EEE 3121 - Signals & Systems

Lecture 4: Differential Equations

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References

Our main reference text book in this course is

[1] B. P. Lathi and R. A. Green, Linear Systems and Signals, 3rd Ed., 2018, Oxford University Press, New York. ISBN 978-0-19-020017-6

[2] Kuo Franklin, F., Network Analysis and Synthesis, 3rd Ed., 1986, J. Wiley (SE), ISBN 0-471-51118-8.

[3] Sundararanjan, D., A Practical Approach to Signals and Systems, 2008, John Wiley & Sons (Asia) Pte Ltd, ISBN 978-0-470-82353-8.

However, feel free to use pretty much any additional text which you might find relevant to our course.

4.1 Introduction

- In this part of the course, we will concern ourselves with a brief study of ordinary differential equations (ODEs). The ODEs considered have the general form

$$F\left[x(t), x'(t), \ldots, x^{(n)}(t), t\right] = 0 \tag{4.1}$$

where t is the independent variable and $x(t)$ is a function dependent upon t. The superscripted terms $x^{(i)}(t)$ indicate the ith derivative of $x(t)$ with respect to t, namely,

$$x^{(i)}(t) = \frac{d^{(i)}x(t)}{dt^i} \tag{4.2}$$

- The solution of Eq. 4.1 is $x(t)$ and must be obtained as an explicit function of t. If we substitute $x(t)$ into F, the equation must equal zero.

4.1 Introduction Cont'd

- If F in Eq. 4.1 is a linear ODE, it is given by the general equation

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \cdots + a_1 x'(t) + a_0 x(t) = f(t) \tag{4.3}$$

- The order of the equation is n, the order of the highest derivative term. The term $f(t)$ on the right-hand side of the equation is the forcing function or driver, and is independent of $x(t)$. When $f(t)$ is identically zero, the equation is said to be homogeneous; otherwise, the equation is non-homogeneous.
- Here, we will restrict ourselves to linear ODEs with constant coefficients.
- Ordinary. An ordinary differential equation is one in which there is only one independent variable, thus there is no need for partial derivatives.
- Constant coefficients. The coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are constant, independent of the variable t.

4.1 Introduction Cont'd

$$f(t) = af_1(t) + bf_2(t) \quad \textbf{(4.6)}$$

the solution would be

$$x(t) = ax_1(t) + bx_2(t) \quad \textbf{(4.7)}$$

here a and b are arbitrary constants.

- It is worth noting that the superposition property is of primary importance in any discussion of linear DEs.

4.2 Homogeneous Linear Differential Equations

- We now focus our attention on some methods to solve homogeneous linear DEs. First, let us find the solution to

$$x'(t) - 2x(t) = 0 \tag{4.8}$$

- Using intuition and/or scientific guess assume the solution to be of the form

$$x(t) = Ce^{2t} \tag{4.9}$$

where C is any arbitrary constant. We check if $x(t) = Ce^{2t}$ is truly a solution of Eq. 4.8. Thus, substituting the assumed solution in Eq. 4.8, gives

$$2Ce^{2t} - 2Ce^{2t} = 0 \tag{4.10}$$

- In general, solutions of homogeneous , linear DEs consist of exponential terms of the form $C_i e^{p_i t}$. Thus, to obtain the solution of any DE, we substitute Ce^{pt} for $x(t)$ in the equation and determine values of p for which the equation is zero.

4.2 Homogeneous Linear Differential Equations Cont'd

- Simply put, given the general equation

$$a_n x^{(n)}(t) + \cdots + a_1 x'(t) + a_0 x(t) = 0 \quad \textbf{(4.11)}$$

we let $x(t) = Ce^{pt}$, so that Eq. 4.11 becomes

$$Ce^{pt}\left(a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0\right) = 0 \quad \textbf{(4.12)}$$

- Since e^{pt} cannot be zero except at $p = -\infty$, the only nontrivial solutions for Eq. 4.12 occur if the polynomial

$$H(p) = a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0 = 0 \quad \textbf{(4.13)}$$

- Equation 4.13 is often called the characteristic equation, which is zero only at its roots. Thus, let us factor $H(p)$ to give

$$H(p) = a_n (p - p_0)(p - p_1)\cdots(p - p_{n-1}) \quad \textbf{(4.14)}$$

From Eq. 4.14, we note that $C_0 e^{p_0 t}, C_1 e^{p_1 t}, \ldots, C_{n-1} e^{p_{n-1} t}$ are all solutions of Eq. 4.11.

4.2 Homogeneous Linear Differential Equations Cont'd

- By the superposition principle, the total solution is a linear combination of all the individual solutions, that is,

$$x(t) = C_0 e^{p_0 t} + C_1 e^{p_1 t} + \cdots + C_{n-1} e^{p_{n-1} t} \quad \textbf{(4.15)}$$

here $C_0, C_1, \ldots, C_{n-1}$ are generally complex. The solution $x(t)$ in Eq. 4.15 is not unique unless the constants $C_0, C_1, \ldots, C_{n-1}$ are uniquely specified.

- Initial conditions, are additional pieces of information needed to determine the constants C_i, i.e., $x(0+), x'(0+), \ldots, x^{(n-1)}(0+)$ for $t = 0+$.
- Note that, if the initial values are given as $x(0-), x'(0-), \ldots, x^{(n-1)}(0-)$ for $t = 0-$, we must first determine the values at $t = 0+$.
- For example, in Eq. 4.9, given that $x(0+) = 4$, then we obtain the constant from the equation

$$x(0+) = Ce^0 = C \quad \textbf{(4.16)}$$

so that $x(t)$ is uniquely determined to be $x(t) = 4e^{2t}$.

[Example 4.1] Homogeneous Linear Differential Equations

Find the solution for

$$x''(t) + 5x'(t) + 4x(t) = 0 \quad \textbf{(4.17)}$$

given the initial conditions $x(0+) = 2$, $x'(0+) = -1$.

