EEE 3121 - Signals & Systems

Lecture 3: The Frequency domain: Fourier analysis

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References

Our main reference text book in this course is

[1] B. P. Lathi and R. A. Green, Linear Systems and Signals, 3rd Ed., 2018, Oxford University Press, New York. ISBN 978-0-19-020017-6

[2] Kuo Franklin, F., Network Analysis and Synthesis, 3rd Ed., 1986, J. Wiley (SE), ISBN 0-471-51118-8.

[3] Sundararanjan, D., A Practical Approach to Signals and Systems, 2008, John Wiley & Sons (Asia) Pte Ltd, ISBN 978-0-470-82353-8.

However, feel free to use pretty much any additional text which you might find relevant to our course.

3.1 Introduction

- One of the most common classes of signals encountered are periodic signals. If T_0 is the period of the signal, then

$$s(t) = s(t \pm nT_0) \qquad n = 0, 1, 2, \ldots \tag{3.1}$$

- In addition, if $s(t)$ has only a finite number of discontinuities in any finite period and if the integral

$$\int_{\alpha}^{\alpha+T_0} \left| s(t) \right| dt < \infty$$

is finite (where α is an arbitrary real number), then $s(t)$ can be expanded into the infinite trigonometric series

$$\begin{aligned} s(t) = a_0 &+ a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + \cdots \\ &+ b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + \cdots \end{aligned} \tag{3.2}$$

3.1 Introduction Cont'd

- Here $\omega_0 = 2\pi/T_0$. This series is known as the Fourier series. In compact form, the Fourier series is

$$s(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \textbf{(3.3)}$$

- Thus, when $s(t)$ can be described completely in terms of the coefficients of its harmonic terms. These coefficients constitute a frequency domain description of the signal.
- Our task now is to derive the equations for the coefficients a_n, b_n in terms of the given signal function $s(t)$.
- Lets us first focus our energy and discussion on the mathematical basis of Fourier series, the theory of orthogonal sets.

3.2 Orthogonal functions

- Consider any two functions $f_1(t)$ and $f_2(t)$ that are not identically zero. Then if

$$\int_{T_1}^{T_2} f_1(t) f_2(t)\, dt = 0 \qquad \textbf{(3.4)}$$

- We say that $f_1(t)$ and $f_2(t)$ are orthogonal over the interval $[T_1, T_2]$. For instance, the functions $\sin t$ and $\cos t$ are orthogonal over the interval $n2\pi \le t \le (n+1)2\pi$. Consider next a set of real functions $\{\phi_1(t), \phi_2(t), \ldots, \phi_n(t)\}$. If the functions obey the condition

$$\langle \phi_i, \phi_j \rangle \equiv \int_{T_1}^{T_2} \phi_i(t)\phi_j(t)\, dt = \begin{cases} 0, & \text{for } i \ne j \\ \alpha, & \text{for } i = j \end{cases} \qquad \textbf{(3.5)}$$

where $\alpha \ne 0$, then the set $\{\phi_i(t)\}$ forms an orthogonal set over $[T_1, T_2]$. In Eq. 3.5 the integral is denoted by the inner product $\langle \phi_i, \phi_j \rangle$. For convenience here, we use the inner product notation in our discussions.

3.2 Orthogonal functions Cont'd

- The set $\{\phi_i(t)\}$ is orthonormal over $[T_1, T_2]$ if

$$\langle \phi_i, \phi_j \rangle = \begin{cases} 0, & \forall \;\; i \neq j \\ 1, & \forall \;\; i = j \end{cases} \qquad \textbf{(3.6)}$$

- The norm of an element ϕ_k in the set $\{\phi_i(t)\}$ is defined as

$$\|\phi_k\| = \langle \phi_k, \phi_k \rangle^{1/2} = \left[\int_{T_1}^{T_2} \phi_k^2(t)\, dt \right]^{1/2} \qquad \textbf{(3.7)}$$

- Any orthogonal set $\{\phi_1(t), \phi_2(t), \ldots, \phi_n(t)\}$ can be normalized by dividing each term ϕ_k by its norm $\|\phi_k\|$.

Example 3.1 Orthogonal functions

- The Laguerre set, which has been shown to be very useful in time domain approximation, has the first four terms of as

$$\phi_1(t) = e^{-at}\ ; \quad \phi_2(t) = e^{-at}\left[1-2(at)\right]; \quad \phi_3(t) = e^{-at}\left[1-4(at)+2(at)^2\right];$$

$$\phi_4(t) = e^{-at}\left[1-6(at)+6(at)^2-\frac{4}{3}(at)^3\right] \quad \cdots \tag{3.8}$$

- Show that the Laguerre set is orthogonal over $[0,\ \infty]$.

[Solution]

- To show that the set is orthogonal, let us consider the integral

$$\int_0^\infty \phi_1(t)\phi_3(t)\,dt = \int_0^\infty e^{-2at}\left[1-4(at)+2(at)^2\right]dt \tag{3.9}$$

[Example 3.1] Orthogonal functions Cont'd

- Letting $\tau = at$, we have

$$\left\langle \phi_1, \phi_3 \right\rangle = \frac{1}{a}\int_0^\infty e^{-2\tau}(1-4\tau+2\tau^2)d\tau$$
$$= \frac{1}{a}\left[\frac{1}{2} - 4\left(\frac{1}{4}\right) + 2\left(\frac{2}{8}\right)\right] = 0 \quad \textbf{(3.10)}$$

- The norms of $\phi_1(t)$ and $\phi_2(t)$ are

$$\left\|\phi_1\right\| = \left(\int_0^\infty e^{-2at}dt\right)^{1/2} = \frac{1}{\sqrt{2a}} \quad \textbf{(3.11)}$$

$$\left\|\phi_2\right\| = \left(\int_0^\infty e^{-2at}\left[1-4(at)+4(at)^2\right]dt\right)^{1/2}$$
$$= \frac{1}{\sqrt{2a}} \quad \textbf{(3.12)}$$

Example 3.1 Orthogonal functions Cont'd

- It is trivial to verify that the norms of all the elements in the set are also equal to $1/\sqrt{2a}$. Therefore, to render the Laguerre set orthonomal, we divide each element ϕ_i by $1/\sqrt{2a}$.

3.3 Approx. Using Orthogonal functions

- We now explore some uses of orthogonal functions in linear approximation of functions. The key problem is approximating a function $f(t)$ by a sequence of functions $f_n(t)$ such that the mean squared error (MSE) is

$$\epsilon = \lim_{n\to\infty} \int_{T_1}^{T_2} \left[f(t) - f_n(t) \right]^2 dt = 0 \qquad \textbf{(3.13)}$$

- When Eq. 3.13 is satisfied, we say that $\{f_n(t)\}$ converges in the mean to $f(t)$.
- To examine the concept of convergence in the mean more closely, we must consider the following definitions:

Definition 3.1 Given a function $f(t)$ and constant $p > 0$ for which

$$\int_{T_1}^{T_2} |f(t)|^p \, dt < \infty$$

we say that $f(t)$ is integrable L^p in $[T_1, T_2]$, we write $f(t) \in L^p$ in $[T_1, T_2]$.

3.3 Approx. Using Orthogonal functions Cont'd

Definition 3.2 If $f(t) \in L^p$ in $[T_1, T_2]$, and $\{f_n(t)\}$ is a sequence of functions integrable L^p in $[T_1, T_2]$, we say that if

$$\lim_{n\to\infty} \int_{T_1}^{T_2} \left[f(t) - f_n(t) \right]^p dt = 0$$

then $\{f_n(t)\}$ converges in the mean order p to $f(t)$. Specifically, when $p = 2$ we say that $\{f_n(t)\}$ converges in the mean to $f(t)$.

- The principle of least squares. Consider the case when $f_n(t)$ consists of a linear combination of orthonormal functions $\phi_1, \phi_2, \ldots, \phi_n$.

$$f_n(t) = \sum_{i=1}^{n} a_i \phi_i(t) \tag{3.14}$$

3.3 Approx. Using Orthogonal functions Cont'd

- Our problem is to determine the constants a_i such that the integral squared error

$$\left\| f - f_n \right\|^2 = \int_{T_1}^{T_2} \left[f(t) - f_n(t) \right]^2 dt \qquad \textbf{(3.15)}$$

is a minimum. The principle of least squares states that in order to attain minimum squared error, the constants a_i must have values

$$c_i = \int_{T_1}^{T_2} f(t) \phi_i(t) dt \qquad \textbf{(3.16)}$$

Proof. We show that in order for $\left\| f - f_n \right\|^2$ to be minimum, we must set $a_i = c_i$ for every $i = 1, 2, \ldots, n$.

$$\begin{aligned} \left\| f - f_n \right\|^2 &= \langle f, f \rangle - 2 \langle f, f_n \rangle + \langle f_n, f_n \rangle \\ &= \left\| f \right\|^2 - 2 \sum_{i=1}^{n} a_i \langle f, \phi_i \rangle + \sum_{i=1}^{n} a_i^{\,2} \left\| \phi_i \right\|^2 \end{aligned} \qquad \textbf{(3.17)}$$

3.3 Approx. Using Orthogonal functions Cont'd

- Since the set $\{\phi_i\}$ is orthonormal, $\|\phi_i\|^2 = 1$, and by definition $c_i = \langle f, \phi_i \rangle$. We thus have

$$\|f - f_n\|^2 = \|f\|^2 - 2\sum_{i=1}^{n} a_i c_i + \sum_{i=1}^{n} a_i^2 \quad \textbf{(3.18)}$$

- Adding and subtracting $\sum_{i=1}^{n} c_i^2$ yields

$$\begin{aligned}\|f - f_n\|^2 &= \|f\|^2 - 2\sum_{i=1}^{n} a_i c_i + \sum_{i=1}^{n} c_i^2 + \sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} c_i^2 \\ &= \|f\|^2 + \sum_{i=1}^{n} (c_i - a_i)^2 - \sum_{i=1}^{n} c_i^2\end{aligned} \quad \textbf{(3.19)}$$

- We see that in order to attain minimum integral squared error, we must set $a_i = c_i$. The coefficients c_i , defined in Eq. 3.16 are called the Fourier Coefficients of $f(t)$ with respect to orthonormal set $\{\phi_i(t)\}$.

