EEE 3121 - Signals & Systems

Lecture 5: System Analysis I

Instructor: Jerry MUWAMBA
Email: jerry.muwamba@unza.zm
jerrymuwamba@yahoo.com

March/April 2021

University of Zambia
School of Engineering,
Department of Electrical & Electronic Engineering

References

Our main reference text book in this course is

[1] B. P. Lathi and R. A. Green, Linear Systems and Signals, 3rd Ed., 2018, Oxford University Press, New York. ISBN 978-0-19-020017-6

[2] Kuo Franklin, F., Network Analysis and Synthesis, 3rd Ed., 1986, J. Wiley (SE), ISBN 0-471-51118-8.

[3] Sundararanjan, D., A Practical Approach to Signals and Systems, 2008, John Wiley & Sons (Asia) Pte Ltd, ISBN 978-0-470-82353-8.

However, feel free to use pretty much any additional text which you might find relevant to our course.

5.1 Introduction

- Here we apply our knowledge of differential equations to analyse Linear Time-Invariant continuous-time (LTIC) systems.
- Typically at time $t = 0$ a switch is closed which connects an energy source (voltage or current) source to a system as shown in Fig. 5.1.
- This is analogous with applying the energy signal $s(t) = e(t)u(t)$.
- Before the switch is closed, the currents and voltages in the systems have known values (initial conditions) at $t = 0-$.
- Thus, we need the initial conditions just after the switch closes at $t = 0+$, to solve the network equations

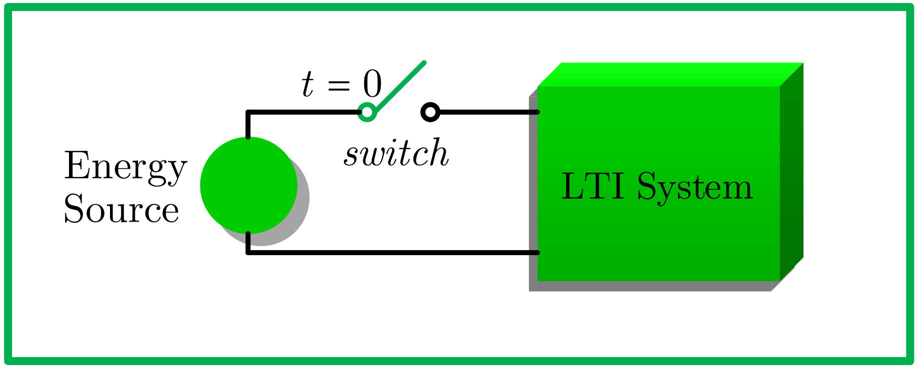


Fig. 5.1: Switching action

5.1 Introduction Cont'd

- By the superposition principle, the current through any element in a linear circuit with n voltage and m current sources is equal to the algebraic sum of currents through the same element resulting from the sources taken one at a time, the other sources having been suppressed.

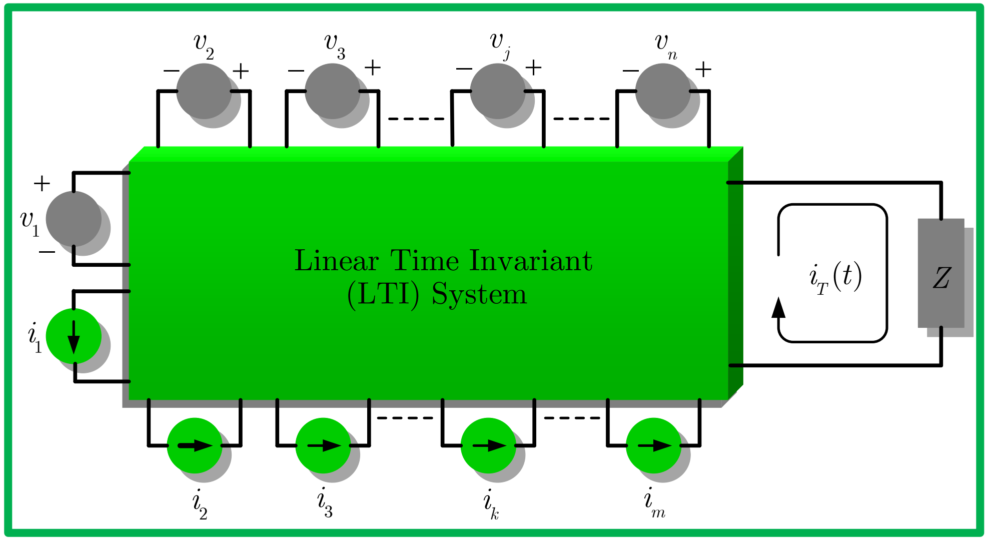


Fig. 5.2*a*

- For the LTI system in Fig. 5.2*a*, to find the current $i_T(t)$ through an element Z shown:
- We open-circuit all current sources and short-circuit $n-1$ voltage sources, to leave only $v_j(t)$, see Fig. 5.2*b*.
- Let $i_{v_j}(t)$ denote the current through Z due to $v_j(t)$ alone.

5.1 Introduction

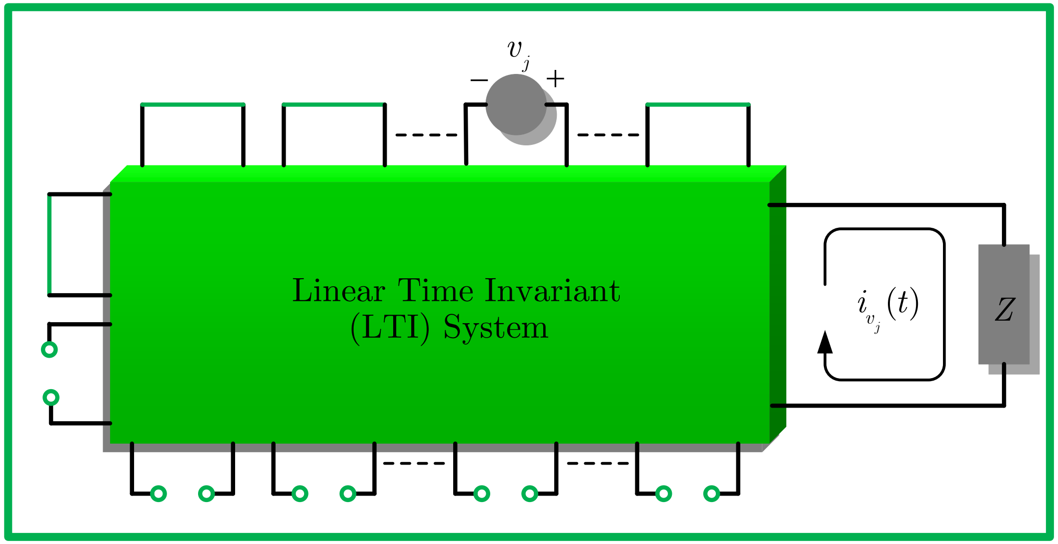


Fig. 5.2*b*

- Similarly, let $i_{c_k}(t)$ denote the current through Z due to the current source $i_k(t)$ acting alone, see Fig. 5.2*c*.
- By the superposition principle, the total current $i_T(t)$ due to all of the sources is equal to the algebraic sum, see Eq. 5.1.

