(t) = tu(t) \tag{2.40}$$

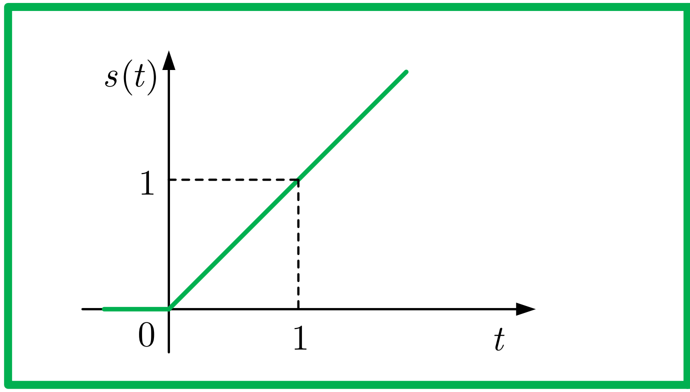

Fig. 2.20: Unit ramp.

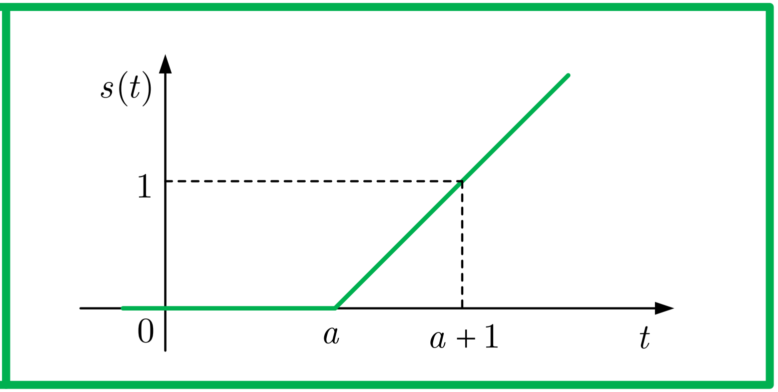

Fig. 2.21: Ramp shifted by $t = a$.

- Let us replace the variable t by a new variable $t' = t - a$. Then

$$\rho(t') = t'u(t') \tag{2.41}$$

2.3 The Step Function and Associated Waveforms Cont'd

- When $\rho(t')$ is plotted against t', the resulting curve is identical to the plot in Fig. 2.20. If, however, we substitute $t - a = t'$ in $\rho(t')$, we then have

$$\rho(t') = (t-a)u(t-a) \tag{2.42}$$

- Plotting $\rho(t')$ against t, we have a the delayed version in Fig. 2.21.
- Vividly, if any signal $f(t)u(t)$ is delayed by a time T, the resultant signal is given by

$$f(t') = f(t-T)u(t-T) \tag{2.43}$$

- **Example.** Let us delay the function $(\sin \pi t/T)\, u(t)$ by a period T. Then

$$s(t') = \left[\sin \frac{\pi}{T}(t-T)\right] u(t-T) \tag{2.44}$$

2.3 The Step Function and Associated Waveforms

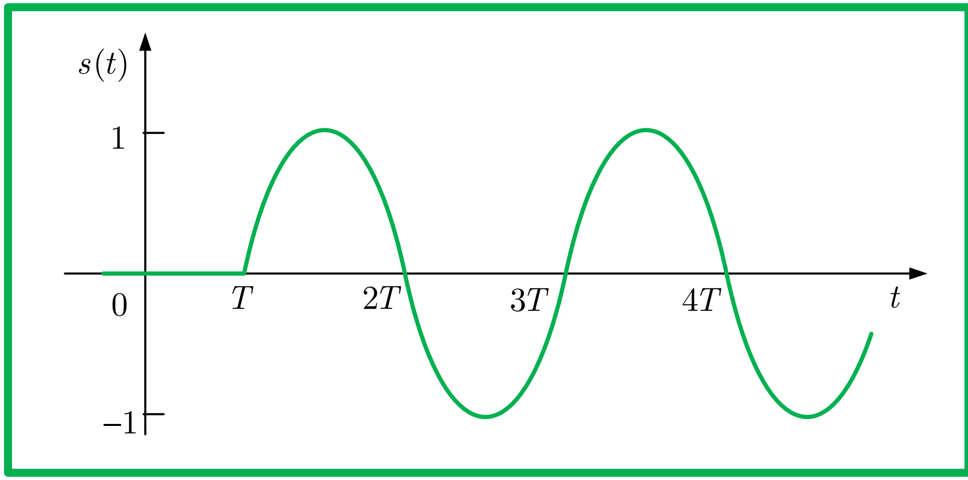

Fig. 2.22: Shifted sine wave.