3.3 Approx. Using Orthogonal functions Cont'd

- Parseval's equality. Consider $f_n(t)$ given in Eq. 3.14. We see that

$$\int_{T_1}^{T_2} \left[f_n(t) \right]^2 dt = \sum_{i=1}^{n} c_i^{\,2} \qquad \textbf{(3.20)}$$

since ϕ_i are orthonormal functions. This result is known as *Parseval's equality*, and is important in determining the energy of a periodic signal.

3.4 Fourier Series

- Let us return to the Fourier series as defined earlier.

$$s(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad \textbf{(3.21)}$$

- By the approximation using orthonormal functions just discussed, we see that a periodic function $s(t)$ with period T_0 can be approximated by a Fourier series $s_n(t)$ such that $s_n(t)$ converges in the mean to $s(t)$, that is,

$$\lim_{\alpha \to \infty} \int_{\alpha}^{\alpha+T} \left[s(t) - s_n(t) \right]^2 dt \to 0 \qquad \textbf{(3.22)}$$

where α is any real number. We know, that if n is finite, the mean squared error $\left\| s(t) - s_n(t) \right\|^2$ is minimized when the constants a_i, b_i are Fourier coefficients of $s(t)$ with respect to the orthonormal set

$$\left\{ \frac{\cos k\omega_0 t}{(T_0/2)^{1/2}}, \frac{\sin k\omega_0 t}{(T_0/2)^{1/2}} \right\}, \qquad k = 0, 1, 2, \ldots$$

3.4 Fourier Series Cont'd

In explicit form the Fourier coefficients, according to the definition given earlier, are obtained from the equations

$$a_0 = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} s(t)\,dt \qquad \textbf{(3.23)}$$

$$a_k = \frac{2}{T_0}\int_{\alpha}^{\alpha+T_0} s(t)\cos k\omega_0 t\,dt \qquad \textbf{(3.24)}$$

$$b_k = \frac{2}{T_0}\int_{\alpha}^{\alpha+T_0} s(t)\sin k\omega_0 t\,dt \qquad \textbf{(3.25)}$$

3.4 Fourier Series Cont'd

- It is worth noting that because the Fourier series $s_n(t)$ only converges to $s(t)$ in the mean, when $s(t)$ contains a jump discontinuity, for example, at t_0

$$s_n(t_0) = \frac{s(t_0+) + s(t_0-)}{2} \quad \textbf{(3.26)}$$

- At any point t_1 that $s(t)$ is differentiable $s_n(t_1)$ converges to $s(t_1)$.

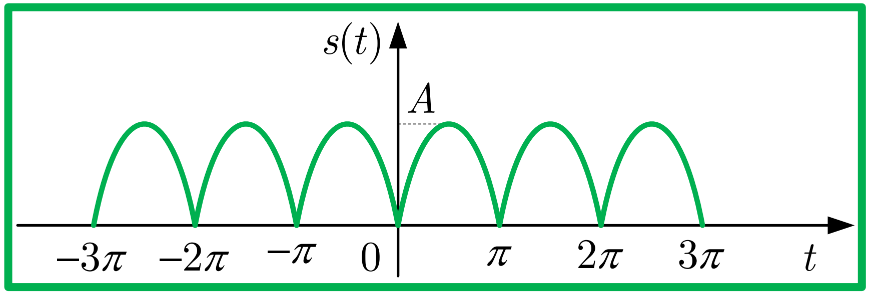


Fig. 3.1: Rectified sine wave.

[Example 3.2] Fourier Series

Determine the Fourier coefficients of the fully rectified sine wave in Fig. 3.1. The period is $T_0 = \pi$ so that the fundamental frequency is $\omega_0 = 2$. The signal is given as

$$s(t) = A\left|\sin t\right| \quad \textbf{(3.27)}$$

[Solution]

Lets us take $\alpha = 0$ and evaluate between 0 and π. Using the formulae just derived, we have

$$b_n = \frac{2}{\pi}\int_0^{\pi} s(t)\sin 2nt\,dt = \frac{2A}{\pi}\int_0^{\pi} \sin t \sin 2nt\,dt = 0 \quad \textbf{(3.28)}$$

$$a_0 = \frac{A}{\pi}\int_0^{\pi} \sin t\,dt = -\frac{A}{\pi}\cos t\Big|_0^{\pi} = \frac{2A}{\pi} \quad \textbf{(3.29)}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} s(t)\cos 2nt\,dt = \frac{2}{\pi}\int_0^{\pi} A\sin t\cos 2nt\,dt$$

[Example 3.2] Fourier Series Cont'd

- Use of the trigonometric identity $A\sin t\cos 2nt = \frac{A}{2}[\sin(1+2n)t + \sin(1-2n)t]$ yields

$$a_n = \frac{2}{\pi}\int_0^{\pi}\frac{A}{2}[\sin(1+2n)t + \sin(1-2n)t]\,dt\,;\ \text{ i.e.,}$$

$$a_n = -\frac{A}{\pi}\left[\frac{\cos(1+2n)t}{1+2n} + \frac{\cos(1-2n)t}{1-2n}\right]_0^{\pi};\qquad \therefore\ a_n = \frac{1}{1-4n^2}\frac{4A}{\pi} \tag{3.30}$$

- Thus the Fourier series of the rectified sine wave is

$$s(t) = \frac{4A}{2\pi}\left\{1 + \sum_{n=1}^{\infty}\frac{2}{1-4n^2}\cos 2nt\right\} \tag{3.31}$$

3.5 Evaluation of Fourier Coefficients

- We now consider two other useful forms of Fourier series. From Eqs. 3.23-3.25, let $\alpha = -T_0/2$ and we represent the integrals as the sum of two separate parts, that is

$$a_n = \frac{2}{T_0}\left[\int_0^{T_0/2} s(t)\cos n\omega_0 t\,dt + \int_{-T_0/2}^{0} s(t)\cos n\omega_0 t\,dt\right]$$
$$b_n = \frac{2}{T_0}\left[\int_0^{T_0/2} s(t)\sin n\omega_0 t\,dt + \int_{-T_0/2}^{0} s(t)\sin n\omega_0 t\,dt\right] \quad \textbf{(3.32)}$$

- Since the variable (t) in the above integrals is a dummy variable, let us substitute $x = t$ in the integrals with limits $(0;\ T_0/2)$, and let $x = -t$ in the integrals with limits $(-T_0/2;\ 0)$. Then we have

$$a_n = \frac{2}{T_0}\int_0^{T_0/2}\left[s(x) + s(-x)\right]\cos n\omega_0 x\,dx$$
$$b_n = \frac{2}{T_0}\int_0^{T_0/2}\left[s(x) - s(-x)\right]\sin n\omega_0 x\,dx \quad \textbf{(3.33)}$$

3.5 Evaluation of Fourier Coefficients Cont'd

- Suppose now the function is odd, that is, $s(x) = -s(-x)$, then we see that $a_n = 0$ for all n, and

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} s(x)\sin n\omega_0 x\,dx \tag{3.34}$$

- The implication is that, the Fourier series of an odd function will only contain *sine terms*.

- Suppose the function is even, that is, $s(x) = s(-x)$, then $b_n = 0$ and

$$a_n = \frac{4}{T_0}\int_0^{T_0/2} s(x)\cos n\omega_0 x\,dx \tag{3.35}$$

3.5 Evaluation of Fourier Coefficients Cont'd

- Consequently, the Fourier series of an even function will contain only cosine terms.

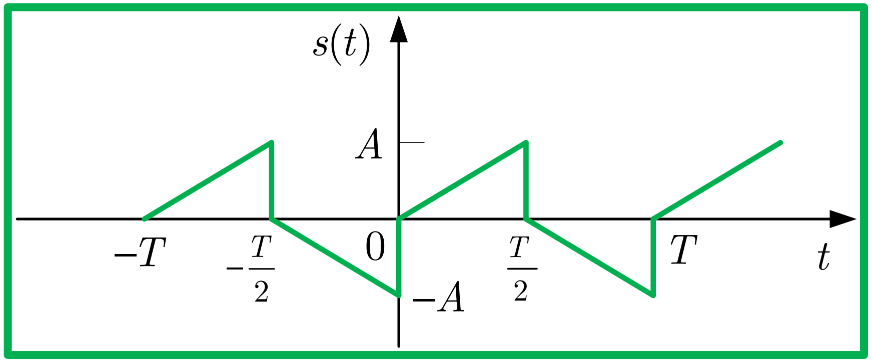


Fig. 3.2

- Suppose next, the function $s(t)$ obeys the condition

$$s\left(t \pm \frac{T}{2}\right) = -s(t) \qquad \textbf{(3.36)}$$

as given by the example in Fig. 3.2.

3.5 Evaluation of Fourier Coefficients Cont'd

- We show that $s(t)$ given in Fig. 3.2 contains only odd harmonic terms, that is,

$$\begin{aligned} a_n &= b_n = 0; \quad n \text{ even} \\ a_n &= \frac{4}{T_0}\int_0^{T_0/2} s(t)\cos n\omega_0 t\, dt \\ b_n &= \frac{4}{T_0}\int_0^{T_0/2} s(t)\sin n\omega_0 t\, dt, \quad n \text{ odd} \end{aligned} \tag{3.37}$$

and

- With knowledge of symmetry conditions, let us examine how we can approximate an arbitrary time function $s(t)$ by a Fourier series within an interval $[0, T]$. Outside this interval, the Fourier series $s_n(t)$ is not required to fit $s(t)$.

3.5 Evaluation of Fourier Coefficients Cont'd

- Consider the signal $s(t)$ in Fig. 3.3. We can approximate $s(t)$ by any periodic functions shown in Fig. 3.4. Observe that each periodic waveform exhibits some sort of symmetry.