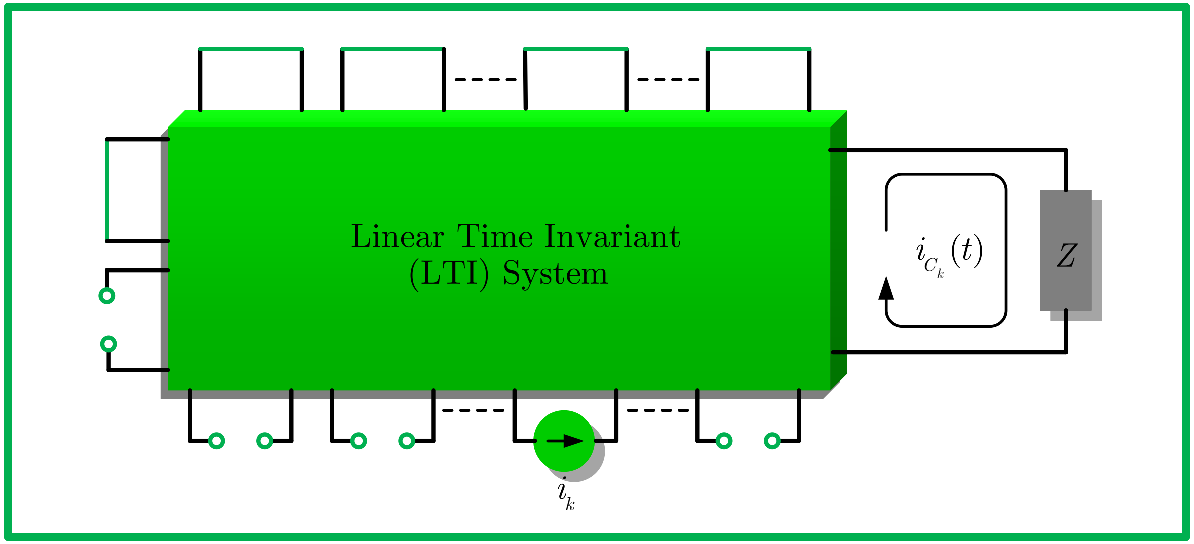


Fig. 5.2*c*

5.2 System Elements

$$i_T = \sum_{j=1}^{n} i_{v_j} + \sum_{k=1}^{m} i_{c_k} \qquad (5.1)$$

- Here, we will assume that the positive polarity for voltage is the tail of the current arrow, see Fig. 5.3. A recap of the voltage-current relations for R, L and C.

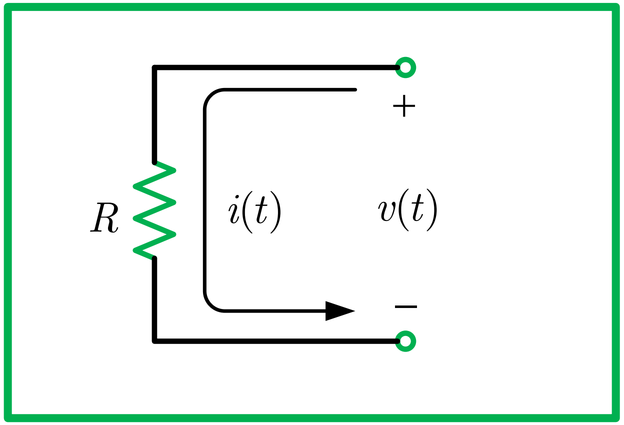


Fig. 5.3: Resistor.

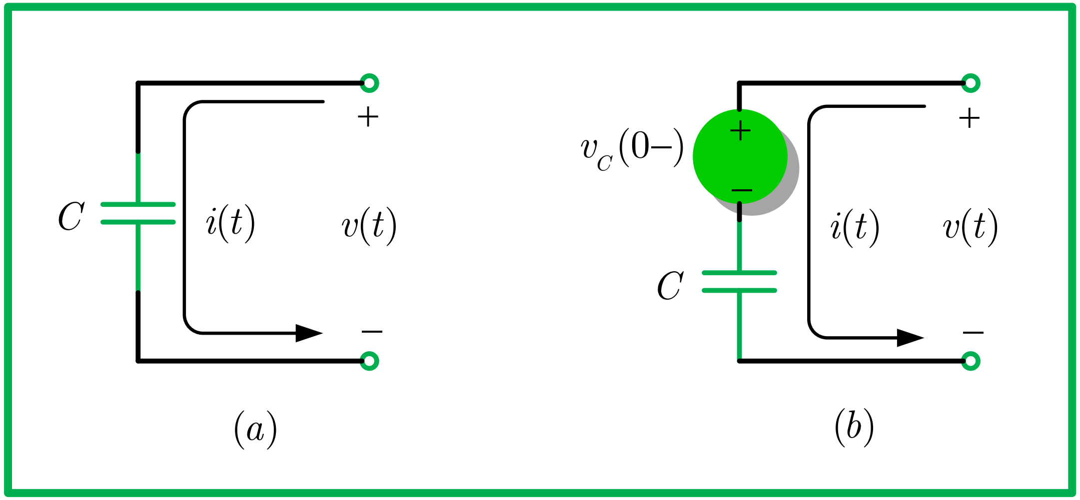


Fig. 5.4: (*a*) Capacitor; (*b*) Capacitor with initial voltage.