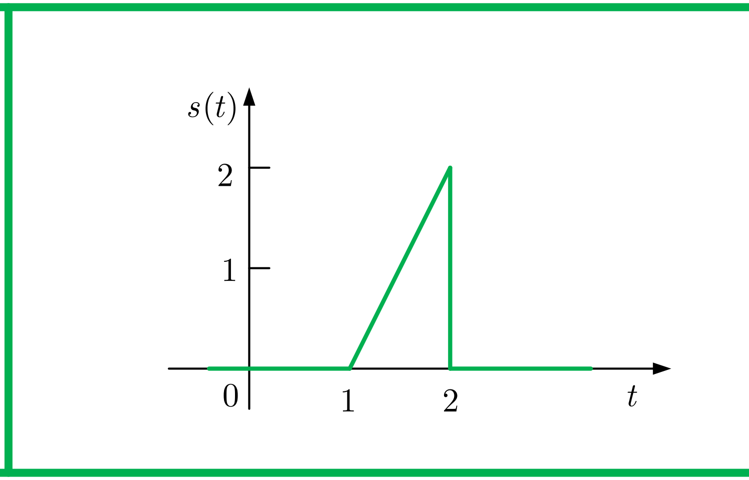

Fig. 2.23: Triangular pulse.

- **Example**. Consider Fig. 2.23, it follows that the equation of the given function in terms of its components is of the form

$$s(t) = 2(t-1)u(t-1) - 2(t-2)u(t-2) - 2u(t-2) \tag{2.45}$$

- The plot of the components is shown in Fig. 2.24.

2.3 The Step Function and Associated Waveforms Cont'd

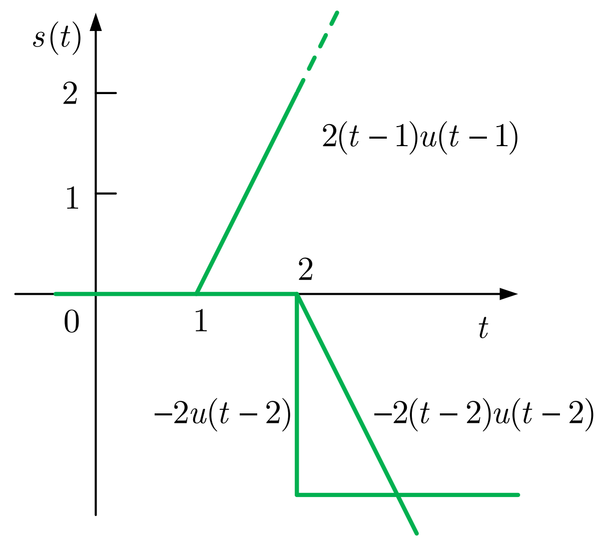

Fig. 2.24: Decomposition of triangular pulse in Fig. 2.23.

2.4 The Unit Impulse

- The unit impulse, or delta function, is a mathematical anomaly, P.A.M. Dirac defined the delta function $\delta(t)$ by the equations

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \tag{2.46}$$

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0 \tag{2.47}$$

- Its most important property is the shifting property, expressed by

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0) \tag{2.48}$$

- From theory, the unit impulse is the derivative of the unit step

$$\delta(t) = u'(t) \tag{2.49}$$

- The above statement is doubtful at first glance. Consider the function $g_{\in}(t)$ in Fig. 2.25. Clearly,