[Solution] We first obtain the characteristic equation

$$H(p) = p^2 + 5p + 4 = 0 \quad \textbf{(4.18)}$$

which factors into $(p+4)(p+1) = 0$ **(4.19)**

The roots are $p = -1; \quad p = -4$. The $x(t)$ takes the form

$$x(t) = C_1 e^{-t} + C_2 e^{-4t} \quad \textbf{(4.20)}$$

From the initial condition $x(0+) = 2$ we obtain

$$x(0+) = 2 = C_1 + C_2 \quad \textbf{(4.21)}$$

[Example 4.1] Homogeneous Linear Differential Equations Cont'd

- Using the other initial condition $x'(0+) = -1$, and taking the derivative of Eq. 4.20, we get

$$x'(t) = -C_1 e^{-t} - 4C_2 e^{-4t} \quad \textbf{(4.22)}$$

At $t = 0+$, $x'(t)$ is

$$x'(0+) = -1 = -C_1 - 4C_2 \quad \textbf{(4.23)}$$

Solving Eqns. 4.21 and 4.23 simultaneously, gives $C_1 = 7/3$; $C_2 = -1/3$

- Thus the final solution is

$$x(t) = \frac{7}{3}e^{-t} - \frac{1}{3}e^{-4t} \quad \textbf{(4.24)}$$

4.2 Homogeneous Linear Differential Equations Cont'd

- We now examine the case when the characteristic equations $H(p)$ has multiple roots, i.e., consider the case when $H(p)$ has a root $p = p_0$ of multiplicity k as given by

$$H(p) = a_n (p - p_0)^k (p - p_1) \cdots (p - p_n) \tag{4.25}$$

- Thus, the solution contains k terms involving $e^{p_0 t}$ of the form

$$\begin{aligned} x(t) = C_{00} e^{p_0 t} + C_{01} t e^{p_0 t} + C_{02} t^2 e^{p_0 t} + \cdots + C_{0k-1} t^{k-1} e^{p_0 t} \\ + C_1 e^{p_1 t} + C_2 e^{p_2 t} + \cdots + C_n e^{p_n t} \end{aligned} \tag{4.26}$$

here, the double-scripted terms denote the terms due to the multiple root, $(p - p_0)^k$.

[Example 4.2] Homogeneous Linear Differential Equations

Solve the equation

$$x''(t) - 8x'(t) + 16x(t) = 0 \qquad \textbf{(4.27)}$$

given the initial conditions $x(0+) = 2$, $x'(0+) = 4$.

[Solution]: The characteristic equation is of the form

$$H(p) = p^2 - 8p + 16 = (p-4)^2 \qquad \textbf{(4.28)}$$

Since $H(p)$ has a double root at $p = 4$, the solution take the form

$$x(t) = C_1 e^{4t} + C_2 t e^{4t} \qquad \textbf{(4.29)}$$

To determine C_1 and C_2, we evaluate $x(t)$ and $x'(t)$ at $t = 0+$ to get

$$x(0+) = C_1 = 2$$
$$x'(0+) = 4C_1 + C_2 = 4; \quad C_2 = -4 \qquad \textbf{(4.30)}$$

Thus, the solution is

$$x(t) = 2e^{4t} - 4te^{4t} \qquad \textbf{(4.31)}$$

4.2 Homogeneous Linear Differential Equations Cont'd

- Another case is when $H(p)$ has complex conjugate roots. Consider the equation

$$H(p) = a_2(p - p_1)(p - p_2) \tag{4.32}$$

where p_1 and p_2 are complex conjugate roots, that is

$$p_1, p_2 = \sigma \pm j\omega \tag{4.33}$$

- The solution is of the form

$$x(t) = C_1 e^{(\sigma + j\omega)t} + C_2 e^{(\sigma - j\omega)t} \tag{4.34}$$

- Using Euler's equation to expand $e^{j\omega t}$, we have

$$x(t) = C_1 e^{\sigma t}(\cos \omega t + j \sin \omega t) + C_2 e^{\sigma t}(\cos \omega t - j \sin \omega t) \tag{4.35}$$

which reduces to

$$x(t) = (C_1 + C_2)\, e^{\sigma t} \cos \omega t + j(C_1 - C_2)\, e^{\sigma t} \sin \omega t \tag{4.36}$$

4.2 Homogeneous Linear Differential Equations Cont'd

- To express $x(t)$ in a more convenient form, we use new constants, M_1 and M_2, so that

$$x(t) = M_1 e^{\sigma t}\cos\omega t + M_2 e^{\sigma t}\sin\omega t \qquad \textbf{(4.37)}$$

here, the constants are related as follows

$$M_1 = C_1 + C_2; \qquad M_2 = j(C_1 - C_2) \qquad \textbf{(4.38)}$$

- The constants M_1 and M_2 are determined using initial conditions.
- Another convenient form for the solution can be obtained by introducing yet another pair of constants, M and ϕ, defined as

$$M_1 = M\sin\phi; \quad M_2 = M\cos\phi \qquad \textbf{(4.39)}$$

- Using the new constants, we obtain another form

$$x(t) = Me^{\sigma t}\sin(\omega t + \phi) \qquad \textbf{(4.40)}$$

[Example 4.3] Homogeneous Linear Differential Equations

Solve the equation

$$x''(t) + 2x'(t) + 5x(t) = 0 \qquad \textbf{(4.41)}$$

with the initial conditions $x(0+) = 1$, $x'(0+) = 0$.

[Solution] The characteristic equation is of the form

$$H(p) = p^2 + 2p + 5 = (p + 1 + j2)(p + 1 - j2) \qquad \textbf{(4.42)}$$

From, Eqn. 4.37, we have $\sigma = -1$ and $\omega = 2$. Then

$$x(t) = M_1 e^{-t} \cos 2t + M_2 e^{-t} \sin 2t \qquad \textbf{(4.43)}$$

At $t = 0+$ $x(0+) = 1 = M_1$ **(4.44)**

The derivative of $x(t)$ is

$$x'(t) = M_1(-e^{-t} \cos 2t - 2e^{-t} \sin 2t) + M_2(-e^{-t} \sin 2t + 2e^{-t} \cos 2t) \qquad \textbf{(4.45)}$$

[Example 4.3] Homogeneous Linear Differential Equations Cont'd

At $t = 0+$ we have

$$x'(0+) = 0 = -M_1 + 2M_2 \tag{4.46}$$

Solving Eqns. 4.44 and 4.46 simultaneously, gives $M_1 = 1;\ M_2 = 1/2$

Thus, the solution is

$$x(t) = e^{-t}\left\{\cos 2t + \frac{1}{2}\sin 2t\right\} \tag{4.47}$$

By means of Eq. 4.40, the solution can be of the form

$$x(t) = \sqrt{\frac{5}{4}}e^{-t}\sin\left[2t + \tan^{-1}(2)\right] \tag{4.48}$$

- Now, we consider an example that illustrates all that has been discussed.