5.2 System Elements Cont'd

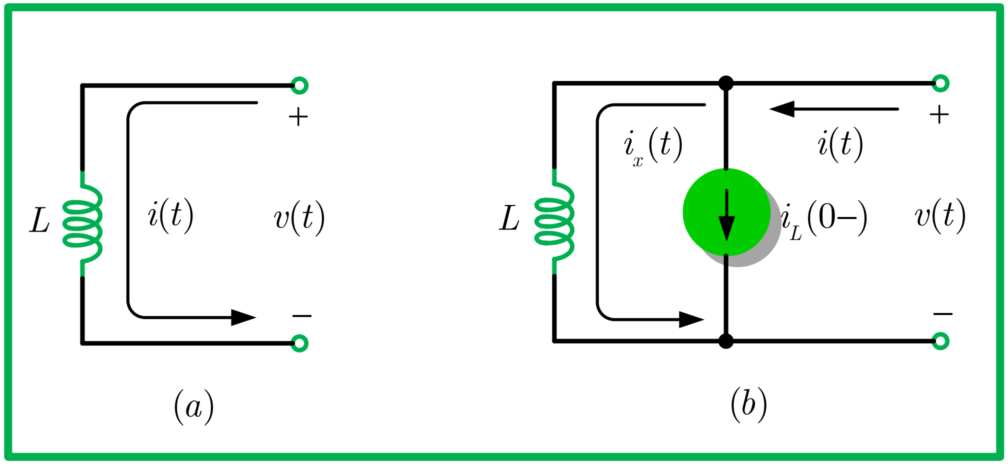


Fig. 5.5: (*a*) Inductor. (*b*) Inductor with initial current.

- **Resistor.** The resistor relates voltage and current as follows, see Fig. 5.3.

$$v(t) = Ri(t) \tag{5.2}$$

$$i(t) = Gv(t)\ ;\ \text{with } G = \frac{1}{R}$$

here R is given in ohms and G in mhos.

- **Capacitor.** For the capacitor shown in Fig. 5.4*a* the v-i relationships are

$$i(t) = C\frac{dv(t)}{dt} \qquad v(t) = \frac{1}{C}\int_{0-}^{t} i(\tau)\,d\tau + v_C(0-) \tag{5.3}$$

where C is given in farads. The initial value $v_C(0-)$ is the voltage across the capacitor just before the switching action.

5.2 System Elements Cont'd

- Note that, $v_C(0-) = v_C(0+)$ for all excitations except impulses and their derivatives.
- **Inductor.** The inductor in Fig. 5.5a describes the dual relationship between voltage and current when compared to a capacitor. Thus,

$$v(t) = L\frac{di(t)}{dt} \qquad i(t) = \frac{1}{L}\int_{0-}^{t} v(\tau)\,d\tau + i_L(0-) \qquad \textbf{(5.4)}$$

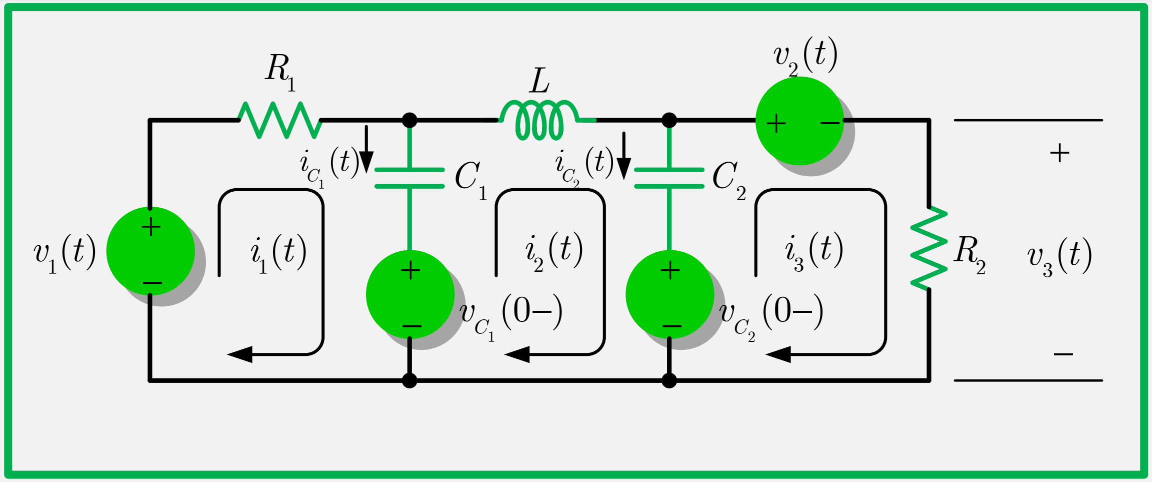
where L is in henrys. The initial current $i_L(0-)$ can be regarded as an independent current source, as shown in Fig. 5.5*b*.

- Note once again that, $i_L(0-) = i_L(0+)$ for all excitations except for impulses and their derivatives.

5.2 System Elements Cont'd

- If system elements are interconnected, integrodifferential equations arise relating the excitation to the response.
- Kirchhoff's voltage and current laws will be exploited to formulate equations.
- Recall, if the number of branches in the circuit is B, and if the number of nodes is N, the number of independent loop equations for the circuit is $B - N + 1$.

[Example 5.1] System Elements



For the system in Fig. 5.6,

a) Determine the number of independent loop equations.

b) Hence, formulate these loop equations.

Fig. 5.6.

[Example 5.1] System Elements Cont'd

[Solution]

(a) Clearly the circuit of Fig. 5.6 has seven branches and five nodes. Thus, there are $7-5+1=3$ independent mesh equations.

(b) Noting that capacitors in the circuit have associated initial voltages, the mesh equations are:

Mesh i_1 : KVL yields

$$v_1(t)-v_{C_1}(0-)=R_1 i_1(t)+\frac{1}{C_1}\int_{0-}^{t} i_1(\tau)d\tau-\frac{1}{C_1}\int_{0-}^{t} i_2(\tau)d\tau$$

Mesh i_2 : KVL yields

$$v_{C_1}(0-)-v_{C_2}(0-)=-\frac{1}{C_1}\int_{0-}^{t} i_1(\tau)d\tau+L\frac{di_2(t)}{dt}$$
$$+\left\{\frac{1}{C_1}+\frac{1}{C_2}\right\}\int_{0-}^{t} i_2(\tau)d\tau-\frac{1}{C_2}\int_{0-}^{t} i_3(\tau)d\tau$$

[Example 5.1] System Elements Cont'd

- *Mesh* i_3 : KVL yields

$$-v_2(t) + v_{C_2}(0-) = -\frac{1}{C_2}\int_{0-}^{t} i_2(\tau)d\tau + \frac{1}{C_2}\int_{0-}^{t} i_3(\tau)d\tau + R_2 i_3(t) \qquad \textbf{(5.5)}$$

- Upon determining i_1, i_2 and i_3 , we can determine branch currents and voltages across the elements, for instance,

$$\begin{aligned} i_{C_1}(t) &= i_1(t) - i_2(t) \\ i_{C_2}(t) &= i_2(t) - i_3(t) \end{aligned} \qquad \textbf{(5.6)}$$

- If the voltage $v_3(t)$ in Fig. 5.6 is our objective, we see that

$$v_3(t) = i_3(t)R_2 \qquad \textbf{(5.7)}$$

5.2 System Elements

- Let us use KCL for nodal analysis of a system. If the number of nodes in the network is *N*, the number of independent node equations required is $N-1$.