2.4 The Unit Impulse Cont'd

$$\lim_{\epsilon \to 0} g_{\epsilon}(t) = u(t) \qquad \textbf{(2.50)}$$

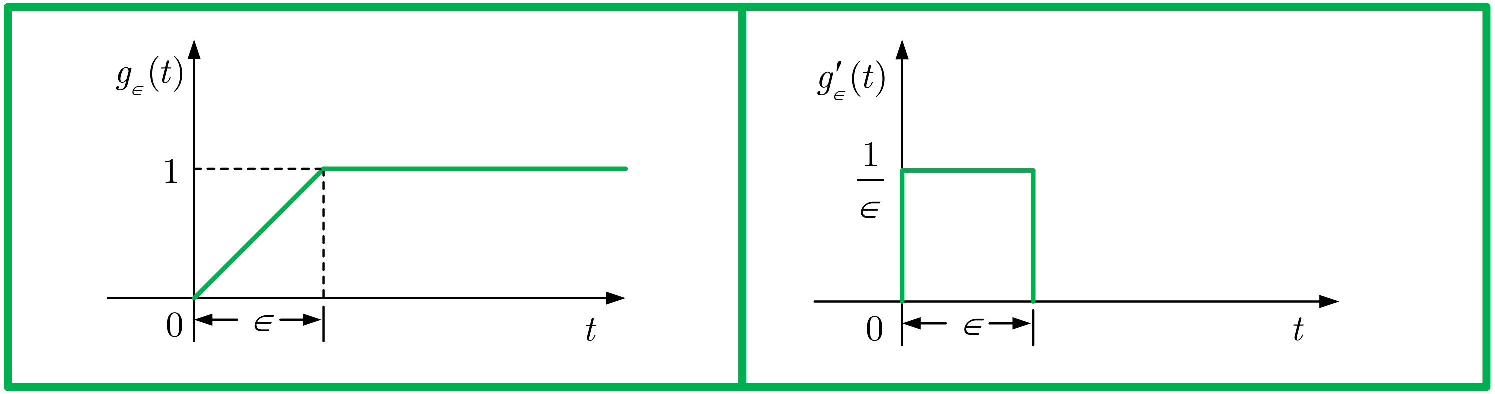

Fig. 2.25: Unit step when $\epsilon \to 0$. **Fig. 2.26**: Derivative of $g_{\epsilon}(t)$ in Fig. 2.25.

- Taking the derivative of $g_{\epsilon}(t)$, we obtain $g'_{\epsilon}(t)$ shown in Fig. 2.26, which is given as

$$g'_{\epsilon}(t) = \begin{cases} \dfrac{1}{\epsilon}; & \text{for} \quad 0 \le t \le \epsilon \\ 0; & \text{for} \quad t < 0, \ t > \epsilon \end{cases} \qquad \textbf{(2.51)}$$

2.4 The Unit Impulse Cont'd

- Now let $\in$ take on a sequence of values $\in_i$ such that $\in_i > \in_{i+1}$. Consider the sequence of functions $\{g'_{\in_i}(t)\}$ for decreasing values of $\in_i$, shown in Fig. 2.27. The sequence has the following property:

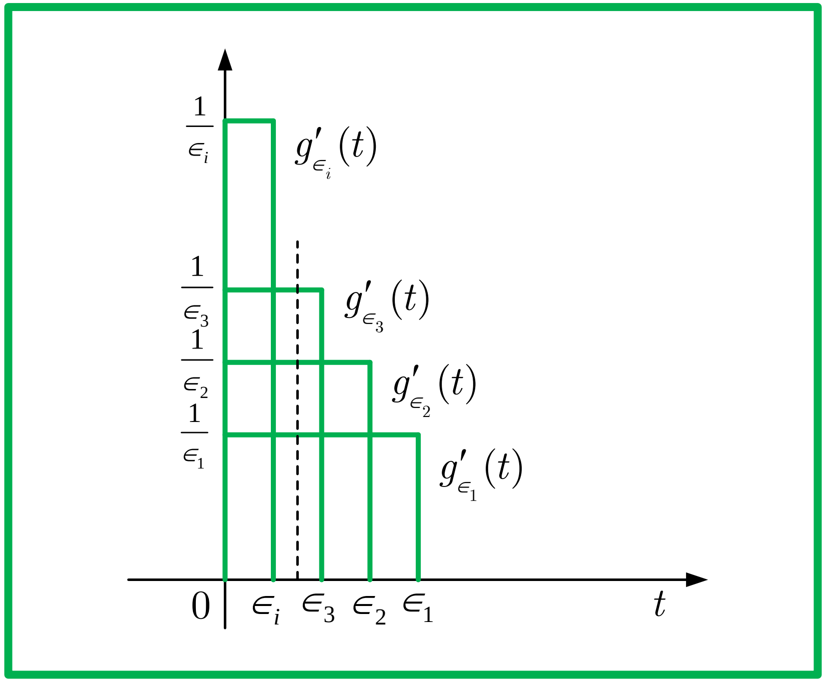

Fig. 2.27: The sequence $\{g'_{\in_i}(t)\}$.

2.4 The Unit Impulse Cont'd

$$\lim_{\epsilon \to 0} \int_{t_1<0}^{t_2>0} g'_{\epsilon_i}(t)\,dt = 1 \qquad \textbf{(2.52)}$$

where t_1 and t_2 are arbitrary real numbers. For every nonzero value of ϵ there corresponds a well-behaved function (i.e., it does not blow up) $g'_{\epsilon_i}(t)$. As ϵ_i approaches zero

$$g'_{\epsilon_i}(0+) \underset{\epsilon_i \to 0}{\longrightarrow} \infty \qquad \textbf{(2.53)}$$

- Another sequence of functions which obeys the property given in Eq. 2.52 is $\{f_{\epsilon_i}(t)\}$ in Fig. 2.28. Thus, we define the unit impulse $\delta(t)$ as the class of all sequences of functions which obey Eq. 2.52. In particular, defined as

2.4 The Unit Impulse Cont'd

$$\int_{t_1<0}^{t_2>0} \delta(t)\,dt \stackrel{\Delta}{=} \lim_{\epsilon\to 0} \int_{t_1<0}^{t_2>0} g'_{\epsilon_i}(t)\,dt \stackrel{\Delta}{=} \lim_{\epsilon\to 0} \int_{t_1<0}^{t_2>0} f_{\epsilon_i}(t)\,dt \qquad (2.54)$$