[Example 4.4] Homogeneous Linear Differential Equations

- The differential equation is

$$x^{(5)}(t) + 9x^{(4)}(t) + 32x^{(3)}(t) + 58x''(t) + 56x'(t) + 24x(t) = 0 \quad \textbf{(4.49)}$$

with the initial conditions

$$x^{(4)}(0+) = 0;\ x^{(3)}(0+) = 1$$

$$x''(0+) = -1;\ x'(0+) = 0;\ \ x(0+) = 1$$

[Solution] The characteristic equation is of the form

$$H(p) = p^5 + 9p^4 + 32p^3 + 58p^2 + 56p + 24 = 0 \quad \textbf{(4.50)}$$

- Which factors into

$$H(p) = (p + 1 + j1)(p + 1 - j1)(p + 2)^2(p + 3) = 0 \quad \textbf{(4.51)}$$

- Thus

$$x(t) = M_1 e^{-t}\cos t + M_2 e^{-t}\sin t + C_0 e^{-2t} + C_1 t e^{-2t} + C_2 e^{-3t} \quad \textbf{(4.52)}$$

[Example 4.4] Homogeneous Linear Differential Equations Cont'd

- Since there are five coefficients, we need five equations to evaluate the unknowns. That is

$$\begin{aligned} x(0+) &= M_1 + C_0 + C_2 = 1 \\ x'(0+) &= -M_1 + M_2 - 2C_0 - 3C_2 + C_1 = 0 \\ x''(0+) &= -2M_2 + 4C_0 - 4C_1 + 9C_2 = -1 \\ x^{(3)}(0+) &= 2M_1 + 2M_2 - 8C_0 + 12C_1 - 27C_2 = 1 \\ x^{(4)}(0+) &= -4M_1 + 16C_0 + 81C_2 - 32C_1 = 0 \end{aligned} \qquad \textbf{(4.53)}$$

- Solving the five simultaneous equations, gives

$$M_1 = 0; \quad M_2 = 3/2; \quad C_0 = 1; \quad C_1 = 1/2; \quad C_2 = 0$$

so that the final solution is

$$x(t) = \frac{3}{2}e^{-t}\sin t + e^{-2t} + \frac{1}{2}te^{-2t} \qquad \textbf{(4.54)}$$

4.3 Nonhomogeneous Differential Equations

- A nonhomogeneous DE has a forcing function $f(t)$ not zero for all *t*. It is of the form

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \cdots + a_1 x'(t) + a_0 x(t) = f(t) \quad \textbf{(4.55)}$$

- From Eq. 4.55, let $x_p(t)$ be a particular solution and $x_c(t)$ be the solution of the homogeneous equation, that is, when $f(t) = 0$. Thus

$$x(t) = x_p(t) + x_c(t) \quad \textbf{(4.56)}$$

is also a solution of Eq. 4.55. By the uniqueness theorem, the solution in Eq. 4.56 is unique if it satisfies the specified initial conditions at $t = 0+$.

- In Eq. 4.56, $x_p(t)$ is the particular integral; $x_c(t)$ is the complementary function; and $x(t)$ is the total solution.
- We now know how to find $x_c(t)$, so we look at how to find $x_p(t)$. A reliable rule of thumb is that $f(t)$ usually takes the same form as the forcing function $x_p(t)$.

4.3 Nonhomogeneous Differential Equations Cont'd

- Specifically, $x_p(t)$ assumes the form of $f(t)$ plus its derivatives. For example, for $f(t) = \alpha \sin \omega t$, then $x_p(t)$ takes the form

$$x_p(t) = A \sin \omega t + B \cos \omega t$$

- The only unknowns to be determined are coefficients *A* and *B*. The method of obtaining $x_p(t)$ is appropriately called the method of undetermined coefficients or unknown coefficients.
- Let us take $f(t)$ to be

$$f(t) = \alpha e^{\beta t} \tag{4.57}$$

where α and β are arbitrary constants. We assume $x_p(t)$ to be of the form

$$x_p(t) = A e^{\beta t} \tag{4.58}$$

here *A* is the unknown coefficient. We thus substitute $x_p(t)$ into the DE.

4.3 Nonhomogeneous Differential Equations Cont'd

- Thus, $$A e^{\beta t}(a_n \beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1 \beta + a_0) = \alpha e^{\beta t} \quad (4.59)$$

- Clearly, the polynomial within the parentheses is the characteristic equation $H(p)$ with $p = \beta$. As such

$$A = \frac{\alpha}{H(\beta)} \quad (4.60)$$

provided that $H(\beta) \neq 0$.

[Example 4.5] Nonhomogeneous Linear Differential Equations

Determine the solution of the equation

$$x''(t) + 3x'(t) + 2x(t) = 4e^{t} \quad (4.61)$$

with the initial conditions, $x(0+) = 1$, $x'(0+) = -1$.

[Example 4.5] Nonhomogeneous Linear Differential Equations Cont'd

[Solution] The characteristic equation is of the form

$$H(p) = p^2 + 3p + 2 = (p+2)(p+1)$$

So that the complementary function is

$$x_c(t) = C_1 e^{-t} + C_2 e^{-2t}$$

For the forcing function $f(t) = 4e^t$, the constants in Eq. 4.60 are $\alpha = 4$, $\beta = 1$. Thus

$$A = \frac{\alpha}{H(\beta)} = \frac{4}{H(1)} = \frac{2}{3}$$

It follows that, $x_p(t) = \frac{2}{3}e^t$, such that the total solution is

$$x(t) = x_c(t) + x_p(t) = C_1 e^{-t} + C_2 e^{-2t} + \frac{2}{3}e^t \qquad \textbf{(4.62)}$$

[Example 4.5] Nonhomogeneous Linear Differential Equations Cont'd

- We now evaluate the constants using the given initial conditions.

$$x(0+) = 1 = C_1 + C_2 + \frac{2}{3}; \; x'(0+) = -1 = -C_1 - 2C_2 + \frac{2}{3} \qquad \textbf{(4.63)}$$

- Solving Eq. 4.63, gives $C_1 = -1$, $C_2 = 4/3$. Therefore,

$$x(t) = -e^{-t} + \frac{4}{3}e^{-2t} + \frac{2}{3}e^{t} \qquad \textbf{(4.64)}$$

- It is worth noting that the constants are obtained using the initial conditions for the total solution.

- Next, consider an example of a constant forcing function $f(t) = \alpha$. We may use Eq. 4.60 if we write the function in the form

$$f(t) = \alpha = \alpha e^{t0} \qquad \textbf{(4.65)}$$

4.3 Nonhomogeneous Differential Equations Cont'd

that is, $\beta = 0$. For the DE in Example 4.5 with $f(t) = 4$, we see that

$$x_p(t) = A = \frac{4}{H(0)} = 2 \tag{4.66}$$

and

$$x(t) = C_1 e^{-t} + C_2 e^{-2t} + 2 \tag{4.67}$$

- If the forcing function is a sine or cosine function, we can still use the exponential form and exploit the method of undetermined coefficients .
- Suppose

$$f(t) = \alpha e^{j\omega t} = \alpha(\cos \omega t + j \sin \omega t) \tag{4.68}$$

then the particular integral $x_{p1}(t)$ can be written as

$$x_{p1}(t) = \operatorname{Re} x_{p1}(t) + j \operatorname{Im} x_{p1}(t) \tag{4.69}$$

4.3 Nonhomogeneous Differential Equations Cont'd

- By the superposition principle, we can show that

if $f(t) = \alpha \cos \omega t \quad \text{then} \quad x_p(t) = \operatorname{Re} x_{p1}(t)$

if $f(t) = \alpha \sin \omega t \quad \text{then} \quad x_p(t) = \operatorname{Im} x_{p1}(t)$

- Thus, whether the excitation is a cosine or sine function, we can still use the exponential driver $f(t) = \alpha e^{j\omega t}$, then take the real or imaginary part of the resulting particular integral.