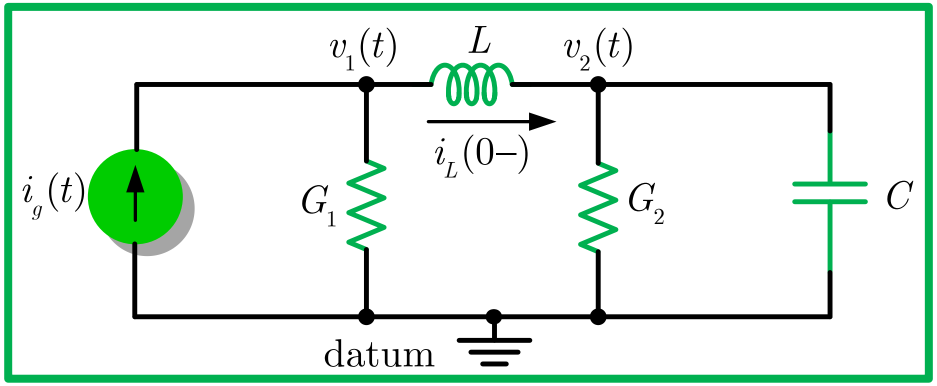


Fig. 5.7.

- Consider the network in Fig. 5.7. We write a set of node equations in the network with the datum node shown.
- Since the number of nodes in the network $N=3$, we need $N-1=2$ independent equations.

- With respect to the nodes v_1 and v_2, the node equations are
- *Node* v_1 :

$$i_g(t) - i_L(0-) = G_1 v_1(t) + \frac{1}{L}\int_{0-}^{t} v_1(\tau)\,d\tau - \frac{1}{L}\int_{0-}^{t} v_2(\tau)\,d\tau$$

- *Node* v_2 :

$$i_L(0-) = -\frac{1}{L}\int_{0-}^{t} v_1(\tau)\,d\tau + \frac{1}{L}\int_{0-}^{t} v_2(\tau)\,d\tau + C\frac{dv_2(t)}{dt} + G_2 v_2(t) \tag{5.8}$$

5.3 Initial and Final Conditions

- We consider some methods for obtaining initial conditions for circuit differential equations.
- Examine ways to obtain particular integrals for systems with constant (dc) or sinusoidal (ac) excitations.
- In the solution of system ODEs, the complementary function is called the transient solution or free response.
- The particular integral is known as the forced response. In the case of constant or periodic excitations, the forced response at $t = \infty$ is the steady-state or final solution.
- Initial conditions at $t = 0+$ for a system are gotten through the ODEs describing the system or through the knowledge of the physical behavior of the R, L, and C elements in the system.

5.3 Initial and Final Conditions Cont'd

- **Initial conditions for a capacitor**. For a capacitor, the v-i relationship at $t = 0+$ is

$$v_C(0+) = \frac{1}{C}\int_{0-}^{0+} i(\tau)\,d\tau + v_C(0-) \qquad \textbf{(5.9)}$$

- If $i(t)$ contains no impulses or their derivatives, $v_C(0+) = v_C(0-)$. If q is the charge on the capacitor at $t = 0-$, the initial voltage is

$$v_C(0+) = v_C(0-) = \frac{q}{C} \qquad \textbf{(5.10)}$$

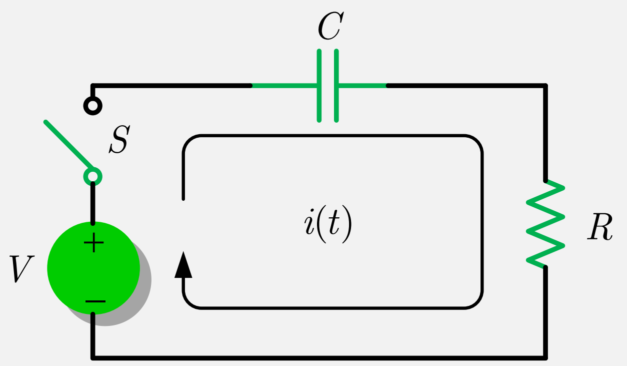
- At $t = 0+$, we can replace the capacitor by a voltage source if the initial charge exist, or by a short circuit if there is no initial charge.

[Example 5.2] Initial and Final Conditions

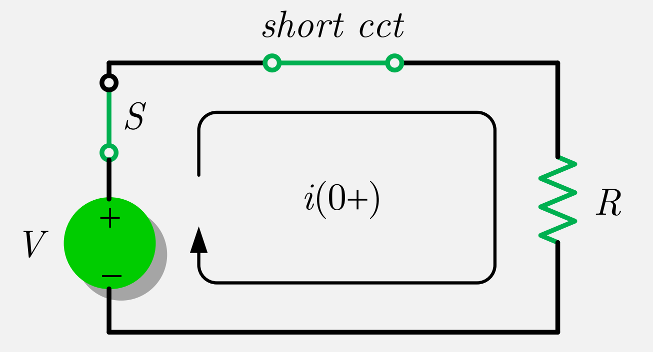
Consider the *R*-*C* network in Fig. 5.8*a*. The switch is closed at $t = 0$, and we assume there is no initial charge in the capacitor. Let us find the initial conditions $i(0+)$ and $i'(0+)$ for the ODE of the circuit.

[Example 5.2] Initial and Final Conditions Cont'd

[Solution]



R-C Network
(a)



Equivalent circuit at $t = 0+$
(b)

Fig. 5.8.