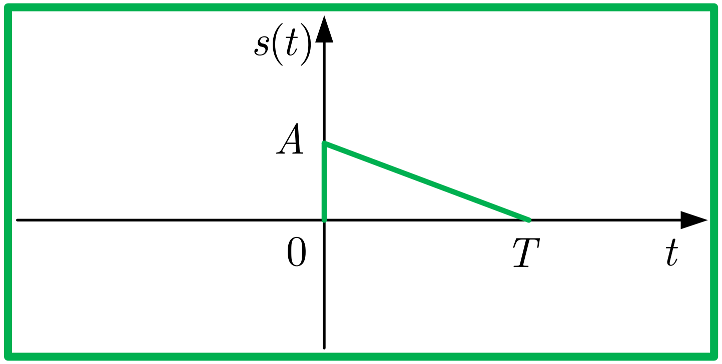
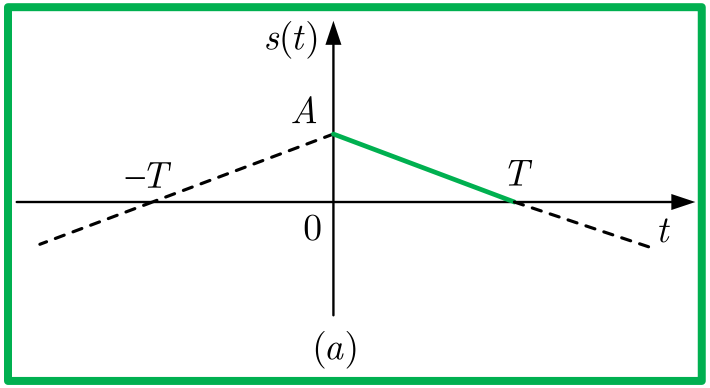


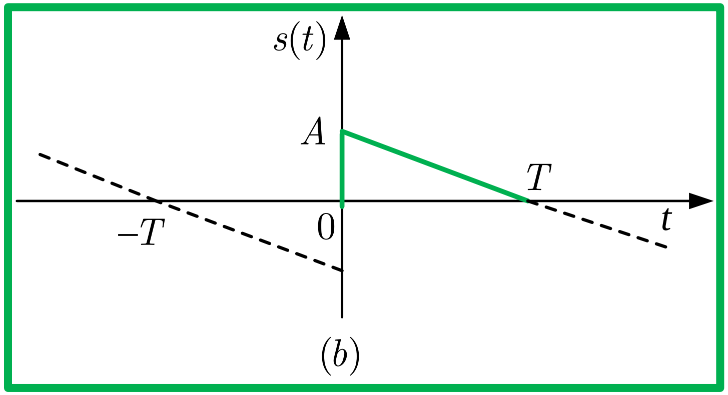
Fig. 3.3: Signal to be approximated.

- Now let us consider two other useful forms of Fourier series. The first is the Fourier cosine series, based on trigonometric identity,

$$C_n \cos(n\omega_0 t + \theta_n) = C_n \cos n\omega_0 t \cos\theta_n - C_n \sin n\omega_0 t \sin\theta_n \quad \textbf{(3.38)}$$

3.5 Evaluation of Fourier Coefficients Cont'd





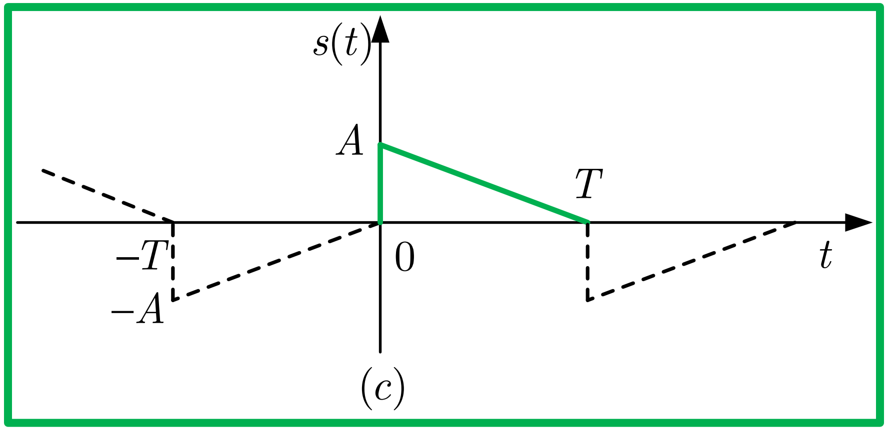


Fig. 3.4: (*a*) Even function cosine terms only. (*b*) Odd function sine terms only. (*c*) Odd harmonics only with both sine and cosine terms.

3.5 Evaluation of Fourier Coefficients Cont'd

- We can derive the form of Fourier cosine series by setting

$$a_n = C_n \cos\theta_n \qquad \textbf{(3.39)}$$

and

$$b_n = -C_n \sin\theta_n \qquad \textbf{(3.40)}$$

- We then obtain C_n and θ_n in terms of a_n and b_n, as

$$C_n = \left[a_n^{\,2} + b_n^{\,2}\right]^{1/2}; \quad C_0 = a_0; \quad \theta_n = \tan^{-1}\left\{-\frac{b_n}{a_n}\right\} \qquad \textbf{(3.41)}$$

- If we combine the cosine and sine terms of each harmonic in the original series, we readily obtain from Eqs. 3.38 - 3.41 the Fourier cosine series

3.5 Evaluation of Fourier Coefficients Cont'd

$$f(t) = C_0 + C_1 \cos(\omega_0 t + \theta_1) + C_2 \cos(2\omega_0 t + \theta_2) + C_3 \cos(3\omega_0 t + \theta_3) + \cdots + C_n \cos(n\omega_0 t + \theta_n) \quad (3.42)$$

- Note, that the coefficients C_n are usually taken to be positive. If however, a term such as $-3\cos 2\omega_0 t$ carries a negative sign, then we can use the equivalent form

$$-3\cos 2\omega_0 t = 3\cos(2\omega_0 t + \pi) \quad (3.43)$$

- Example, the Fourier series of the fully rectified sine wave in Fig. 3.1 was shown to be

$$s(t) = \frac{4A}{2\pi}\left\{1 + \sum_{n=1}^{\infty} \frac{2}{1-4n^2} \cos 2nt\right\} \quad (3.44)$$

- Expressed as a Fourier cosine series, $s(t)$ is

3.5 Evaluation of Fourier Coefficients Cont'd

$$s(t) = \frac{4A}{2\pi}\left\{1 + \sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}\cos(2nt + \pi)\right\} \qquad \textbf{(3.45)}$$

- Next, consider the complex form of a Fourier series. If we express $\cos n\omega_0 t$ and $\sin n\omega_0 t$ in terms of complex exponentials, then the Fourier series can be written as

$$\begin{aligned} s(t) &= a_0 + \sum_{n=1}^{\infty}\left\{a_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} + b_n \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right\} \\ &= a_0 + \sum_{n=1}^{\infty}\left\{\frac{a_n - jb_n}{2} e^{jn\omega_0 t} + \frac{a_n + jb_n}{2} e^{-jn\omega_0 t}\right\} \end{aligned} \qquad \textbf{(3.46)}$$

- If we define

$$\beta_n = \frac{a_n - jb_n}{2}, \quad \beta_{-n} = \frac{a_n + jb_n}{2}, \quad \beta_0 = a_0 \qquad \textbf{(3.47)}$$

3.5 Evaluation of Fourier Coefficients Cont'd

- Thus the complex form of the Fourier series is

$$s(t) = \beta_0 + \sum_{n=1}^{\infty} (\beta_n e^{jn\omega_0 t} + \beta_{-n} e^{-jn\omega_0 t})$$
$$= \sum_{n=-\infty}^{\infty} \beta_n e^{jn\omega_0 t} \qquad \textbf{(3.48)}$$

- We can readily express the coefficients β_n as a function of $s(t)$, since

$$\beta_n = \frac{a_n - jb_n}{2}$$
$$= \frac{1}{T_0}\int_0^{T_0} s(t)(\cos n\omega_0 t - j\sin n\omega_0 t)\, dt \qquad \textbf{(3.49)}$$
$$= \frac{1}{T_0}\int_0^{T_0} s(t) e^{-jn\omega_0 t} dt$$

3.5 Evaluation of Fourier Coefficients Cont'd

- Equation 3.49 is sometimes called the discrete Fourier transform of $s(t)$ and Eq. 3.48 is the inverse transform of $\beta_n(n\omega_0) = \beta_n$.

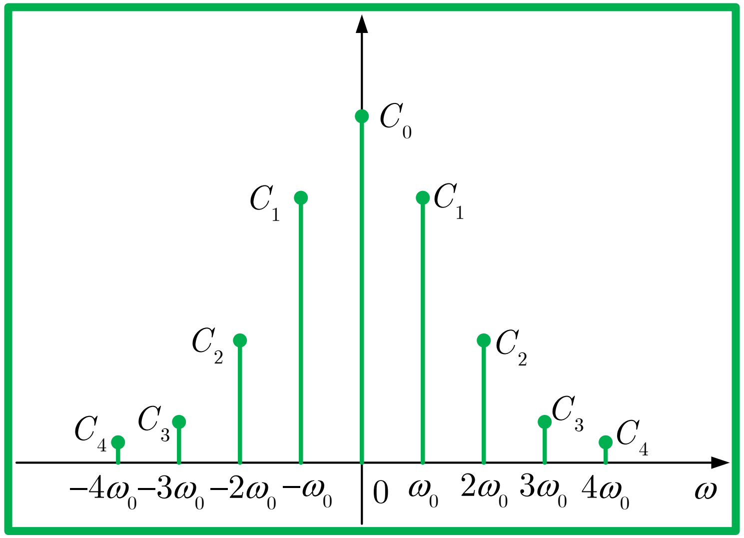


Fig. 3.5: Amplitude spectrum.