[Example 4.6] Nonhomogeneous Linear Differential Equations

Find the particular integral for the equation

$$x''(t) + 5x'(t) + 4x(t) = 2\sin 3t \tag{4.70}$$

[Solution] We, let the excitation be of the form

$$f(t) = 2e^{j3t} \tag{4.71}$$

[Example 4.6] Nonhomogeneous Linear Differential Equations Cont'd

- It follows that the particular integral $x_{p1}(t)$ is of the form

$$x_{p1}(t) = Ae^{j3t} \tag{4.72}$$

- From the characteristic equation

$$H(p) = p^2 + 5p + 4$$

we determine the coefficient *A* to be

$$A = \frac{2}{H(j3)} = \frac{2}{-5 + j15} = \frac{2}{5\sqrt{10}} e^{j[\tan^{-1}(3) - \pi]} \tag{4.73}$$

- Then

$$x_{p1}(t) = \frac{2}{5\sqrt{10}} e^{j[\tan^{-1}(3) + 3t - \pi]} \tag{4.74}$$

so that the particular integral $x_p(t)$ for $f(t) = 2\sin 3t$ is

$$x_p(t) = \operatorname{Im} x_{p1}(t) = \frac{2}{5\sqrt{10}} \sin[3t + \tan^{-1}(3) - \pi] \tag{4.75}$$

4.3 Nonhomogeneous Differential Equations Cont'd

- Note that the method of undetermined coefficients has certain limitations .
- For example, if $f(t)$ were a Bessel function $J_0(t)$, we could not assume $x_p(t)$ to be a Bessel function of the same form.
- For the purpose of linear network analysis, the method is more than adequate.
- Suppose the forcing function were $f(t) = At^k e^{pt}; \quad p = \sigma + j\omega$
- The particular integral can be written as

$$x_p(t) = [A_k t^k + A_{k-1} t^{k-1} + \cdots + A_1 t + A_0] e^{pt} \qquad \textbf{(4.76)}$$

where the coefficients $A_k, A_{k-1}, \ldots, A_1, A_0$ are to be determined.

4.4 Step & Impulse Response

- We now discuss solutions of differential equations with step or impulse forcing functions.
- As physical quantities, the step and impulse responses of a linear system are highly significant measures of system performance.
- Thus, a reliable measure of the transient behavior of the system is given by its step and impulse response.
- Here, we concern ourselves with the mathematical problem of solving for impulse and step response, given a linear DE with initial conditions at $t = 0-$,
- Recall that the unit step function is given by

$$u(t) = \begin{cases} 1 & \forall \;\; t \geq 0 \\ 0 & \forall \;\; t < 0 \end{cases}$$

4.4 Step & Impulse Response Cont'd

and the unit impulse was shown to have the properties:

$$\delta(t) = \begin{cases} \infty & \forall\ t = 0 \\ 0 & \forall\ t \neq 0 \end{cases}$$

and

$$\int_{0-}^{0+} \delta(t)dt = 1$$

- Furthermore, we have the relationship

$$\delta(t) = \frac{du(t)}{dt}$$

- Definitions of $\delta(t)$ and $u(t)$ indicate that both functions have discontinuities at $t = 0$. In dealing with initial conditions, the following may be the prevailing conditions

$$x^{(n)}(0-) \neq x^{(n)}(0+);\ \ x^{(n-1)}(0-) \neq x^{(n-1)}(0+);\ldots\ \ x(0-) \neq x(0+)$$

4.4 Step & Impulse Response Cont'd

- In many physical problems, the initial conditions are given at $t = 0-$. However, to evaluate the unknown constants of the total solution, we must have the initial conditions at $t = 0+$.
- To do this, we use a method called "integrating through a Green's function."
- Consider the differential equation with an impulse forcing function

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \cdots + a_0 x(t) = A\delta(t) \qquad \textbf{(4.77)}$$

- By inspection, to ensure the RHS of Eq. 4.77 equals the LHS, the highest derivative $x^{(n)}(t)$ must contain the impulse. Thus, $x^{(n-1)}(t)$ would contain a step and $x^{(n-2)}(t)$, a ramp.
- We conclude that, for an impulse forcing function, the two highest derivative terms are discontinuous at $t = 0$.

4.4 Step & Impulse Response Cont'd

- For a step forcing function, only the highest derivative term is discontinuous at $t = 0$.
- Since initial conditions are usually given at $t = 0-$, *our task* is to determine the values $x^{(n)}(0+)$ and $x^{(n-1)}(0+)$ for an impulse forcing function. Let us integrate Eq. 4.77 between $t = 0-$ and $t = 0+$, namely

$$a_n \int_{0-}^{0+} x^{(n)}(t)dt + a_{n-1}\int_{0-}^{0+} x^{(n-1)}(t)dt + \cdots + a_0 \int_{0-}^{0+} x(t)dt = A\int_{0-}^{0+} \delta(t)dt \quad \textbf{(4.78)}$$

following integration, we obtain

$$a_n \left[x^{(n-1)}(0+) - x^{(n-1)}(0-)\right] + a_{n-1}\left[x^{(n-2)}(0+) - x^{(n-2)}(0-)\right] + \cdots = A \quad \textbf{(4.79)}$$

since all derivative terms below $(n-1)$ are continuous at $t = 0$, Eq. 4.79 simplifies to

$$a_n \left[x^{(n-1)}(0+) - x^{(n-1)}(0-)\right] = A \quad \textbf{(4.80)}$$

4.4 Step & Impulse Response Cont'd

- Thus.

$$x^{(n-1)}(0+) = \frac{A}{a_n} + x^{(n-1)}(0-) \qquad \textbf{(4.81)}$$

We must next determine $x^{(n)}(0+)$. At $t = 0+$, the differential equation in Eq. 4.77 is

$$a_n x^{(n)}(0+) + a_{n-1} x^{(n-1)}(0+) + \cdots + a_0 x(0+) = 0 \qquad \textbf{(4.82)}$$