KVL applied to Fig. 5.8a yields,

$$Vu(t) = Ri(t) + \frac{1}{C}\int_{0-}^{t} i(\tau)\, d\tau \qquad \textbf{(5.11)}$$

The equivalent circuit at $t = 0+$ is shown in Fig. 5.8*b*, from which we get

$$i(0+) = \frac{V}{R} \qquad \textbf{(5.12)}$$

[Example 5.2] Initial and Final Conditions Cont'd

- To obtain $i'(0+)$, we must refer to the ODE

$$V\delta(t) = Ri'(t) + \frac{i(t)}{C} \qquad \textbf{(5.13)}$$

- At $t = 0+$ we have $$0 = Ri'(0+) + \frac{i(0+)}{C} \qquad \textbf{(5.14)}$$

- We the obtain $$i'(0+) = -\frac{i(0+)}{RC} = -\frac{V}{R^2C} \qquad \textbf{(5.15)}$$

- The final condition, or steady-state solution, for the current in Fig. 5.8*a* is obtained from our knowledge of dc circuits. For a dc excitation, a capacitor is an open circuit for dc current. Thus the steady-state current is

$$i_p(t) = i(\infty) = 0$$

5.3 Initial and Final Conditions

- **Initial conditions for an inductor**. For an inductor, the v-i relationship at $t = 0+$ is

$$i_L(0+) = \frac{1}{L}\int_{0-}^{0+} v(\tau)\,d\tau + i_L(0-) \qquad (5.16)$$

- If $v(t)$ contains no impulses or their derivatives, $i_L(0+) = i_L(0-)$. If there is no initial current, $i_L(0+) = 0$, which corresponds to an open circuit at $t = 0+$.
- Note that the current through an inductor cannot change instantaneously due to the conservation of flux linkage.

[Example 5.3] Initial and Final Conditions

In Fig. 5.9a, the switch closes at $t = 0$. Let us find the initial conditions $i(0+)$ and $i'(0+)$ for the differential equation

$$Vu(t) = L\frac{di(t)}{dt} + Ri(t) \qquad (5.17)$$

[Example 5.3] Initial and Final Conditions Cont'd

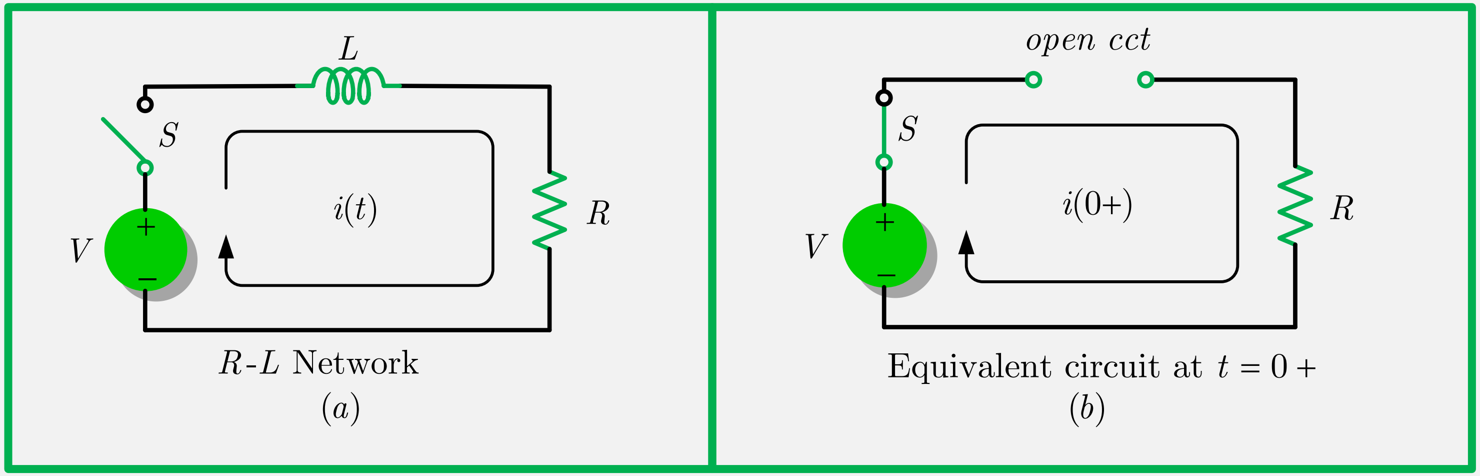


Fig. 5.9.

From the equivalent circuit at $t = 0+$, shown in Fig. 5.9*b*, we see that

$$i(0+) = 0 \tag{5.18}$$

We then refer to the differential equation to obtain $i'(0+)$.

$$V = Li'(0+) + Ri(0+) \tag{5.19}$$

Thus

$$i'(0+) = \frac{V}{L} - \frac{R}{L} i(0+) = \frac{V}{L} \tag{5.20}$$

[Example 5.3] Initial and Final Conditions Cont'd

The steady-state solution for the circuit in Fig. 5.9a is obtained through the knowledge that for a dc source, an inductor is a short circuit.

$$i_p(t) = i(\infty) = \frac{V}{R} \quad \textbf{(5.21)}$$

[Example 5.4] Initial and Final Conditions

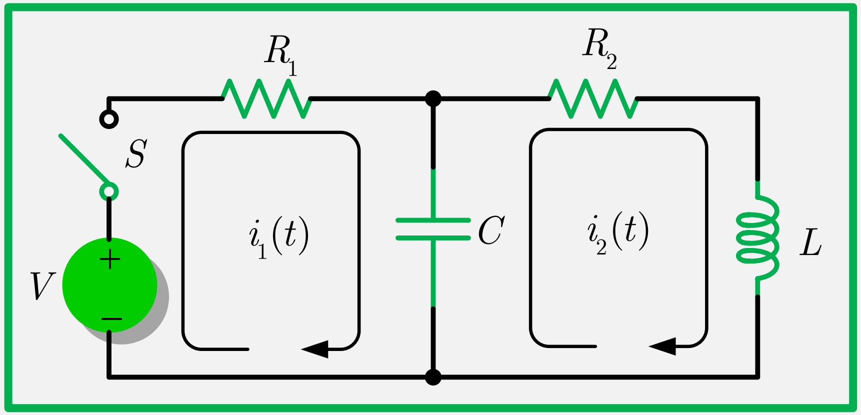


Fig. 5.10.

For the system of Fig. 5.10 the switch is closed at $t = 0$. Use the equivalent circuit models at $t = 0+$ and $t = \infty$ to obtain the initial conditions and steady-state solutions.

[Example 5.4] Initial and Final Conditions Cont'd

[Solution]

- At $t = 0+$ the circuit is as shown in Fig. 5.11a and at $t = \infty$ as shown in Fig. 5.11b.