- It is worth noting that the this is not a rigorous definition, it is as a matter of fact a heuristic one. A more rigorous approach is found in Appendix B of our reference book.
- From the previous definition we can think of the delta function as having the additional properties,

$$\begin{aligned} \delta(0) &= \infty \\ \delta(t) &= 0 \quad \text{for} \quad t \neq 0 \end{aligned} \qquad (2.55)$$

2.4 The Unit Impulse Cont'd

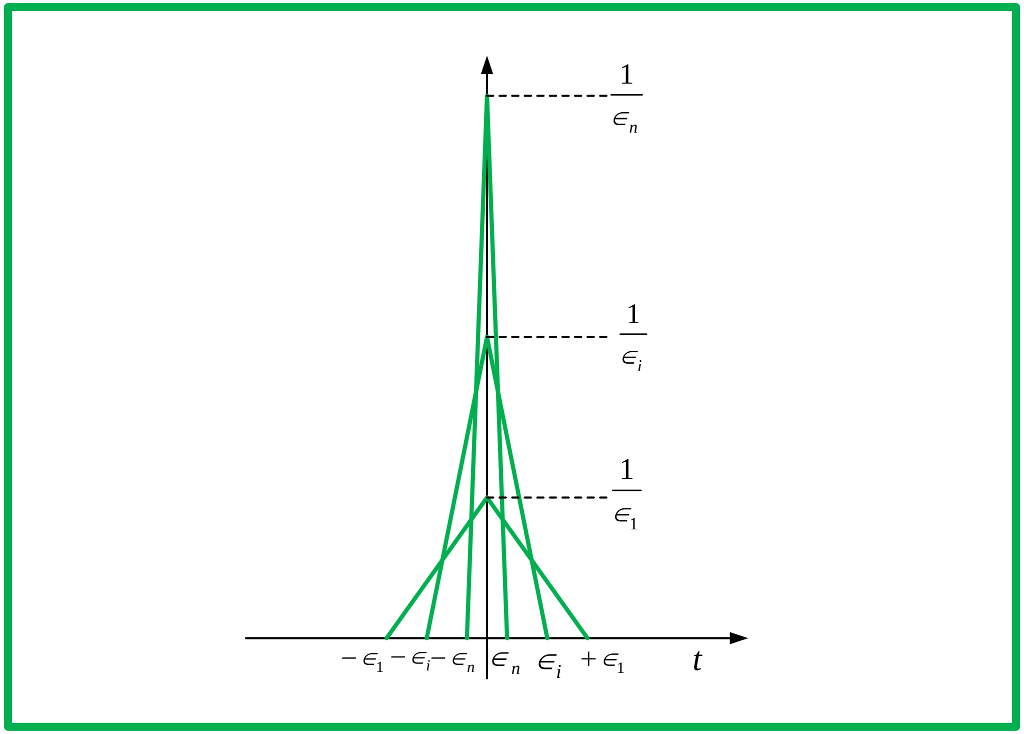

Fig. 2.28: The sequence $\{f_{\epsilon_i}(t)\}$.

- Continuing with the heuristic approach, we say that the area under the impulse is unity, and, since the impulse is zero for $t \neq 0$, we have

2.4 The Unit Impulse Cont'd

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \int_{0-}^{0+} \delta(t)\,dt = 1 \qquad (2.56)$$

- Thus the entire area of the impulse is concentrated at $t = 0$. Thus, any integral that does not integrate through $t = 0$ is zero, as seen by

$$\int_{-\infty}^{0-} \delta(t)\,dt = \int_{0+}^{+\infty} \delta(t)\,dt = 0 \qquad (2.57)$$

- The change of scale and time shift properties discussed earlier also apply for the impulse function. The derivative of a step function yields an impulse function. For Example, given