- Since all derivative terms below $(n-1)$ are continuous, and since we already have solved for $x^{(n-1)}(0+)$, we find that

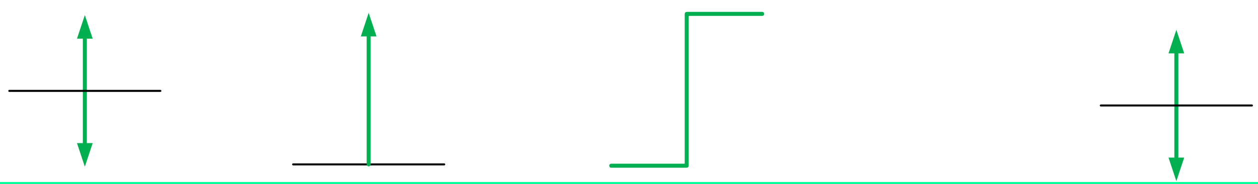
$$x^{(n)}(0+) = -\frac{1}{a_n}\left[a_{n-1} x^{(n-1)}(0+) + \cdots + a_1 x'(0+) + a_0 x(0+)\right] \qquad \textbf{(4.83)}$$

- For a step forcing function $Au(t)$, all derivative terms except $x^{(n)}(t)$, are continuous at $t = 0$. To determine $x^{(n)}(0+)$, we derive in a manner similar to Eq. 4.83, the expression

4.4 Step & Impulse Response Cont'd

$$x^{(n)}(0+) = \frac{A}{a_n} - \frac{1}{a_n}\left[a_{n-1}x^{(n-1)}(0+) + \cdots + a_0 x(0+)\right] \qquad \textbf{(4.84)}$$

- The process of determining initial conditions when the forcing function is an impulse or one of its higher derivatives can be simplified by the visual process in Eqs. 4.85 and 4.86.



$$a_n x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + a_{n-2}x^{(n-2)}(t) + \cdots + a_0 x(t) = \delta'(t) \qquad \textbf{(4.85)}$$

$$a_n x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + a_{n-2}x^{(n-2)}(t) + \cdots + a_0 x(t) = \delta(t) \qquad \textbf{(4.86)}$$

4.4 Step & Impulse Response Cont'd

- It is worth noting that, if a derivative term contains a certain singularity- for instance, a doublet, it also contains all lower derivative terms. For example, in the Equation.

$$x''(t) + 3x'(t) + 2x(t) = 4\delta'(t) \quad (4.87)$$

we assume the following forms for the derivative terms at $t = 0$:

$$\begin{aligned} x''(t) &= A\delta'(t) + B\delta(t) + Cu(t) \\ x'(t) &= A\delta(t) + Bu(t) \\ x(t) &= Au(t) \end{aligned} \quad (4.88)$$

- Substituting Eq. 4.88 into Eq. 4.87, we obtain

4.4 Step & Impulse Response Cont'd

$$A\delta'(t) + B\delta(t) + Cu(t) + 3A\delta(t) + 3Bu(t) + 2Au(t) = 4\delta'(t) \qquad \textbf{(4.89)}$$

or in a more convenient form, we have

$$A\delta'(t) + (B + 3A)\delta(t) + (C + 3B + 2A)u(t) = 4\delta'(t) \qquad \textbf{(4.90)}$$

- Equating like coefficients on both sides of Eq. 3.90 gives

$$\begin{aligned} A &= 4 \\ B + 3A &= 0 \\ C + 3B + 2A &= 0 \end{aligned} \qquad \textbf{(4.91)}$$

from which we obtain $B = -12$ and $C = 28$. Therefore, at $t = 0$, it is true that

$$\begin{aligned} x''(t) &= 4\delta'(t) - 12\delta(t) + 28u(t) \\ x'(t) &= 4\delta(t) - 12u(t) \\ x(t) &= 4u(t) \end{aligned} \qquad \textbf{(4.92)}$$

4.4 Step & Impulse Response Cont'd

- The $u(t)$ terms in Eq. 4.92 give rise to the discontinuities in the initial conditions at $t = 0$. We are given the initial conditions at $t = 0-$. Upon evaluating A, B, C in Eq. 4.88, we can obtain initial conditions at $t = 0+$. by referring to coefficients of the step terms.
- For example, given

$$x(0-) = -2$$
$$x'(0-) = -1$$
$$x''(0-) = 7$$

Then from Eq. 4.92 we obtain

$$\begin{aligned} x(0+) &= -2 + 4 = 2 \\ x'(0+) &= -1 - 12 = -13 \\ x''(0+) &= 7 + 28 = 35 \end{aligned} \tag{4.93}$$

4.4 Step & Impulse Response Cont'd

- The total solution of Eq. 4.87 is obtained as though it were a homogeneous equation, since $\delta'(t) = 0$ for $t \neq 0$.
- The only influence the doublet driver has is to produce discontinuities in the initial conditions at $t = 0$. Having evaluated the initial conditions at $t = 0+$, we can obtain the total solution with ease. Thus,

$$x(t) = C_1 e^{-t} + C_2 e^{-2t} \quad \textbf{(4.94)}$$

From Eq. 4.93 we readily obtain

$$x(t) = [-9e^{-t} + 11e^{-2t}]u(t) \quad \textbf{(4.95)}$$

- For a step forcing function, only the highest derivative term has a discontinuity at $t = 0$. Thus, no need for the initial condition for this term. Hence solve it like a standard nonhomogeneous equation with a constant forcing function.
- For an impulse driver, once we determine the initial conditions at $t = 0+$, the equation is solved in the same manner as a homogeneous equation.

[Example 4.7] Step & Impulse Response

Find the step and impulse response for the equation

$$2x''(t) + 4x'(t) + 10x(t) = f(t)$$

where $f(t) = \delta(t)$ and $f(t) = u(t)$, respectively. The initial conditions at $t = 0-$ are $x(0-) = x'(0-) = x''(0-) = 0$.

[Solution]

Let us first find the impulse response. We note that the $x''(t)$ term has an impulse; the $x'(t)$ term has a step; the term $x(t)$ contains a ramp, and is therefore continuous at $t = 0$. Thus $x(0+) = x(0-) = 0$. To obtain $x'(0+)$, we use Eq. 4.81

$$x'(0+) = \frac{K}{a_2} + x'(0-) = \frac{1}{2} + 0 = \frac{1}{2} \qquad \textbf{(4.96)}$$

Note, need only $x(0+)$ and $x'(0+)$ to evaluate the constants for the second-order DE.