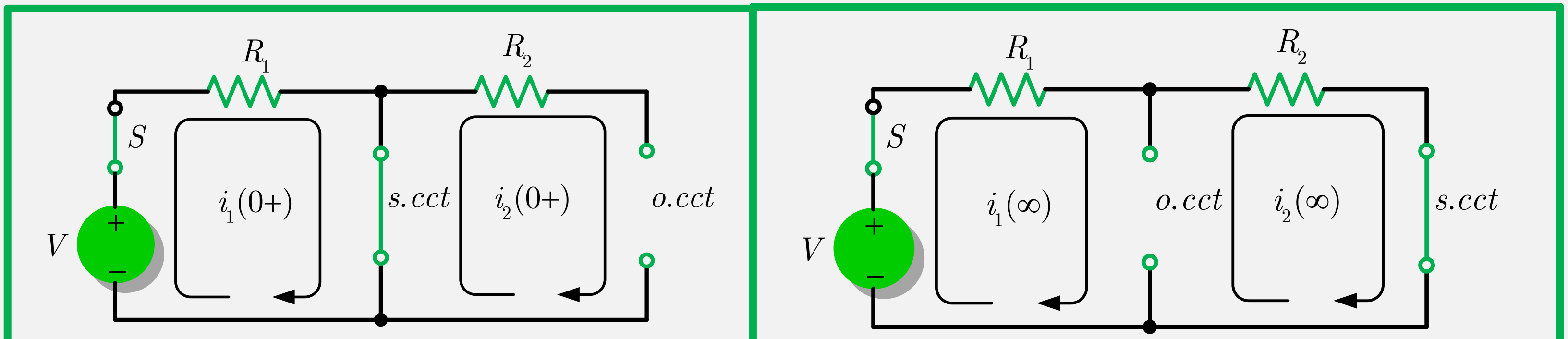


Fig. 5.11. Equivalent circuit at $t = 0+$ (a); Equivalent circuit at $t = \infty$ (b)

- Vividly the initial currents are:

$$i_1(0+) = \frac{V}{R_1}; \qquad i_2(0+) = 0 \qquad \textbf{(5.22)}$$

- The steady-state solutions are

$$i_1(\infty) = \frac{V}{R_1 + R_2} = i_2(\infty) \qquad \textbf{(5.23)}$$

5.3 Initial and Final Conditions Cont'd

- **Final conditions for sinusoidal excitations**
- For a pure sinusoidal excitation, the steady-state currents and voltages in the circuit are also sinusoids of the same frequency as the excitation.
- If the unknown is a voltage, for example, $v_1(t)$, the steady-state solution would take the form of Eq. 5.24.

$$v_{1p}(t) = \left|V(j\omega_0)\right| \sin\left[\omega_0 t - \phi(\omega_0)\right] \qquad \textbf{(5.24)}$$

where ω_0 is the frequency of the excitation, and $\left|V(j\omega_0)\right|$ and $\phi(\omega_0)$ are the magnitude and phase of $v_{1p}(t)$ respectively.

- A similar expression would hold if the unknown were a current.

5.3 Initial and Final Conditions Cont'd

- Standard procedures in ac circuit analysis are exploited to obtain the magnitude and the phase. For example, consider the *R*-*C* circuit in Fig. 5.12.
- The current generator is

$$i_g(t) = (I_0 \sin \omega_0 t)u(t) \quad \textbf{(5.25)}$$

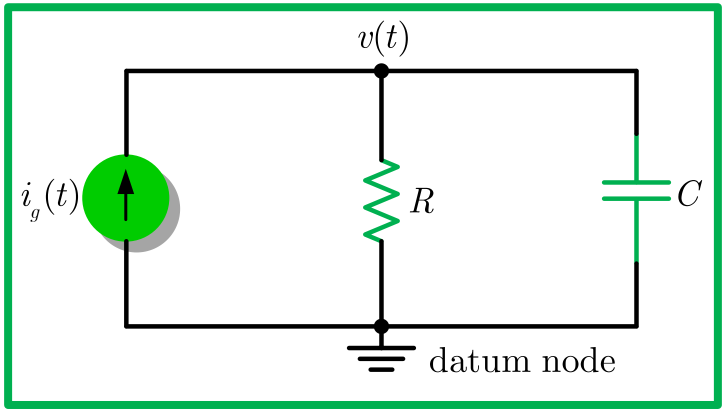


Fig. 5.12.

- If the steady-state voltage takes the form shown in Eq. 5.24,

$$\left|V(j\omega_0)\right| = \frac{I_0}{\left|Y(j\omega_0)\right|} = \frac{I_0}{\left(G^2 + \omega_0{}^2C^2\right)^{1/2}} \quad \textbf{(5.26)}$$

- and

$$\phi(\omega_0) = -\tan^{-1}\frac{\omega_0 C}{G} \quad \textbf{(5.27)}$$

- so that

$$v_p(t) = \frac{I_0}{\left(G^2 + \omega_0{}^2C^2\right)^{1/2}} \sin\left\{\omega_0 t - \tan^{-1}\frac{\omega_0 C}{G}\right\} \quad \textbf{(5.28)}$$

5.4 Step and Impulse Response

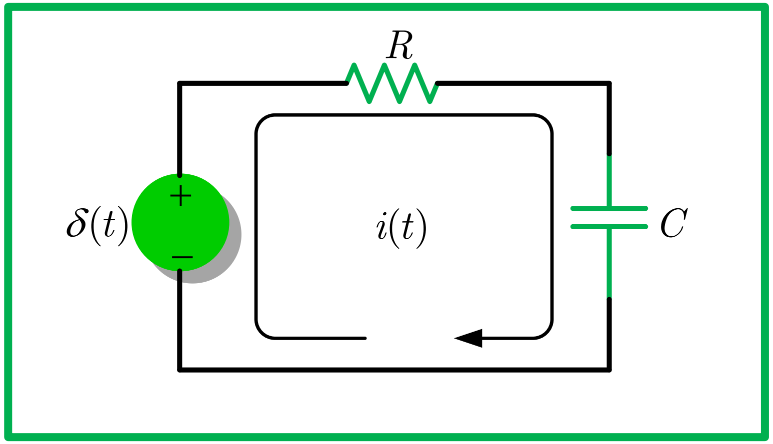


Fig. 5.13.

- If the excitation is a step voltage, the physical analogy is that of a switch-closing at time $t = 0$, which connects a 1 V battery to a circuit.
- The physical analogy of an impulse excitation is that of a very short pulse (compared to the time constants of the circuit) with large amplitude.

[Example 5.5] Step and Impulse Response

For the series *R*-*C* circuit in Fig. 5.13 find the impulse response.