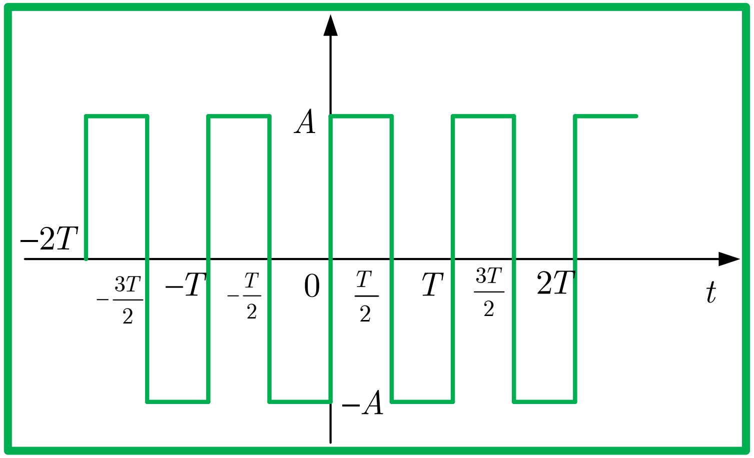


Fig. 3.6: Square wave.

- β_n is usually complex and can be represented as.

$$\beta_n = \mathrm{Re}\,\beta_n + j\,\mathrm{Im}\,\beta_n \qquad \textbf{(3.50)}$$

3.5 Evaluation of Fourier Coefficients Cont'd

- The real part of β_n is obtained from Eq. 3.49 as

$$\operatorname{Re}\beta_n = \frac{1}{T_0}\int_0^{T_0} s(t)\cos n\omega_0 t\,dt \qquad \textbf{(3.51)}$$

and the imaginary part of β_n is

$$j\operatorname{Im}\beta_n = \frac{-j}{T_0}\int_0^{T_0} s(t)\sin n\omega_0 t\,dt \qquad \textbf{(3.52)}$$

- Clearly, $\operatorname{Re}\beta_n$ is an even function in n, whereas $\operatorname{Im}\beta_n$ is an odd function in n. The amplitude spectrum of the Fourier series is defined as

$$\left|\beta_n\right| = \left\{\operatorname{Re}^2\beta_n + \operatorname{Im}^2\beta_n\right\}^{1/2} \qquad \textbf{(3.53)}$$

and the phase spectrum is defined as

$$\phi_n = \arctan\left\{\frac{\operatorname{Im}\beta_n}{\operatorname{Re}\beta_n}\right\} \qquad \textbf{(3.54)}$$

3.5 Evaluation of Fourier Coefficients Cont'd

- It is easily seen that the amplitude spectrum is an even function and the phase spectrum is an odd function in n.
- The amplitude spectrum gives an insight as to where to truncate the infinite series and still maintain a good approximation to the original waveform.
- Clearly, for the amplitude spectrum in Fig. 3.5, we see that a good approximation can be obtained if we disregard any harmonic above the third.

[Example 3.3] Complex Fourier Coefficients

Obtain the complex Fourier coefficients for the square wave in Fig. 3.6. Also find the amplitude and phase spectra of the square wave.

[Solution]

From Fig. 3.6, we note that $s(t)$ is an odd function. Moreover, since $s(t - T/2) = -s(t)$, the series has only odd harmonics. Thus from Eq. 3.49 we obtain the coefficients of the complex Fourier series as

[Example 3.3] Complex Fourier Coefficients Cont'd

$$\beta_n = \frac{1}{T_0}\int_0^{T_0/2} A e^{-jn\omega_0 t}\, dt - \frac{1}{T_0}\int_{T_0/2}^{T_0} A e^{-jn\omega_0 t}\, dt$$

$$= \frac{A}{jn\omega_0 T_0}\left(1 - 2e^{-(jn\omega_0 T_0/2)} + e^{-jn\omega_0 T_0}\right) \qquad \textbf{(3.55)}$$

- Since $n\omega_0 T_0 = n2\pi$, β_n can be simplified to

$$\beta_n = \frac{A}{j2n\pi}\left(1 - 2e^{-jn\pi} + e^{-j2n\pi}\right) \qquad \textbf{(3.56)}$$

- Simplifying β_n one step further, we obtain

$$\beta_n = \begin{cases} \dfrac{2A}{jn\pi} & \forall \text{ odd } n \\ 0 & \forall \text{ even } n \end{cases} \qquad \textbf{(3.57)}$$

[Example 3.3] Complex Fourier Coefficients Cont'd

The amplitude and phase spectra of the square wave are thus as given in Fig. 3.7.

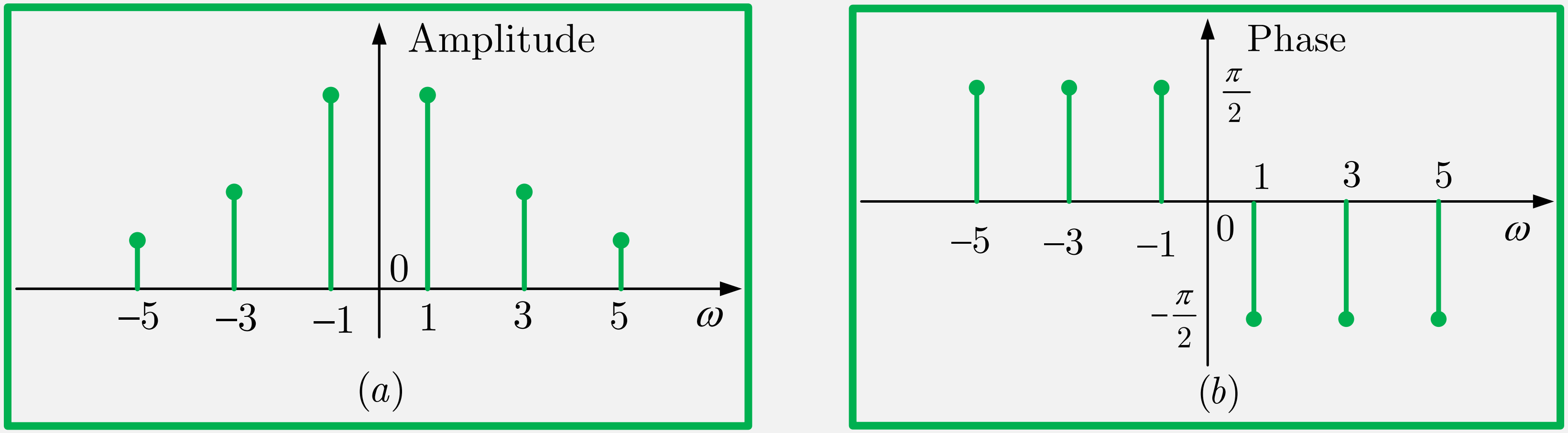


Fig. 3.7: Discrete spectra of square wave. (*a*) Amplitude. (*b*) Phase.

3.6 Evaluation of Fourier Coefficients Using Unit Impulses

- Here, we make use of a basic property of impulse functions to simplify the calculation of complex Fourier coefficients. This method is applicable to functions consisting of straight-line components only. Thus the method applies for the square wave in Fig. 3.6.
- The method is based on the relation

$$\int_{-\infty}^{\infty} f(t)\delta(t - T_1)dt = f(T_1) \qquad \textbf{(3.58)}$$

- Let us use Eq. 3.58 to evaluate the complex Fourier coefficients for the impulse train in Fig. 3.8. For $f(t) = e^{-jn\omega_0 t}$, we have

$$\beta_n = \frac{A}{T_0}\int_0^{T_0} \delta\left(t - \frac{T_0}{2}\right) e^{-jn\omega_0 t} dt = \frac{A}{T_0} e^{-(jn\omega_0 T_0/2)} \qquad \textbf{(3.59)}$$

3.6 Evaluation of Fourier Coefficients Using Unit Impulses Cont'd

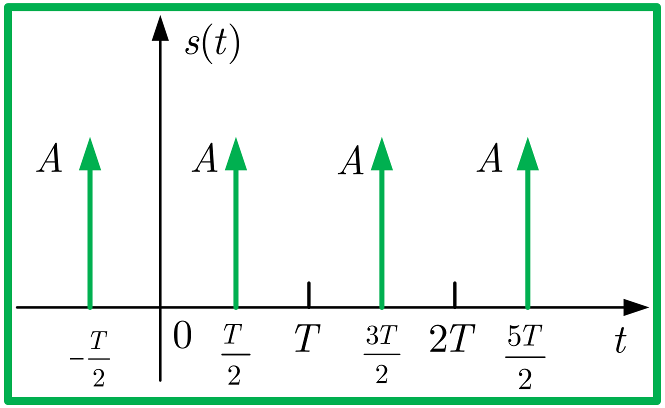
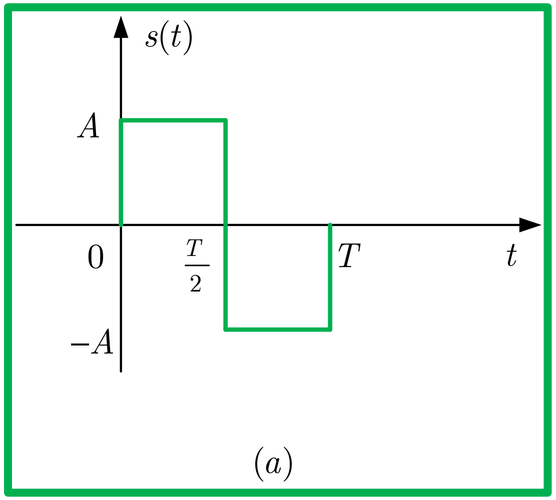


Fig. 3.8: Impulse train.

- The complex Fourier coefficients for the impulse functions are obtained by simply substituting the time at which the impulses occur into the expression $e^{-jn\omega_0 t}$.
- In the evaluation of Fourier coefficients, we must remember that the limits for the β_n integral are taken over one period only, i.e., we consider only a single period of the signal in the analysis.
- Consider, as an example, the square wave in Fig. 3.6. To evaluate β_n, we consider only a single period of the square wave as shown in Fig. 3.9*a*.