[Example 4.7] Step & Impulse Response Cont'd

- Next, we proceed to the complementary function $x_c(t)$. The characteristic equation is

$$H(p) = 2(p^2 + 2p + 5) = 2(p + 1 + j2)(p + 1 - j2) \quad \textbf{(4.97)}$$

- Thus, the complementary function is of the form

$$x_c(t) = Me^{-t}\sin(2t + \phi) \quad \textbf{(4.98)}$$

- Substituting the initial conditions at $t = 0+$, we obtain

$$x(0+) = 0 = M\sin\phi; \quad x'(0+) = \frac{1}{2} = 2M\cos\phi - M\sin\phi \quad \textbf{(4.99)}$$

- we thus find $\phi = 0$ and $M = 1/4$. Thus the impulse response which we denote here as $x_\delta(t)$ is

$$x_\delta(t) = \frac{1}{4}e^{-t}\sin 2t\,u(t) \quad \textbf{(4.100)}$$

[Example 4.7] Step & Impulse Response Cont'd

- Next, we solve the step response $x_u(t)$. Let us write the complementary function as

$$x_c(t) = e^{-t}(A_1 \sin 2t + A_2 \cos 2t) \tag{4.101}$$

- The particular integral is evaluated taking the driver as a constant $f(t) = 1$, so that

$$x_p(t) = \frac{1}{H(0)} = \frac{1}{10} \tag{4.102}$$

- The total solution is then

$$x(t) = (A_1 \sin 2t + A_2 \cos 2t)e^{-t} + \frac{1}{10} \tag{4.103}$$

- Since $x'(t)$ and $x(t)$ must be continuous for a step forcing function

$$x(0+) = x(0-) = 0; \quad x'(0+) = x'(0-) = 0 \tag{4.104}$$

[Example 4.7] Step & Impulse Response Cont'd

- Substituting these initial conditions into $x(t)$ and $x'(t)$, we find that $A_1 = -0.05$, $A_2 = -0.1$. Therefore, the step response is

$$x_u(t) = 0.1[1 - e^{-t}(0.5\sin 2t + \cos 2t)]u(t) \qquad \textbf{(4.105)}$$

$$x_\rho(t) \xrightarrow{d/dt} x_u(t) \xrightarrow{d/dt} x_\delta(t)$$

$$x_\delta(t) \xrightarrow{\int_{0-}^{t}} x_u(t) \xrightarrow{\int_{0-}^{t}} x_\rho(t)$$

Fig. 4.1.

- It is worth noting that the impulse and step responses are related by the eqn.

$$x_\delta(t) = \frac{d}{dt} x_u(t) \qquad \textbf{(4.106)}$$

- Let us demonstrate Eq. 4.106, by substituting $x_u(t)$ into the original eq.

$$2\frac{d^2}{dt^2} x_u(t) + 4\frac{d}{dt} x_u(t) + 10x_u(t) = u(t) \qquad \textbf{(4.107)}$$

[Example 4.7] Step & Impulse Response Cont'd

- Differentiating both sides, we have

$$2\frac{d^2}{dt^2}\left\{\frac{d}{dt}x_u(t)\right\}+4\frac{d}{dt}\left\{\frac{d}{dt}x_u(t)\right\}+10\left\{\frac{d}{dt}x_u(t)\right\}=\delta(t) \qquad \textbf{(4.108)}$$

  from which Eq. 4.106 follows.

- Generalizing, we see that, if we have a step response for a DE, we can obtain the impulse response to by differentiating the step response.
- In like manner we can obtain the response to a ramp function $f(t) = A\rho(t)$ (where *A* is the height of the step) by integrating the step response. Fig. 4.1 summarizes the relationships discussed.

4.5 Integrodifferential Equations

- Consider an integrodifferential equation of the form

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \cdots + a_0 x(t) + a_{-1} \int_0^t x(\tau) d\tau = f(t) \qquad \textbf{(4.109)}$$

where the coefficients $\{a_n, a_{n-1}, \ldots, a_{-1}\}$ are constants. In solving an equation of the form of Eq. 4.109 we use two very similar methods.

- The first method is to differentiate both sides of Eq. 4.109 to give

$$a_n x^{(n+1)}(t) + a_{n-1} x^{(n)}(t) + \cdots + a_0 x'(t) + a_{-1} x(t) = f'(t) \qquad \textbf{(4.110)}$$

- The second method consists of a change of variables. We let $y'(t) = x(t)$; Eq. 4.109 then becomes

$$a_n y^{(n+1)}(t) + a_{n-1} y^{(n)}(t) + \cdots + a_0 y'(t) + a_{-1} y(t) = f(t) \qquad \textbf{(4.111)}$$

- From Eq. 4.110 we obtain $x(t)$ directly. From Eq. 4.111, we obtain $y(t)$, which we must then differentiate to obtain $x(t)$.

[Example 4.8] Integrodifferential Equations

Solve the integrodifferential equation

$$x'(t) + 3x(t) + 2\int_0^t x(\tau)d\tau = 5u(t) \quad \textbf{(4.112)}$$

with initial condition $x(0-) = 1$.

[Solution]

Since the characteristic equation of Eq. 4.112 is of second degree, we need an additional initial condition $x'(0+)$. We obtain $x'(0+)$ from the given equation at $t = 0+$;

$$x'(0+) + 3x(0+) + 2\int_0^{0+} x(\tau)d\tau = 5 \quad \textbf{(4.113)}$$

Since $x(t)$ is continuous at $t = 0$,

$$\int_0^{0+} x(\tau)d\tau = 0 \quad \textbf{(4.114)}$$

and

$$x(0+) = x(0-) = 1 \quad \textbf{(4.115)}$$

[Example 4.8] Integrodifferential Equations Cont'd

Therefore, $x'(0+) = 5 - 3x(0+) = 2$ **(4.116)**

Method 1. Differentiating both sides of Eq. 4.112, we obtain

$$x''(t) + 3x'(t) + 2x(t) = 5\delta(t) \quad \textbf{(4.117)}$$

The complementary function is then

$$x_c(t) = C_1 e^{-t} + C_2 e^{-2t} \quad \textbf{(4.118)}$$

Using the initial conditions for $x(0+)$ and $x'(0+)$, we obtain the total solution

$$x(t) = 4e^{-t} - 3e^{-2t} \quad \textbf{(4.119)}$$

Method 2. Letting $y'(t) = x(t)$, the original DE then becomes

$$y''(t) + 3y'(t) + 2y(t) = 5u(t) \quad \textbf{(4.120)}$$

We know that $y'(0+) = x(0+) = 1; \;\; y''(0+) = x'(0+) = 2$ **(4.121)**

[Example 4.8] Integrodifferential Equations Cont'd

- From Eq. 4.120, at $t = 0+$, we obtain

$$y(0+) = \frac{1}{2}\left[5 - y''(0+) - 3y'(0+)\right] = 0 \quad \textbf{(4.122)}$$