[Solution]

The ODE of the circuit is

$$v(t) = \delta(t) = Ri(t) + \frac{1}{C}\int_{0-}^{t} i(\tau)\, d\tau \qquad \textbf{(5.29)}$$

Assuming $v_C(0-) = 0$.

[Example 5.5] Step and Impulse Response Cont'd

- Since Eq. 5.29 contains an integral, we substitute $x'(t)$ for $i(t)$ in the equation, which yields

$$\delta(t) = Rx'(t) + \frac{1}{C}x(t) \tag{5.30}$$

- Integrating both sides between $0-$ and $0+$ gives

$$x(0+) = \frac{1}{R} \tag{5.31}$$

- The characteristic equation is

$$H(p) = Rp + \frac{1}{C} \tag{5.32}$$

and with little effort we have

$$x(t) = \frac{1}{R}e^{-t/RC}u(t) \tag{5.33}$$

[Example 5.5] Step and Impulse Response Cont'd

Thus,

$$i(t) = \frac{1}{R}\left[\delta(t) - \frac{1}{RC}e^{-t/RC}u(t)\right] \qquad (5.34)$$

This is shown in Fig. 5.14*a*.

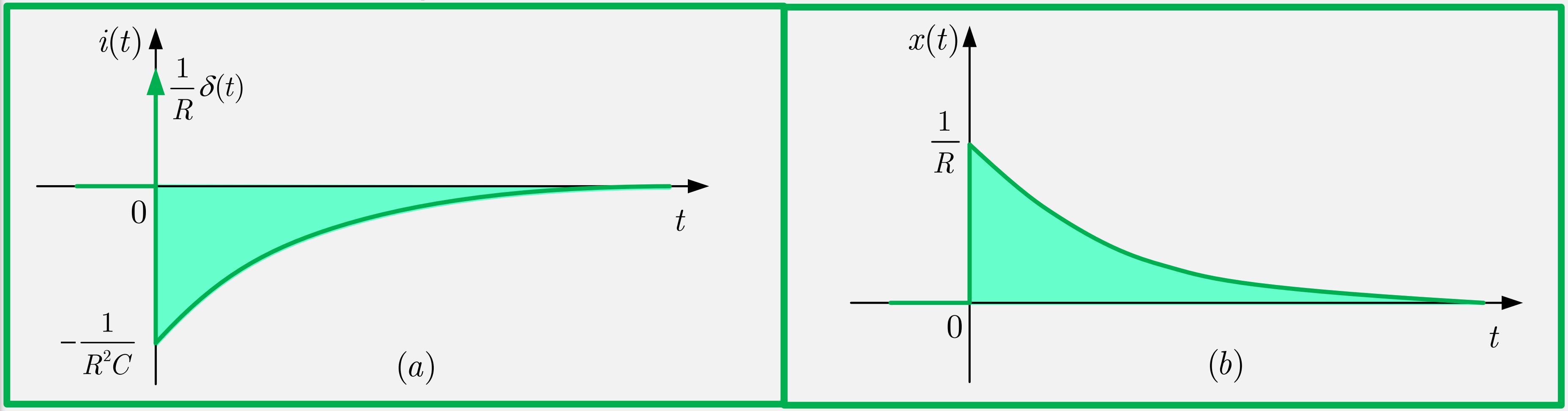


Fig. 5.14. (*a*) Impulse response of *R-C* circuit. (*b*) Step response of *R-C* circuit.

- We arrived at the current impulse response $i(t)$ as the result of an impulse voltage excitation. In the process we obtained the step response $x(t)$ in Fig. 5.14*b*.

[Example 5.6] Step and Impulse Response

- Consider the parallel R-C circuit in Fig. 5.12. Let it be driven by a step current source $i(t) = I_0 u(t)$, where I_0 is a constant.
- Assuming zero initial conditions, determine the step response.

[Solution]

- The ODE is
$$I_0 u(t) = Gv(t) + C\frac{dv(t)}{dt} \quad \textbf{(5.35)}$$
- Thus, we obtain the characteristic equation
$$H(p) = Cp + G \quad \textbf{(5.36)}$$
- The steady-state value of $v(t)$ is
$$v_p(t) = \frac{I_0}{H(0)} = \frac{I_0}{G} = I_0 R \quad \textbf{(5.37)}$$
- Thus the complete solution for the voltage step response is
$$v(t) = (Ke^{-t/RC} + I_0 R)u(t) \quad \textbf{(5.38)}$$

[Example 5.6] Step and Impulse Response Cont'd

From the initial condition $v(0+) = 0$, we obtain $K = -I_0 R$ so that

$$v(t) = I_0 R(1 - e^{-t/RC})u(t) \tag{5.39}$$

Differentiating Eq. 5.39 gives us the voltage impulse response

$$v_\delta(t) = \frac{I_0}{C} e^{-t/RC} u(t) \tag{5.40}$$

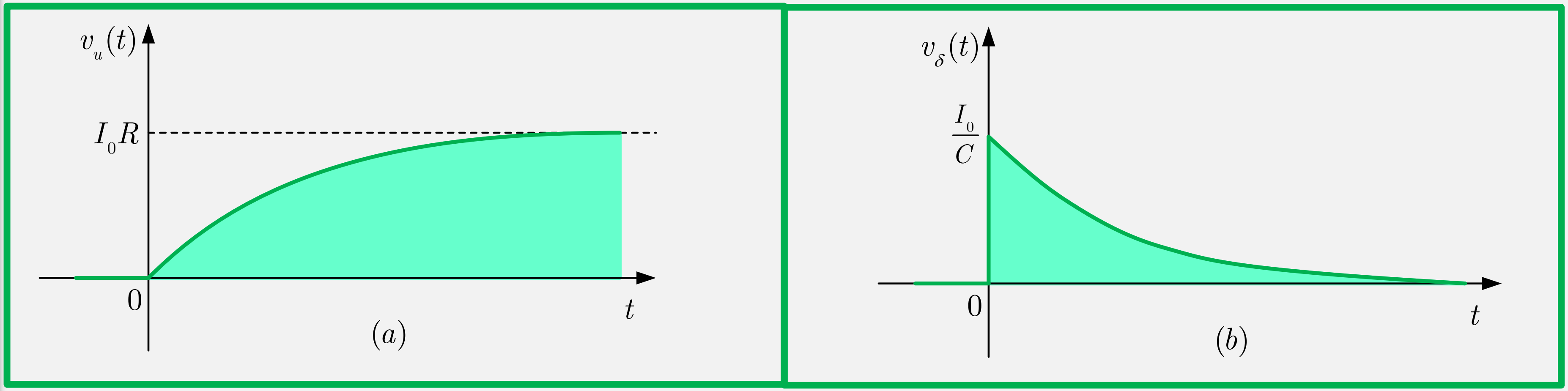


Fig. 5.15. (*a*) Step response of parallel *R*-*C* circuit. (*b*) Impulse response of parallel *R*-*C* circuit.