$$s(t) = Au(t-a) \qquad (2.58)$$

$$s'(t) = A\delta(t-a) \qquad (2.59)$$

2.4 The Unit Impulse Cont'd

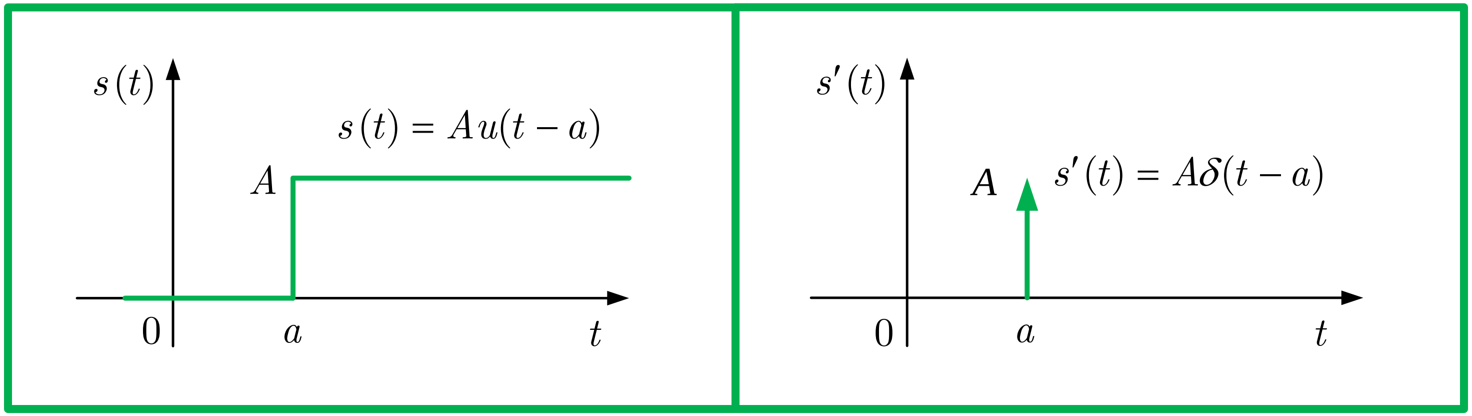

Fig. 2.29: Step and Impulse functions

- Graphically, we represent an impulse function by an arrowhead pointing upward, with the constant multiplier *A* written next to the arrowhead. Note that A is the area under the impulse $A\delta(t-a)$.
- From Eqs. 2.58 and 2.59, the derivative of the step at the jump discontinuity of height *A* yields an impulse of area *A* at the same point $t = T$.
- Generalizing this argument, consider any function $f(t)$ with a jump discontinuity at $t = T$.

2.4 The Unit Impulse Cont'd

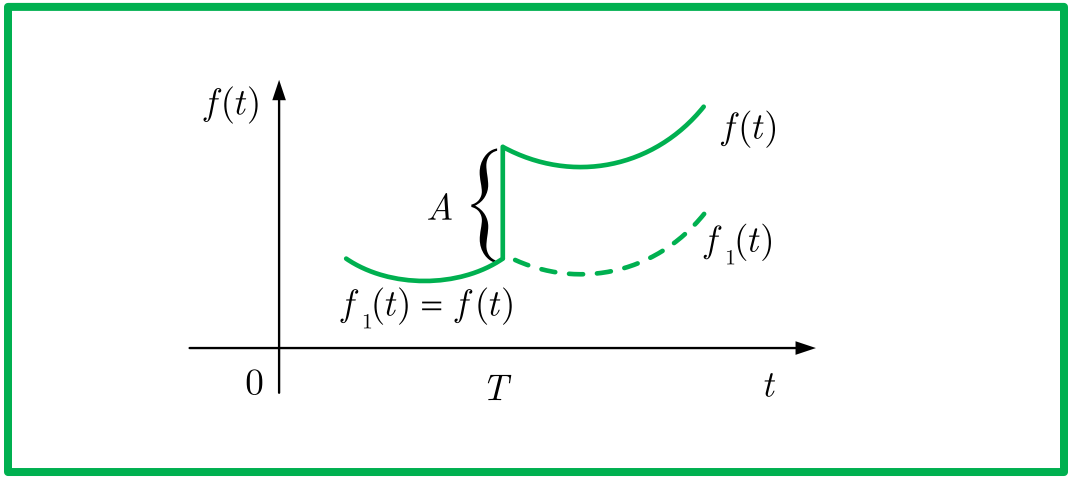

Fig. 2.30: Function with discontinuity at T.

- Then the derivative, $f'(t)$ must have an impulse at $t = T$. As an example, consider $f(t)$ in Fig. 2.30. At $t = T$, $f(t)$ has a discontinuity of height A, which is given as

$$A = f(T+) - f(T-) \qquad \textbf{(2.60)}$$

- Let us define $f_1(t)$ as being equal to $f(t)$ for $t < T$, and having the same shape as the latter, but without the discontinuity for $t < T$, that is,

2.4 The Unit Impulse Cont'd

$$f_1(t) = f(t) - Au(t-T) \tag{2.61}$$

- The derivative $f'(t)$ is then

$$f'(t) = f_1'(t) + A\delta(t-T) \tag{2.62}$$

- Example, to illustrate this point more clearly. In Fig. 2.31*a*, the function $f(t)$ is

$$f(t) = Au(t-a) - Au(t-b) \tag{2.63}$$

Its derivative is $$f'(t) = A\delta(t-a) - A\delta(t-b) \tag{2.64}$$

and is shown in Fig. 2.31*b*. Since $f(t)$ has two discontinuities, at $t = a$ and $t = b$ its derivative must have impulses at those points. The coefficient of the impulse at $t = b$ is negative because