- Eliminating the detailed working, the total solution can be determined as

$$y(t) = -4e^{-t} + \frac{3}{2}e^{-2t} + \frac{5}{2} \quad \textbf{(4.123)}$$

- Differentiating $y(t)$, we have

$$x(t) = y'(t) = 4e^{-t} - 3e^{-2t} \quad \textbf{(4.124)}$$

4.6 Simultaneous Differential Equations

- All this time we considered only DEs with a single dependent variable $x(t)$. We now discuss equations in more than one variable.
- We shall limit us to equations in two unknowns, $x(t)$ and $y(t)$. Nevertheless, the methods described here are applicable to any number of unknowns.
- First, consider the system of homogeneous equations

$$\begin{aligned} \alpha_1 x'(t) + \alpha_0 x(t) + \beta_1 y'(t) + \beta_0 y(t) = 0 \\ \gamma_1 x'(t) + \gamma_0 x(t) + \delta_1 y'(t) + \delta_0 y(t) = 0 \end{aligned} \qquad \textbf{(4.125)}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are arbitrary constants. The complementary function is obtained by assuming that

$$x(t) = C_1 e^{pt}; \qquad y(t) = C_2 e^{pt}$$

4.6 Simultaneous Differential Equations Cont'd

so the characteristic equation is given by the determinant

$$H(p) = \begin{vmatrix} (\alpha_1 p + \alpha_0) & (\beta_1 p + \beta_0) \\ (\gamma_1 p + \gamma_0) & (\delta_1 p + \delta_0) \end{vmatrix} \quad \textbf{(4.126)}$$

- The roots of $H(p)$ are found by setting the determinant equal to zero, that is,

$$(\alpha_1 p + \alpha_0)(\delta_1 p + \delta_0) - (\beta_1 p + \beta_0)(\gamma_1 p + \gamma_0) = 0 \quad \textbf{(4.127)}$$

- It is seen that a nontrivial solution of $H(p) = 0$ exists only if

$$(\alpha_1 p + \alpha_0)(\delta_1 p + \delta_0) \neq (\beta_1 p + \beta_0)(\gamma_1 p + \gamma_0) \quad \textbf{(4.128)}$$

- Assuming the above condition holds, we see that $H(p)$ is a second-degree polynomial in p and can be expressed in factored form as

$$H(p) = C(p - p_0)(p - p_1) \quad \textbf{(4.129)}$$

4.6 Simultaneous Differential Equations Cont'd

- where C is a constant multiplier. The complementary functions are

$$x(t) = K_1 e^{p_0 t} + K_2 e^{p_1 t}$$
$$y(t) = K_3 e^{p_0 t} + K_4 e^{p_1 t} \qquad \textbf{(4.130)}$$

and the constants K_1, K_2, K_3, K_4 are determined from initial conditions. As in the case of a single unknown, if $H(p)$ has a pair of double roots; i.e., if $p_0 = p_1$, then

$$x(t) = (K_1 + K_2 t)e^{p_0 t}$$
$$y(t) = (K_3 + K_4 t)e^{p_0 t} \qquad \textbf{(4.131)}$$

- If $H(p)$ has a pair of conjugate roots, $p_1 = \sigma + j\omega$; $p_1^* = \sigma - j\omega$

then

$$x(t) = M_1 e^{\sigma t} \sin(\omega t + \phi_1)$$
$$y(t) = M_2 e^{\sigma t} \sin(\omega t + \phi_2) \qquad \textbf{(4.132)}$$

[Example 4.9] Simultaneous Differential Equations

Consider the system of equations

$$2x'(t) + 4x(t) + y'(t) - y(t) = 0$$
$$x'(t) + 2x(t) + y'(t) + y(t) = 0 \qquad \textbf{(4.133)}$$

with initial conditions

$$x'(0+) = 2, \qquad y'(0+) = -3$$
$$x(0+) = 0, \qquad y(0+) = 1 \qquad \textbf{(4.134)}$$

[Solution]

The characteristic equation is

$$H(p) = \begin{vmatrix} 2p+4 & p-1 \\ p+2 & p+1 \end{vmatrix} = 0 \qquad \textbf{(4.135)}$$

Evaluating the determinant, we find that

$$H(p) = p^2 + 5p + 6 = (p+2)(p+3) \qquad \textbf{(4.136)}$$

[Example 4.9] Simultaneous Differential Equations Cont'd

so that

$$y(t) = K_1 e^{-2t} + K_2 e^{-3t}$$
$$x(t) = K_3 e^{-2t} + K_4 e^{-3t} \quad \textbf{(4.137)}$$

From the initial conditions $x'(0+) = 2,\ x(0+) = 0$, $K_3 = 2,\ K_4 = -2$.

From the conditions $y'(0+) = -3,\ y(0+) = 1$, $K_1 = 0,\ K_2 = 1$.

$$x(t) = 2e^{-2t} - 2e^{-3t}$$
$$y(t) = e^{-3t} \quad \textbf{(4.138)}$$

4.6 Simultaneous Differential Equations Cont'd

- Let us now determine the solutions for a set of nonhomogeneous DEs, using undetermined coefficients.
- Consider first an exponential forcing function given by the set of equations

$$\begin{aligned} \alpha_1 x'(t) + \alpha_0 x(t) + \beta_1 y'(t) + \beta_0 y(t) &= Ne^{\theta t} \\ \gamma_1 x'(t) + \gamma_0 x(t) + \delta_1 y'(t) + \delta_0 y(t) &= 0 \end{aligned} \tag{4.139}$$

- We first assume that

$$\begin{aligned} x_p(t) &= Ae^{\theta t} \\ y_p(t) &= Be^{\theta t} \end{aligned} \tag{4.140}$$

Then Eq. 4.139 becomes

$$\begin{aligned} (\alpha_1\theta + \alpha_0)A + (\beta_1\theta + \beta_0)B &= N \\ (\gamma_1\theta + \gamma_0)A + (\delta_1\theta + \delta_0)B &= 0 \end{aligned} \tag{4.141}$$

- The determinant for the set of Eqs. 4.141 is

$$H(\theta) = \Delta(\theta) = \begin{vmatrix} \alpha_1\theta + \alpha_0 & \beta_1\theta + \beta_0 \\ \gamma_1\theta + \gamma_0 & \delta_1\theta + \delta_0 \end{vmatrix} \tag{4.142}$$

4.6 Simultaneous Differential Equations Cont'd

here $H(\theta)$ is characteristic equation with $p = \theta$.