3.6 Evaluation of Fourier Coefficients Using Unit Impulses Cont'd



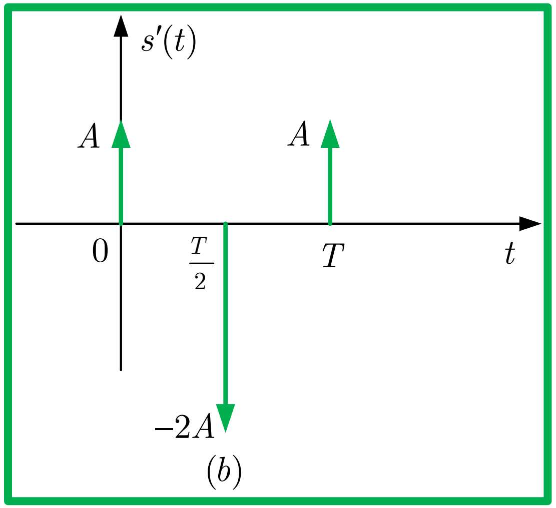


Fig. 3.9: (*a*) Square wave over period $[0, T]$. (*b*) Derivative of square wave over period $[0, T]$.

- Since the square wave is not made up of impulses, let us differentiate the single period of the square wave to give $s'(t)$ as shown in Fig. 3.9*b*.
- We can now evaluate the complex Fourier coefficients for $s'(t)$, which clearly is made up of impulses alone. Analytically, if $s(t)$ is given as

3.6 Evaluation of Fourier Coefficients Using Unit Impulses Cont'd

$$s(t) = \sum_{n=-\infty}^{\infty} \beta_n e^{jn\omega_0 t} \quad \textbf{(3.60)}$$

then the derivative of $s(t)$ is

$$s'(t) = \sum_{n=-\infty}^{\infty} jn\omega_0 \beta_n e^{jn\omega_0 t} \quad \textbf{(3.61)}$$

- Here, we define a new complex coefficient

$$\gamma_n = jn\omega_0 \beta_n \quad \textbf{(3.62)}$$

or

$$\beta_n = \frac{\gamma_n}{jn\omega_0} \quad \textbf{(3.63)}$$

- If the derivative $s'(t)$ is a function which consists of impulse components alone, then we simply evaluate γ_n first and then obtain β_n from Eq. 3.63.

3.6 Evaluation of Fourier Coefficients Using Unit Impulses Cont'd

- For example, the derivative of the square wave yields the impulse train in Fig. 3.9*b*. In the interval $[0, T]$, the signal $s'(t)$ is given as

$$s'(t) = A\delta(t) - 2A\delta\left(t - \frac{T_0}{2}\right) + A\delta(t - T_0) \tag{3.64}$$

- Then the complex coefficients are

$$\begin{aligned}\gamma_n &= \frac{1}{T_0}\int_0^{T_0} s'(t)e^{-jn\omega_0 t}dt \\ &= \frac{A}{T_0}\left(1 - 2e^{-(jn\omega_0 T_0/2)} + e^{-jn\omega T_0}\right)\end{aligned} \tag{3.65}$$

- The Fourier coefficients of the square wave are

$$\begin{aligned}\beta_n &= \frac{\gamma_n}{jn\omega_0} \\ &= \frac{A}{jn\omega_0 T_0}\left(1 - 2e^{-(jn\omega_0 T_0/2)} + e^{-jn\omega_0 T_0}\right)\end{aligned} \tag{3.66}$$

3.6 Evaluation of Fourier Coefficients Using Unit Impulses Cont'd

- The solution obtained in Eq. 3.66 checks with that obtained earlier by the standard way in Eq. 3.55.
- Note, if the first derivative, $s'(t)$, does not contain impulses, then we must differentiate again to yield

$$s''(t) = \sum_{n=-\infty}^{\infty} \lambda_n e^{jn\omega_0 t} \tag{3.67}$$

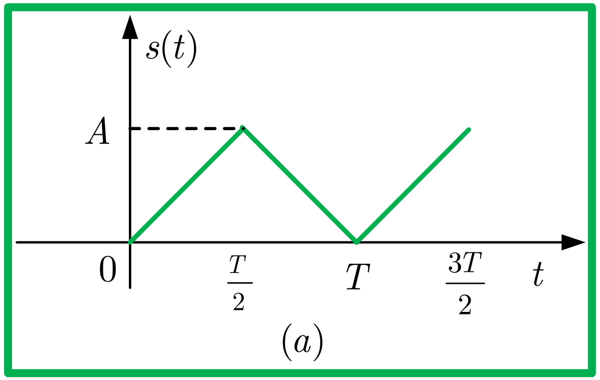
where

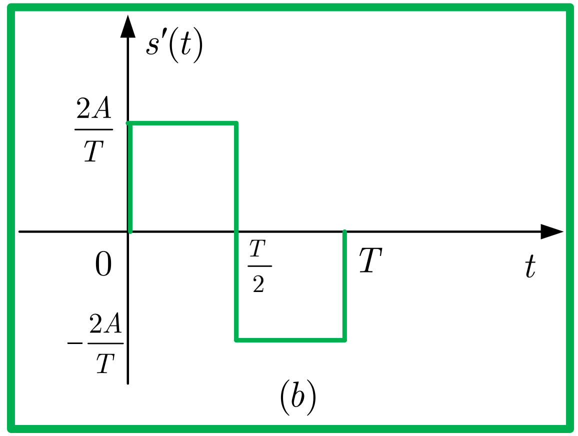
$$\lambda_n = jn\omega_0 \gamma_n = (jn\omega_0)^2 \beta_n \tag{3.68}$$

- For the triangular pulse in Fig. 3.10, the second derivative over the period $[0, T]$ is

$$s''(t) = \frac{2A}{T_0}\left[\delta(t) - 2\delta\left(t - \frac{T_0}{2}\right) + \delta(t - T_0)\right] \tag{3.69}$$

3.6 Evaluation of Fourier Coefficients Using Unit Impulses Cont'd





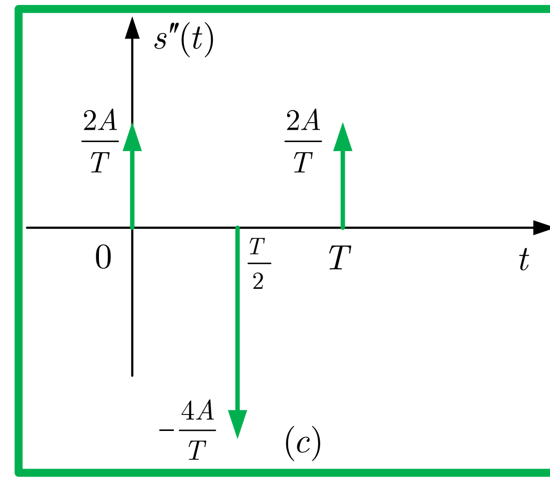


Fig. 3.10: The triangular wave and its derivatives.

- The coefficients λ_n are now obtained as

$$\begin{aligned}\lambda_n &= \frac{1}{T_0}\int_0^{T_0} s''(t)\, e^{-jn\omega_0 t}\, dt \\ &= \frac{2A}{T_0^2}\left(1 - 2e^{-(jn\omega_0 T_0/2)} + e^{-jn\omega_0 T_0}\right)\end{aligned} \qquad \textbf{(3.70)}$$

3.6 Evaluation of Fourier Coefficients Using Unit Impulses Cont'd

- Eq. 3.70 simplifies to give

$$\lambda_n = \begin{cases} \dfrac{8A}{T^2} & \forall\ n \text{ odd} \\ 0 & \forall\ n \text{ even} \end{cases} \tag{3.71}$$

From λ_n we obtain

$$\beta_n = \frac{\lambda_n}{(jn\omega_0)^2} = \begin{cases} -\dfrac{2A}{n^2\pi^2} & \forall\ n \text{ odd} \\ 0 & \forall\ n \text{ even} \end{cases} \tag{3.72}$$

- A slight difficulty arises if the expression for $s'(t)$ contains an impulse in addition to other straight-line terms.

3.6 Evaluation of Fourier Coefficients Using Unit Impulses Cont'd

- Nonetheless, we know that

$$\int_{-\infty}^{\infty} s(t)\delta'(t-T)\,dt = -s'(T) \qquad \textbf{(3.73)}$$

So that

$$\int_{-\infty}^{\infty} \delta'(t-T)e^{-jn\omega t}\,dt = jn\omega e^{-jn\omega T} \qquad \textbf{(3.74)}$$

Thus, doublets or even higher derivatives of impulses can be tolerated.

- Consider the signal $s(t)$ given in Fig. 3.11*a*. Its derivative $s'(t)$, shown in Fig. 3.11*b*, can be expressed as

$$s'(t) = \frac{2}{T}\left[u(t) - u\left(t - \frac{T}{2}\right)\right] + \delta(t) - 2\delta\left(t - \frac{T}{2}\right) \qquad \textbf{(3.75)}$$

- It follows that the second derivative $s''(t)$ is given by Eq. 3.76

3.6 Evaluation of Fourier Coefficients Using Unit Impulses Cont'd

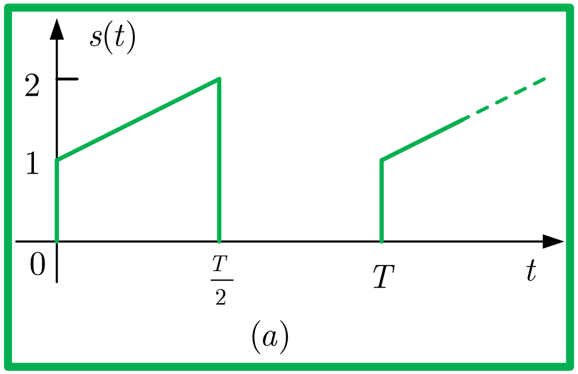
$$s''(t) = \frac{2}{T}\left[\delta(t) - \delta\left(t - \frac{T}{2}\right)\right] + \delta'(t) - 2\delta'\left(t - \frac{T}{2}\right) \qquad \textbf{(3.76)}$$