2.4 The Unit Impulse Cont'd

$$f(b+) - f(b-) = -A \tag{2.65}$$

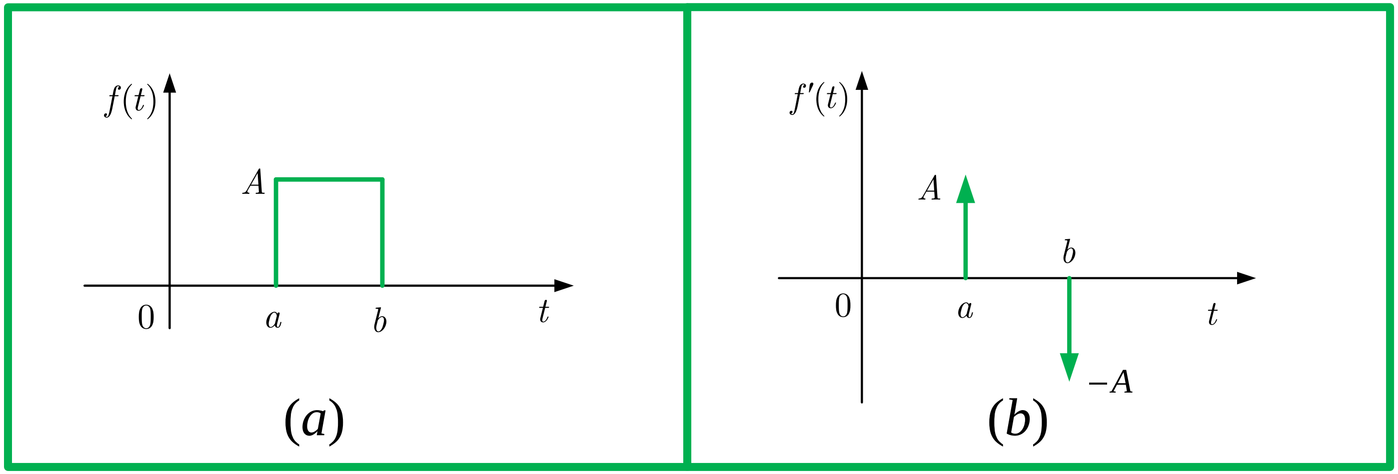

Fig. 2.31: (a) Square pulse. (b) Derivative of the square pulse.

- As a second example, consider the function $g(t)$ shown in Fig. 2.32a. We obtain $g'(t)$ by inspection, and note that the discontinuity at $t = 1$ produces the impulse in $g'(t)$ of area

$$g(1+) - g(1-) = 2 \tag{2.66}$$

2.4 The Unit Impulse Cont'd

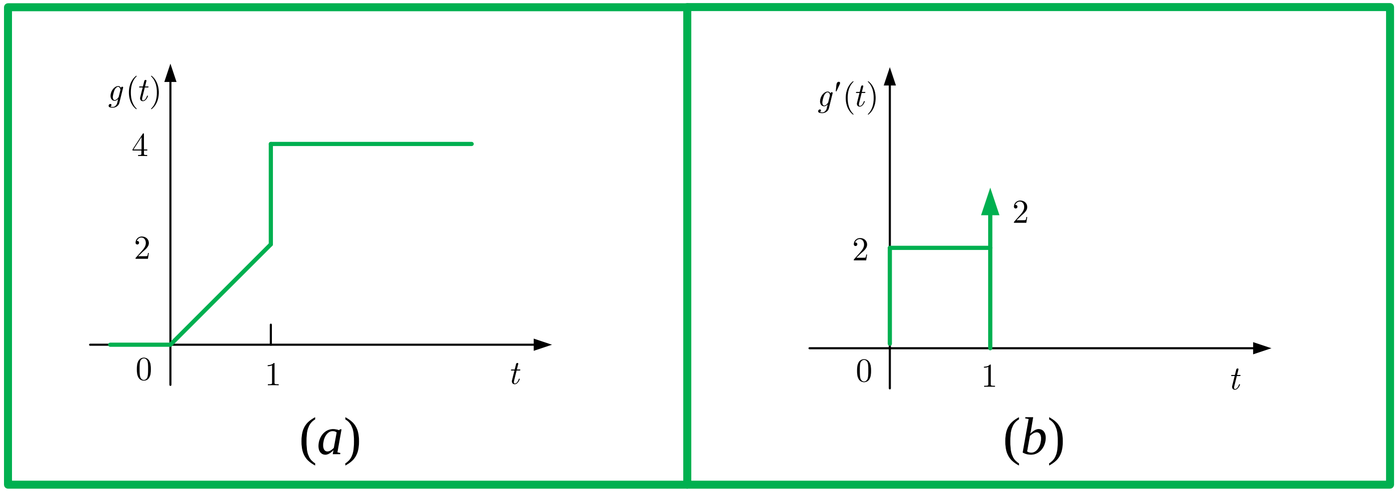

Fig. 2.32: (a) Signal. (b) Derivative.