- We now determine *A* and *B* from $\Delta(\theta)$ and its cofactors, that is

$$A = \frac{N\Delta_{11}(\theta)}{\Delta(\theta)}; \qquad B = \frac{N\Delta_{12}(\theta)}{\Delta(\theta)} \tag{4.143}$$

where Δ_{ij} is the ij^{th} cofactor of $\Delta(\theta)$.

[Example 4.10] Simultaneous ODEs

Solve the set of equations

$$\begin{aligned} 2x'(t) + 4x(t) + y'(t) - y(t) &= 3e^{4t} \\ x'(t) + 2x(t) + y'(t) + y(t) &= 0 \end{aligned} \tag{4.144}$$

given conditions $x'(0+) = 1, x(0+) = 0, y'(0+) = 0, y(0+) = -1$.

[Solution] The complementary functions $x_c(t)$ and $y_c(t)$, as well as the characteristic equation $H(p)$, were determined in Example 4.9.

[Example 4.10] Simultaneous ODEs Cont'd

- We must find A and B in the equations

$$x_p(t) = Ae^{4t}; \quad y_p(t) = Be^{4t} \tag{4.145}$$

- The characteristic equation with $p = 4$ is

$$H(p) = \begin{vmatrix} 2(4)+4 & (4)-1 \\ (4)+2 & (4)+1 \end{vmatrix} = 42 \tag{4.146}$$

- From Eq. 4.143 we obtain the constants as

$$A = \frac{5}{14}; \quad B = -\frac{3}{7}$$

- The incomplete solutions are

$$\begin{aligned} x(t) &= K_1e^{-2t} + K_2e^{-3t} + \frac{5}{14}e^{4t} \\ y(t) &= K_3e^{-2t} + K_4e^{-3t} - \frac{3}{7}e^{4t} \end{aligned} \tag{4.147}$$

[Example 4.10] Simultaneous ODEs Cont'd

Substituting for the initial conditions gives

$$y(t) = \frac{1}{7}\left[-4e^{-3t} - 3e^{4t}\right]$$
$$x(t) = \frac{1}{14}\left[-6e^{-2t} + e^{-3t} + 5e^{4t}\right] \qquad \textbf{(4.148)}$$

[Example 4.11] Simultaneous ODEs

Solve the system of equations

$$2x'(t) + 4x(t) + y'(t) + 7y(t) = 5u(t)$$
$$x'(t) + x(t) + y'(t) + 3y(t) = 5\delta(t) \qquad \textbf{(4.149)}$$

given initial conditions $x(0-) = x'(0-) = y(0-) = y'(0-) = 0$.

[Example 4.11] Simultaneous ODEs Cont'd

[Solution]

- First we find the characteristic equation

$$H(p) = \Delta(p) = \begin{vmatrix} 2p+4 & p+7 \\ p+1 & p+3 \end{vmatrix} \qquad \textbf{(4.150)}$$

which simplifies to give

$$H(p) = [p^2 + 2p + 5] = (p + 1 + j2)(p + 1 - j2) \qquad \textbf{(4.151)}$$

- The complementary functions $x_c(t)$ and $y_c(t)$ are then

$$\begin{aligned} x_c(t) &= A_1 e^{-t} \cos 2t + A_2 e^{-t} \sin 2t \\ y_c(t) &= B_1 e^{-t} \cos 2t + B_2 e^{-t} \sin 2t \end{aligned} \qquad \textbf{(4.152)}$$

- The particular solutions are obtained for the set of equations with $t > 0$,

$$\begin{aligned} 2x'(t) + 4x(t) + y'(t) + 7y(t) &= 5 \\ x'(t) + x(t) + y'(t) + 3y(t) &= 0 \end{aligned} \qquad \textbf{(4.153)}$$

[Example 4.11] Simultaneous ODEs Cont'd

- Using the method of undetermined coefficients, we assume that $x_p(t)$ and $y_p(t)$ are constants: $x_p(t) = C_1$; $y_p(t) = C_2$.
- Let $5 = 5e^{0t}$, so that we solve for the constants using $H(p)$ with $p = 0$.

$$H(0) = \Delta(0) = \begin{vmatrix} 4 & 7 \\ 1 & 3 \end{vmatrix} = 5 \qquad \textbf{(4.154)}$$

and

$$C_1 = \frac{5\Delta_{11}(0)}{\Delta(0)} = \frac{5(3)}{5} = 3; \quad C_2 = -\frac{(5)1}{5} = -1 \qquad \textbf{(4.155)}$$

- The general solution is then

$$\begin{aligned} x(t) &= A_1 e^{-t}\cos 2t + A_2 e^{-t}\sin 2t + 3 \\ y(t) &= B_1 e^{-t}\cos 2t + B_2 e^{-t}\sin 2t - 1 \end{aligned} \qquad \textbf{(4.156)}$$

[Example 4.11] Simultaneous ODEs Cont'd

- We now obtain the initial conditions at $t = 0+$. The values $x(0+)$ and $y(0+)$ are obtained as follows.

$$\int_{0-}^{0+} [2x' + 4x + y' + 7y]dt = \int_{0-}^{0+} 5u(t)dt$$
$$\int_{0-}^{0+} [x' + x + y' + 3y]dt = \int_{0-}^{0+} 5\delta(t)dt \qquad \textbf{(4.157)}$$

- Only the highest derivative terms in both equations contain impulses at $t = 0$. Moreover, both $x(t)$ and $y(t)$ contain, at most, step discontinuities at $t = 0$.
- Therefore,

$$\int_{0-}^{0+} (4x + 7y)dt = 0$$
$$\int_{0-}^{0+} (x + 3y)dt = 0 \qquad \textbf{(4.158)}$$

[Example 4.11] Simultaneous ODEs Cont'd

After integrating, we obtain

$$2x(0+) + y(0+) = 0, \quad x(0+) + y(0+) = 5 \tag{4.159}$$

Solving gives

$$x(0+) = -5, \quad y(0+) = 10 \tag{4.160}$$

We now substitute $x(0+)$ and $y(0+)$ into the original equations, we get

$$\begin{aligned} 2x'(0+) - 20 + y'(0+) + 70 &= 5 \\ x'(0+) - 5 + y'(0+) + 30 &= 0 \end{aligned} \tag{4.161}$$

so that $x'(0+) = -20, \quad y'(0+) = -5$ **(4.162)**

Substituting the initial values into Eq. 4.156, gives the final solutions

$$\begin{aligned} x(t) &= [-8e^{-t}\cos 2t + -14e^{-t}\sin 2t + 3]u(t) \\ y(t) &= [11e^{-t}\cos 2t + 3e^{-t}\sin 2t - 1]u(t) \end{aligned} \tag{4.163}$$

End of Lecture 4

Thank you for your attention!