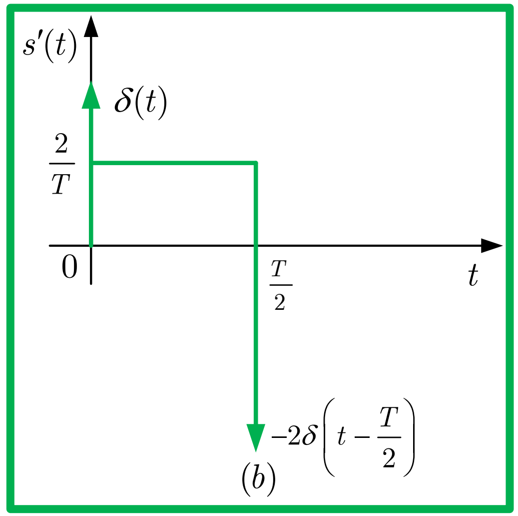
- This is depicted in Fig. 3.11*c*. We therefore evaluate λ_n as

$$\lambda_n = \frac{1}{T_0}\int_0^{T_0} s''(t)\, e^{-jn\omega_0 t}\, dt \qquad \textbf{(3.77)}$$

$$= \frac{2}{T^2}\left(1 - e^{-(jn\omega T/2)}\right) + \frac{jn\omega}{T}\left(1 - 2e^{-(jn\omega T/2)}\right)$$

3.6 Evaluation of Fourier Coefficients Using Unit Impulses Cont'd





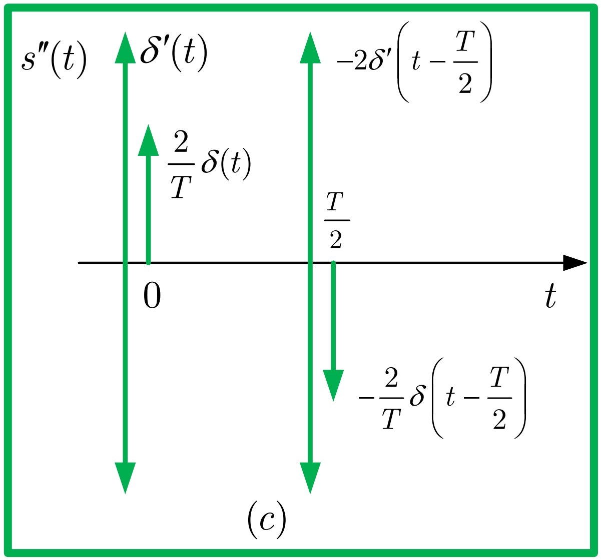


Fig. 3.11

- The complex coefficients β_n are now obtained as

$$\beta_n = \frac{\lambda_n}{(jn\omega)^2} = \frac{2}{(jn\omega T)^2}\left(1 - e^{-(jn\omega T/2)}\right) + \frac{1}{jn\omega T}\left(1 - 2e^{-(jn\omega T/2)}\right) \quad (3.78)$$

3.6 Evaluation of Fourier Coefficients Using Unit Impulses Cont'd

- Simplifying, we get

$$\begin{aligned} \beta_n &= -\frac{1}{n^2\pi^2} + \frac{3}{j2\pi n} \qquad \forall\ n \text{ odd} \\ &= -\frac{1}{j2\pi n} \qquad\qquad\quad \forall\ n \text{ even} \end{aligned} \qquad \textbf{(3.79)}$$

- In conclusion, it is worth noting that the impulse method to evaluate Fourier coefficients does not give the dc component, a_0 or β_0 . Thus, this is obtain through standard means as given by Eq. 3.23.

3.7 The Fourier Integral

- We now extend our signal analysis to the aperiodic case. Generally, aperiodic signals have continuous amplitude and phase spectra.
- In our discussion of Fourier series, the complex coefficient β_n for periodic signals was also called the discrete Fourier transform

$$\beta(nf_0) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} s(t)\, e^{-jn2\pi f_0 t} dt \qquad \textbf{(3.80)}$$

and the inverse (discrete) transform was

$$s(t) = \sum_{n=-\infty}^{\infty} \beta(nf_0)\, e^{jn2\pi f_0 t} \qquad \textbf{(3.81)}$$

- From the discrete Fourier transform we obtain amplitude and phase spectra which consist of discrete lines. The spacing between adjacent lines is

$$\Delta f = (n+1)f_0 - nf_0 = \frac{1}{T} \qquad \textbf{(3.82)}$$

3.7 The Fourier Integral Cont'd

- As the period T becomes larger, the spacing between the harmonic lines in the spectrum becomes smaller. For aperiodic signals, we let T approach infinity so that, in the limit, the discrete spectrum becomes *continuous*.
- We now define the Fourier integral or transform as

$$S(f) = \lim_{\substack{T\to\infty \\ \Delta f\to 0}} \frac{\beta_n(nf_0)}{f_0} = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi ft} dt \quad \text{or} \quad S(\omega) = \int_{-\infty}^{\infty} s(t)\, e^{-j\omega t} dt \qquad \textbf{(3.83)}$$

and the inverse transform is

$$s(t) = \int_{-\infty}^{\infty} S(f)\, e^{j2\pi ft} df \quad \text{or} \quad s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{j\omega t} d\omega \qquad \textbf{(3.84)}$$

- Eqs. 3.83 and 3.84 are sometimes called the Fourier transform pair. If we let $\mathscr{F}$ denote the operation of Fourier transformation and $\mathscr{F}^{-1}$ denote inverse transformation, then

3.7 The Fourier Integral Cont'd

$$S(f) = \mathscr{F} \cdot s(t)$$
$$s(t) = \mathscr{F}^{-1} \cdot S(f)$$
(3.85)

- In general, the Fourier transform $S(f)$ is complex and can be denoted as

$$S(f) = \operatorname{Re} S(f) + j \operatorname{Im} S(f)$$
(3.86)

- The real part of $S(f)$ is obtained through the formula

$$\operatorname{Re} S(f) = \frac{1}{2}\left[S(f) + S(-f)\right] = \int_{-\infty}^{\infty} s(t) \cos 2\pi f t \, dt$$
(3.87)

and the imaginary part through

3.7 The Fourier Integral Cont'd

$$\operatorname{Im} S(f) = \frac{1}{2j}\left[S(f) - S(-f)\right] = -\int_{-\infty}^{\infty} s(t)\sin 2\pi f t\,dt \qquad (3.88)$$

- The amplitude spectrum of $S(f)$ is defined as

$$A(f) = \left[\operatorname{Re} S(f)^2 + \operatorname{Im} S(f)^2\right]^{1/2} \qquad (3.89)$$

and the phase spectrum is

$$\phi(f) = \arctan\frac{\operatorname{Im} S(f)}{\operatorname{Re} S(f)} \qquad (3.90)$$

- By means of the amplitude and phase definition of the Fourier transform, the inverse transform can be expressed as

3.7 The Fourier Integral Cont'd

$$s(t) = \int_{-\infty}^{\infty} A(f)\cos\left[2\pi ft - \phi(f)\right]df \quad \textbf{(3.91)}$$

- Let us examine some examples.

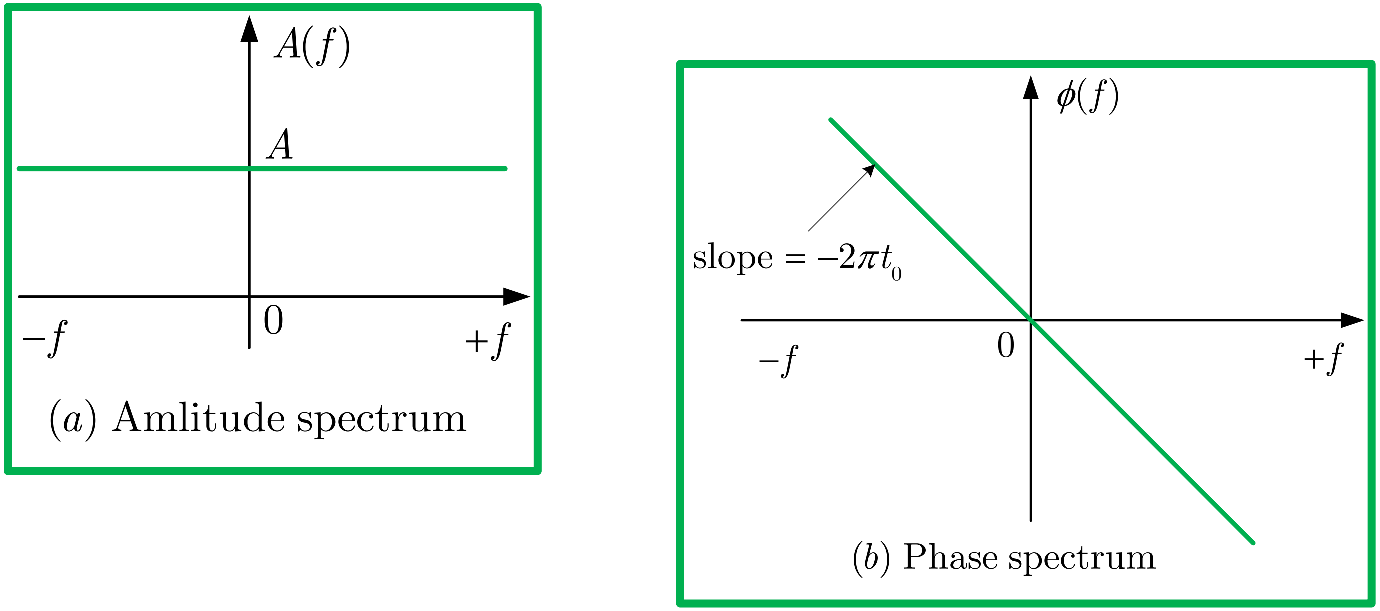


Fig. 3.12: Amplitude and phase spectrum of $A\delta(t-t_0)$.

Example 3.4 The Fourier Integral

Obtain Fig. 3.12 by finding the Fourier transform of $s(t) = A\delta(t - t_0)$.