- Another property of the impulse function is expressed by the integral

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-T)dt = f(T) \tag{2.67}$$

2.4 The Unit Impulse Cont'd

- This integral is easily evaluated if we consider that $\delta(t-T)=0$ for all $t \neq T$. Therefore, the product

$$f(t)\delta(t-T)=0 \quad \forall \ t \neq T \tag{2.68}$$

- If $f(t)$ is single-valued at $t=T$, $f(T)$ can be factored from the integral so that we obtain

$$f(T)\int_{-\infty}^{\infty}\delta(t-T)\,dt = f(T) \tag{2.69}$$

- Figure 2.33 shows $f(t)$ and $\delta(t-T)$, where $f(t)$ is continuous at $t=T$.
  If $f(t)$ has a discontinuity at $t=T$, the integral

$$\int_{-\infty}^{+\infty} f(t)\delta(t-T)\,dt$$

is not defined because the value of $f(T)$ is not uniquely given.

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2.4 The Unit Impulse Cont’d

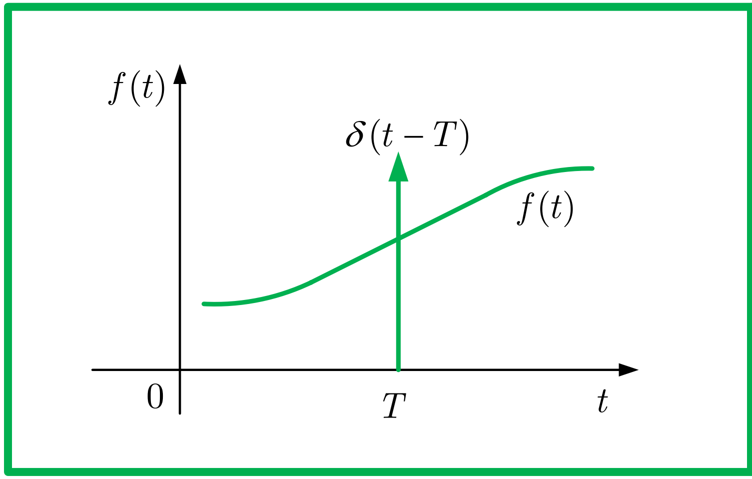

Fig. 2.33

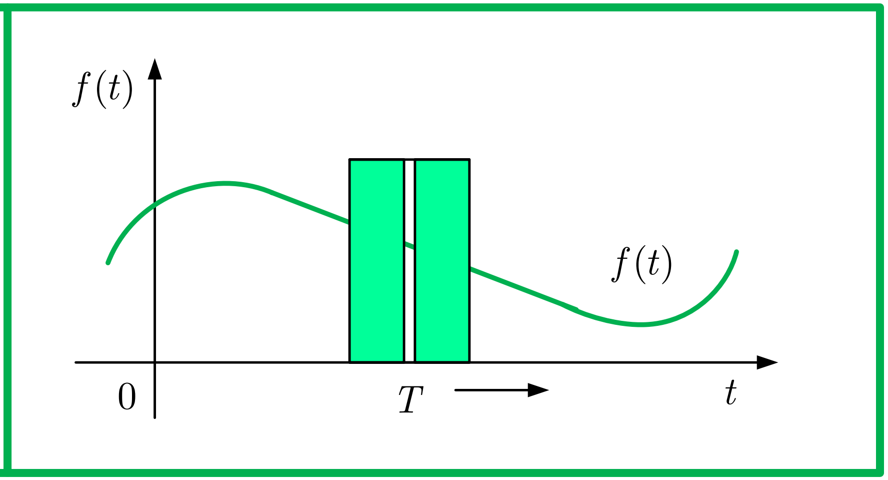

Fig. 2.34: Impulse scanning.

[Example 2.2] Evaluation of Integral Using Impulse

Evaluate the integral having an exponential function $f(t) = e^{j\omega t}$;

[Solution]

Eqn. 2.67 yields

$$\int_{-\infty}^{+\infty} e^{j\omega t}\delta(t-T)\,dt = e^{j\omega T} \tag{2.70}$$

[Example 2.3] Evaluation of Integral Using Impulse

Evaluate the given integral having $f(t) = \sin t;$ $\int_{-\infty}^{+\infty} \sin t \, \delta\left(t - \frac{\pi}{4}\right) dt$

[Solution]

Vividly Eqn. 2.67 yields $\int_{-\infty}^{+\infty} \sin t \, \delta\left(t - \frac{\pi}{4}\right) dt = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ **(2.71)**

- Consider next the case where $f(t)$ is continuous for $-\infty < t < \infty$. Let us direct our attention to the integral

$$\int_{-\infty}^{\infty} f(t)\delta(t - T)\, dt = f(T) \quad \textbf{(2.72)}$$

which holds for all t in this case. If T were varied from $-\infty$ to $+\infty$, then $f(t)$ would be reproduced in its entirety. An operation of this sort corresponds to scanning the function $f(t)$ by moving a sheet of paper with a thin slit across a plot of the function, shown in Fig. 2.34.