[SOLUTION]

$$S(f) = \int_{-\infty}^{\infty} A\delta(t - t_0) e^{-j2\pi ft} dt \\ = Ae^{-j2\pi ft_0} \quad \textbf{(3.92)}$$

Its amplitude spectrum is

$$A(f) = A \quad \textbf{(3.93)}$$

while its phase spectrum is

$$\phi(f) = -2\pi f t_0 \quad \textbf{(3.94)}$$

Example 3.5 The Fourier Integral

- Consider the rectangular function in Fig. 3.13. If formally, we define the function as the *rect* function given by

$$\operatorname{rect} f = \begin{cases} 1 & \forall \; |f| \le \dfrac{W}{2} \\ 0 & \forall \; |f| > \dfrac{W}{2} \end{cases} \qquad \textbf{(3.95)}$$

- The inverse transform of $\operatorname{rect} f$ is defined as $\operatorname{sinc} t$ (pronounced as *sink*),

$$\begin{aligned} \mathscr{F}^{-1}\left[\operatorname{rect} f\right] &= \operatorname{sinc} t \\ &= \int_{-W/2}^{W/2} e^{j2\pi ft} df = \left[\frac{e^{j2\pi ft}}{j2\pi t}\right]_{-W/2}^{W/2} \qquad \textbf{(3.96)} \\ &= \frac{\sin \pi W t}{\pi t} \end{aligned}$$

3.7 The Fourier Integral Cont'd

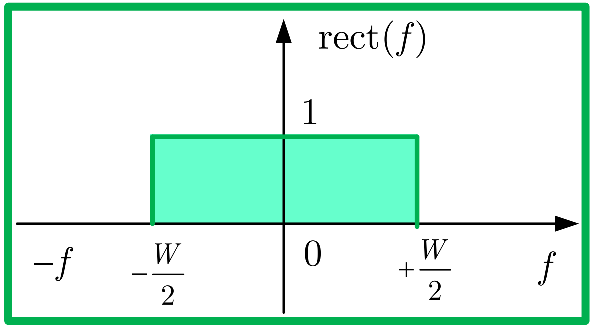


Fig. 3.13: Plot of a rect function.

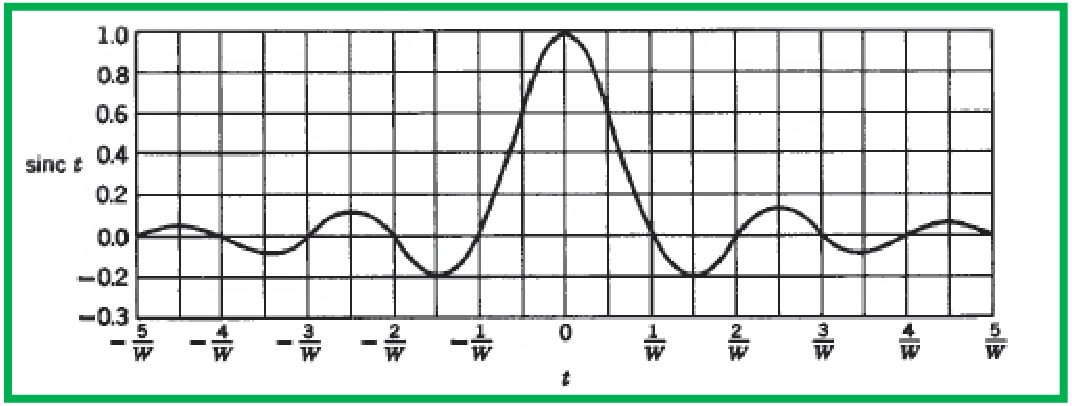


Fig. 3.14: The sinc t curve.

3.7 The Fourier Integral Cont'd

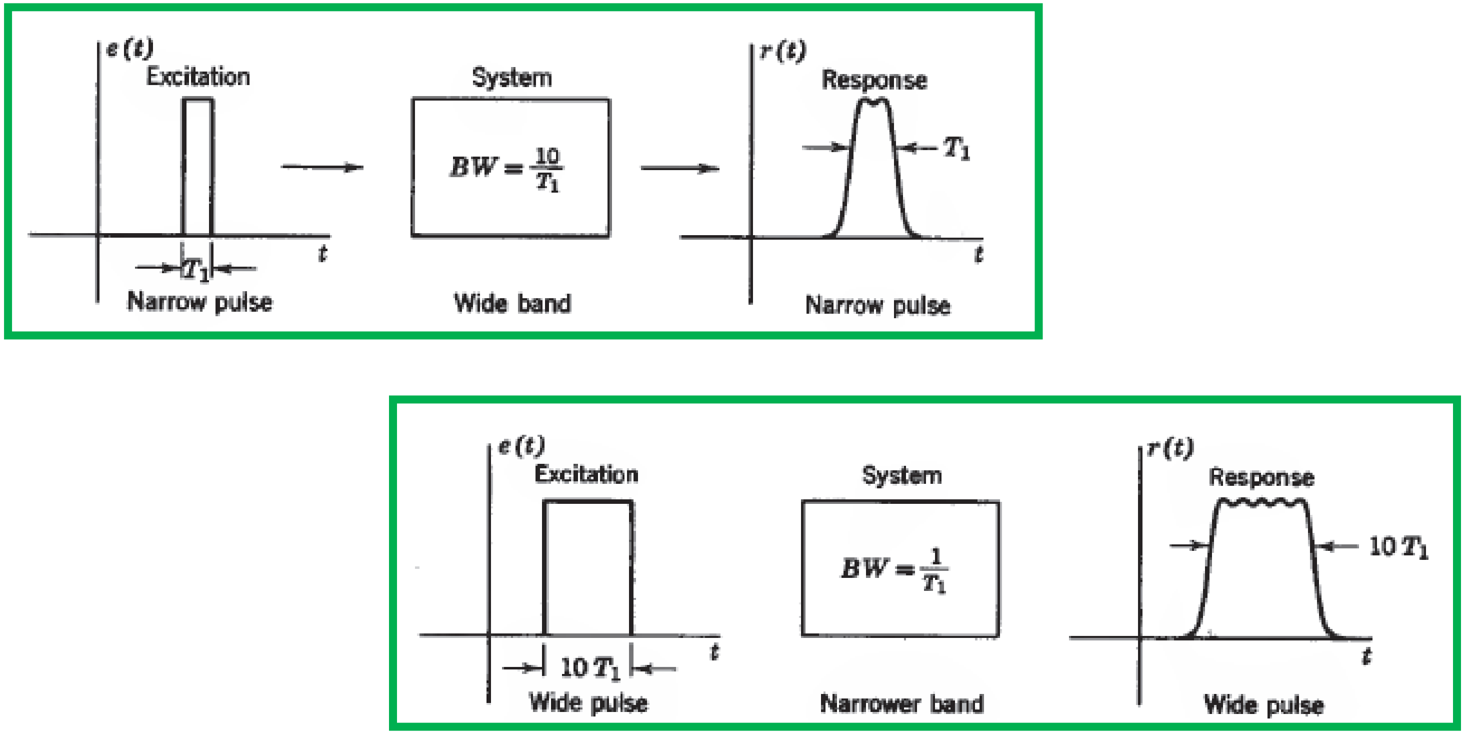


Fig. 3.15: Illustration of the reciprocity relationships between time duration and bandwidth.

3.7 The Fourier Integral Cont'd

- From the plot of $\operatorname{sinc} t$ in Fig. 3.14 we see that $\operatorname{sinc} t$ falls as does $|t|^{-1}$, with zeros at $t = n/W$, $n = 1, 2, 3, \ldots$ Note that most of the energy of the signal is concentrated between the points $-1/W < t < 1/W$.
- Let us define time duration of a signal as that point, t_0, beyond which the amplitude is never greater than a specified value, for example, ϵ_0.
- For the sinc function, the effective time duration is given as $t_0 = \pm 1/W$. The value W, as seen from Fig. 3.13, is the spectral bandwith of the rect function.
- Clearly, if W increases, t_0 decreases. The preceding example illustrates the reciprocal relationship between the time duration of a signal and spectral bandwith of it Fourier transform.
- This concept is quite fundamental. It shows why in pulse transmission, narrow pulses, can only be transmitted through filters with larger bandwidths; whereas, wide pulses do not require wide bandwidths. See Fig. 3.15.

3.8 Properties of Fourier Transforms

- We now focus our energy on some important properties of Fourier transforms.
- Linearity. The linearity property of Fourier transforms states that the Fourier transform of a sum of two signals is the sum of their individual Fourier transforms, that is,

$$\mathscr{F}\left[c_1 s_1(t) + c_2 s_2(t)\right] = c_1 S_1(f) + c_2 S_2(f) \quad \textbf{(3.97)}$$

- Differentiation. This property states that the Fourier transform of the derivative of a signal is $j2\pi f$ times the Fourier transform of the signal itself:

$$\mathscr{F} \cdot s'(t) = j2\pi f S(f) \quad \textbf{(3.98)}$$

more generally,

$$\mathscr{F} \cdot s^{(n)}(t) = (j2\pi f)^n S(f) \quad \textbf{(3.99)}$$

3.8 Properties of Fourier Transforms Cont'd

Proof, is obtained by taking the derivative of both sides of the inverse transform definition,

$$s'(t) = \frac{d}{dt}\int_{-\infty}^{\infty} S(f)e^{j2\pi ft}df \qquad \textbf{(3.100)}$$
$$= \int_{-\infty}^{\infty} j2\pi f S(f)e^{j2\pi ft}df$$

- Similarly, it is easily shown that the transform of the integral of $s(t)$ is

$$\mathscr{F}\left[\int_{-\infty}^{t} s(\tau)d\tau\right] = \frac{1}{j2\pi f}S(f) \qquad \textbf{(3.101)}$$

[Example 3.6] Properties of *F*-Transforms

Consider the following

$$s(t) = e^{-at}u(t) \qquad \textbf{(3.102)}$$

[Example 3.6] Properties of *F*-Transforms Cont'd

- Its Fourier transform is

$$S(f) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j2\pi ft} dt = \int_{0}^{\infty} e^{-at} e^{-j2\pi ft} dt = \frac{1}{a + j2\pi f} \quad \textbf{(3.103)}$$

- The derivative of $s(t)$ is

$$s'(t) = \delta(t) - ae^{-at} u(t) \quad \textbf{(3.104)}$$

- Its Fourier transform is

$$\mathscr{F}\left[s'(t)\right] = 1 - \frac{a}{a + j2\pi f} = \frac{j2\pi f}{a + j2\pi f} = j2\pi f S(f) \quad \textbf{(3.105)}$$

3.8 Properties of Fourier Transforms Cont'd

- Symmetry. The symmetry property of Fourier transforms states that if

$$\mathscr{F} \cdot x(t) = X(f) \quad (3.106)$$

then

$$\mathscr{F} \cdot X(t) = x(-f) \quad (3.107)$$

- This property follows directly from the symmetrical nature of the Fourier transform pair in Eqs. 3.83 and 3.84.