2.4 The Unit Impulse Cont'd

- We examine higher order derivatives of the unit impulse function. Here we let the unit impulse function be $f_\in(t)$ in Fig. 2.28, which as $\in \to 0$, becomes the unit impulse. Fig. 2.35 shows the derivative $f'_\in(t)$.
- Thus, as $\in \to 0$, $f'_\in(t)$ approaches the derivative of the unit impulse $\delta'(t)$, which consists of a pair of impulses as seen in Fig. 2.36. The area under the doublet $\delta'(t)$, is equal to zero. Thus,

$$\int_{-\infty}^{\infty} \delta'(t)\,dt = 0 \qquad \textbf{(2.73)}$$

- The other significant property of the doublet is

$$\int_{-\infty}^{\infty} f(t)\delta'(t-T)\,dt = -f'(T) \qquad \textbf{(2.74)}$$

2.4 The Unit Impulse Cont’d

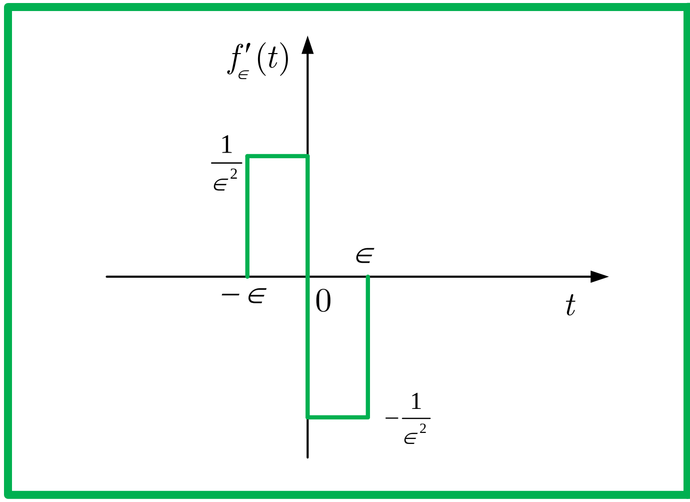

Fig. 2.35: Unit doubles as $\epsilon \rightarrow 0$.

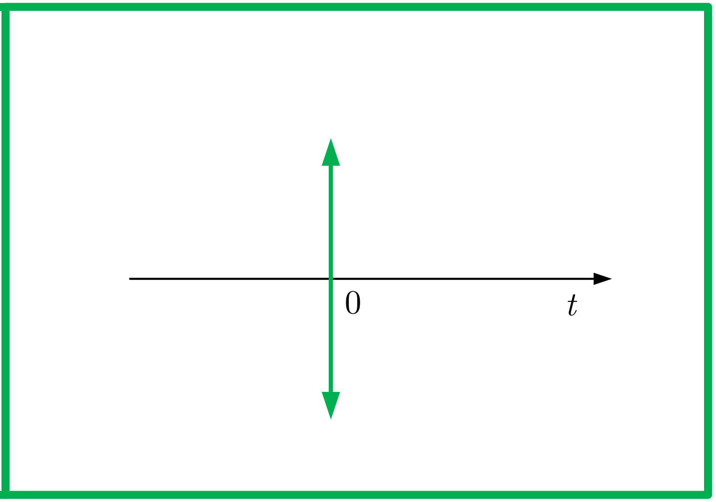

Fig. 2.36: The doublet.

where $f'(T)$ is the derivative of $f(t)$ evaluated at $t = T$ where, again, we assume that $f(t)$ is continuous.

2.4 The Unit Impulse Cont'd

Proof of Eqn. 2.74 using integration by parts

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t-T)\,dt = f(t)\,\delta(t-T)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\,\delta(t-T)\,dt \qquad \textbf{(2.75)}$$

$$= -f'(T)$$

- In general

$$\boxed{\int_{-\infty}^{\infty} f(t)\,\delta^{(n)}(t-T)\,dt = (-1)^n f^{(n)}(T)} \qquad \textbf{(2.76)}$$

where $\delta^{(n)}$ and $f^{(n)}$ denote nth derivatives. The higher order derivatives of $\delta(t)$ can be evaluated in similar fashion.

End of Lecture 2

Thank you for your attention!