[Example 3.7] Properties of *F*-Transforms

We now know that

$$\mathscr{F} \cdot \operatorname{sinc} t = \operatorname{rect} f \quad (3.108)$$

It is then trivial to show that

$$\mathscr{F} \cdot \operatorname{rect} t = \operatorname{sinc}(-f) = \operatorname{sinc} f \quad (3.109)$$

which conforms to the statement of the symmetry property.

[Example 3.8] Properties of *F*-Transforms

- Consider next the *Fourier transform* of the unit impulse, $\mathscr{F}\cdot\delta(t)=1$. From the *symmetry property* we can show that

$$\mathscr{F}\cdot 1=\delta(f) \tag{3.110}$$

as shown in Fig. 3.16.

- The foregoing example is also an extreme illustration of the time-duration and bandwidth reciprocity relationship. It says that zero time duration, $\delta(t)$, gives rise to infinite bandwidth in the frequency domain; while zero bandwidth, $\delta(f)$ corresponds to infinite time duration.

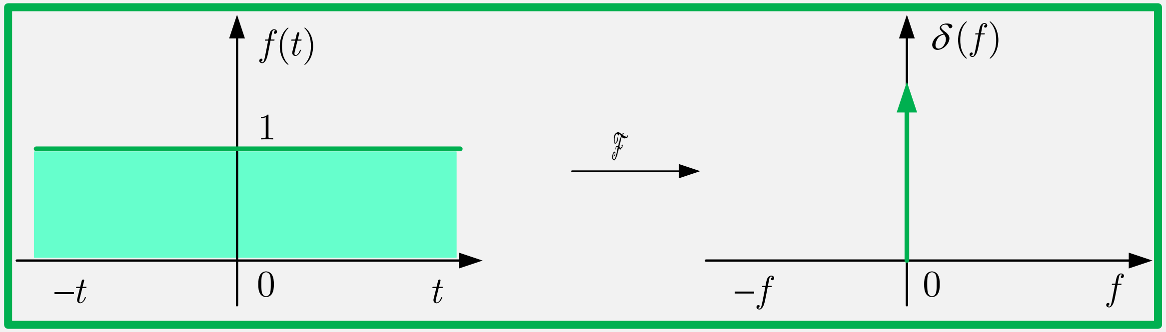


Fig. 3.16: Fourier transform of $f(t)=1.0$.

3.8 Properties of Fourier Transforms Cont'd

- **Scale change.** The scale-change property describes the time-duration and bandwidth reciprocity relationship. It states that

$$\mathscr{F}\left[s\left(\frac{t}{a}\right)\right] = |a|S(af) \qquad \textbf{(3.111)}$$

Proof. We prove this property most easily through the inverse transform

$$\mathscr{F}^{-1}\left[|a|S(af)\right] = |a|\int_{-\infty}^{\infty} S(af)e^{j2\pi ft}df \qquad \textbf{(3.112)}$$

Let $f' = af$, then

$$\mathscr{F}^{-1}\left[|a|S(f')\right] = |a|\int_{-\infty}^{\infty} S(f')e^{j2\pi f'(t/a)}\frac{df'}{a} \qquad \textbf{(3.113)}$$

$$= s\left(\frac{t}{a}\right)$$

[Example 3.9] Properties of *F*-Transforms

Consider

$$\mathscr{F}\left[e^{-at}u(t)\right] = \frac{1}{j2\pi f + a} \quad \textbf{(3.114)}$$

then

$$\mathscr{F}\left[e^{-t}u(t)\right] = \frac{|a|}{j2\pi a f + a} = \frac{1}{j2\pi f + 1} \quad \textbf{(3.115)}$$

if $a > 0$.

- **Folding.** The folding property states that

$$\mathscr{F}\left\{s(-t)\right\} = S(-f) \quad \textbf{(3.116)}$$

3.8 Properties of Fourier Transforms Cont'd

- The proof follows directly from the definition of the Fourier transform. An example is

$$\mathscr{F}\left\{e^{t}u(-t)\right\} = \frac{1}{1 - j2\pi f} \quad \textbf{(3.117)}$$

- Delay. If a signal is delayed by an amount t_0 in the time domain, the corresponding effect in the frequency domain is to multiply the transform of the undelayed signal by $e^{-j2\pi f t_0}$, that is,

$$\mathscr{F}\left\{s(t - t_0)\right\} = e^{-j2\pi f t_0} S(f) \quad \textbf{(3.118)}$$

- For example,

$$\mathscr{F}\left\{e^{-a(t-t_0)}u(t - t_0)\right\} = \frac{e^{-j2\pi f t_0}}{a + j2\pi f} \quad \textbf{(3.119)}$$

3.8 Properties of Fourier Transforms Cont'd

- Modulation. The modulation or frequency shift property of Fourier transforms states that if a Fourier transform is shifted in frequency by an amount f_0, the corresponding effect in time is described by multiplying the original signal by $e^{j2\pi f_0 t}$, that is,

$$\mathscr{F}^{-1}\left[S(f-f_0)\right] = e^{j2\pi f_0 t}s(t) \qquad (3.120)$$

[Example 3.9] Properties of *F*-Transforms

Given $S(f)$ in Fig. 3.17a, let us find the inverse transform of $S_1(f)$ in Fig. 3.17b in terms of $s(t) = \mathscr{F}^{-1}\{S(f)\}$. We know that

$$S_1(f) = S(f-f_0) + S(f+f_0) \qquad (3.121)$$

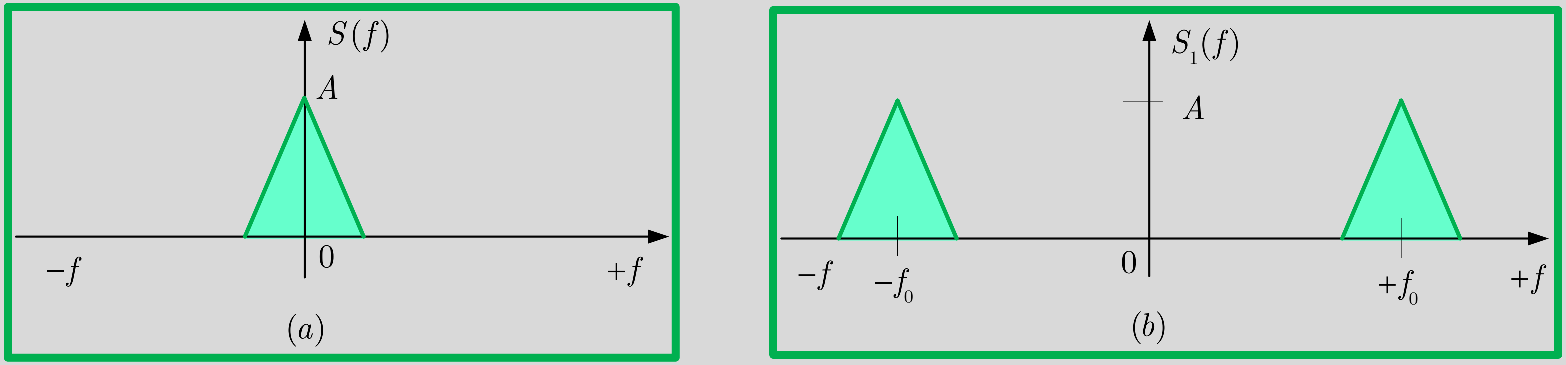


Fig. 3.17: Demonstration of amplitude modulation.

[Example 3.9] Properties of *F*-Transforms Cont'd

Then $\mathscr{F}^{-1}S_1(f) = e^{j2\pi f_0 t}s(t) + e^{-j2\pi f_0 t}s(t) = 2s(t)\cos 2\pi f_0 t$ **(3.122)**

Thus we see that multiplying a signal by a cosine or sine wave in the time domain corresponds to shifting its spectrum by an amount $\pm f_0$. In transmission terminology f_0 is the carrier frequency, and the process of multiplying $s(t)$ by $\cos 2\pi f_0 t$ is called amplitude modulation.

- Parseval's theorem. An important theorem which relates energy in the time and frequency domains is Parseval's theorem, which states that

$$\int_{-\infty}^{\infty} s_1(t)s_2(t)\,dt = \int_{-\infty}^{\infty} S_1(f)S_2(-f)\,df \qquad \textbf{(3.123)}$$

Proof

$$\begin{aligned}\int_{-\infty}^{\infty} s_1(t)s_2(t)\,dt &= \int_{-\infty}^{\infty} s_2(t)\,dt\int_{-\infty}^{\infty} S_1(f)e^{j2\pi ft}\,df \\ &= \int_{-\infty}^{\infty} S_1(f)\,df\int_{-\infty}^{\infty} s_2(t)e^{j2\pi ft}\,dt \\ &= \int_{-\infty}^{\infty} S_1(f)S_2(-f)\,df\end{aligned} \qquad \textbf{(3.124)}$$

3.8 Properties of Fourier Transforms Cont'd

- In particular, when $s_1(t) = s_2(t)$, we have a corollary of Parseval's theorem known as Plancheral's theorem.

$$\int_{-\infty}^{\infty} s^2(t)\,dt = \int_{-\infty}^{\infty} \left|S(f)\right|^2 df \qquad \textbf{(3.125)}$$

- If $s(t)$ is equal to the current through, or the voltage across a 1-ohm resistor, the total energy is

$$\int_{-\infty}^{\infty} s^2(t)\,dt$$

- We see from Eq. 3.125 that the total energy is also equal to the area under the curve of $\left|S(f)\right|^2$. Thus $\left|S(f)\right|^2$ is sometimes called an energy density or energy spectrum.

End of Lecture 3

Thank you for your attention!