5.4 Step and Impulse Response Cont'd

- Suppose the excitation in Fig. 5.12 were a pulse shown in Fig. 5.16.

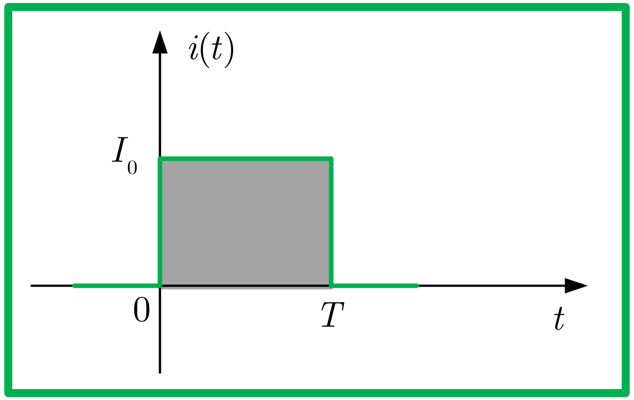


Fig. 5.16. Pulse excitation.

$$i(t) = I_0\left[u(t) - u(t-T)\right] \tag{5.41}$$

- Then by the superposition and time-invariant postulates of linear systems, the response would be

$$v(t) = I_0 R\left[(1 - e^{-t/RC})u(t) - (1 - e^{-(t-T)/RC})u(t-T)\right] \tag{5.42}$$

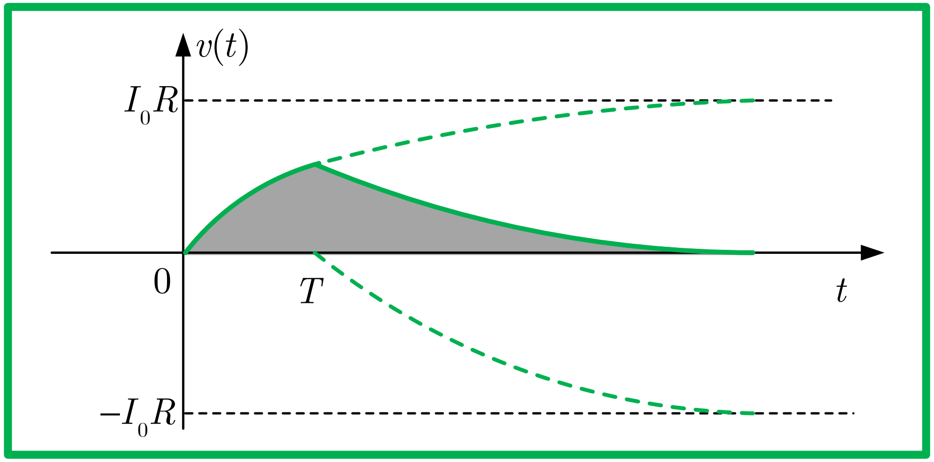


Fig. 5.17. Response of pulse excitation.

5.5 Solution of System Equations

- System equations are made up of mesh, node, or mixed basis.
- The choice between mesh and node equations depends largely upon the unknown quantities.

[Example 5.7] Solution of System Equations

Find the current $i(t)$ for the network in Fig. 5.18, when the voltage source is $e(t) = 2e^{-0.5t}u(t)$ and $v_C(0-) = 0$.

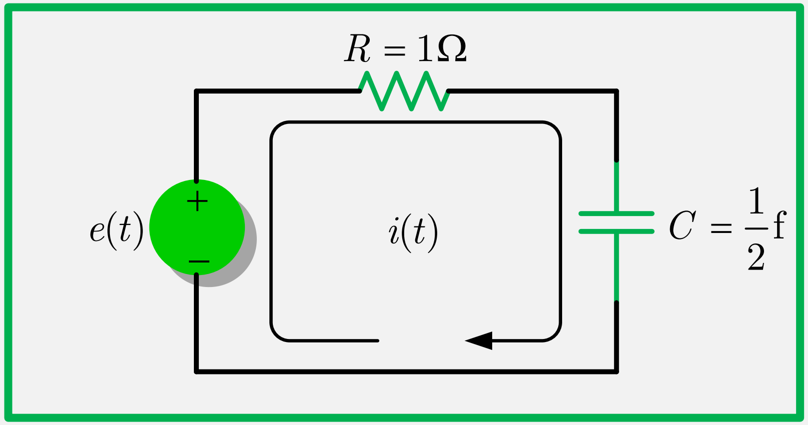


Fig. 5.18.

[Solution]

The DE is

$$e(t) = Ri(t) + \frac{1}{C}\int_{0-}^{t} i(\tau)d\tau + v_C(0-) \quad \textbf{(5.43)}$$

[Example 5.7] Solution of System Equations Cont'd

- Or, in terms of the numerical values, we have

$$2e^{-0.5t}u(t) = i(t) + 2\int_{0-}^{t} i(\tau)\,d\tau \qquad \textbf{(5.44)}$$

- Differentiating both sides of Eq. 5.44, we obtain

$$2\delta(t) - e^{-0.5t}u(t) = \frac{di(t)}{dt} + 2i(t) \qquad \textbf{(5.45)}$$

- To obtain the initial condition $i(0+)$, we must integrate Eq. 5.45 between the limits $t = 0-$ and $t = 0+$ to give $i(0+) = 2$. From the characteristic eq.

$$H(p) = p + 2 = 0 \qquad \textbf{(5.46)}$$

- The complementary function is thus

$$i_C(t) = Ke^{-2t} \qquad \textbf{(5.47)}$$

- Assume the particular integral to be $i_p(t) = Ae^{-0.5t}$, then we obtain

[Example 5.7] Solution of System Equations Cont’d

$$A = -\frac{1}{H(-0.5)} = -\frac{2}{3} \quad \textbf{(5.48)}$$

The complete solution is

$$i(t) = Ke^{-2t} - \frac{2}{3}e^{-0.5t} \quad \textbf{(5.49)}$$

From the initial condition $i(0+) = 2$, we obtain the final solution,

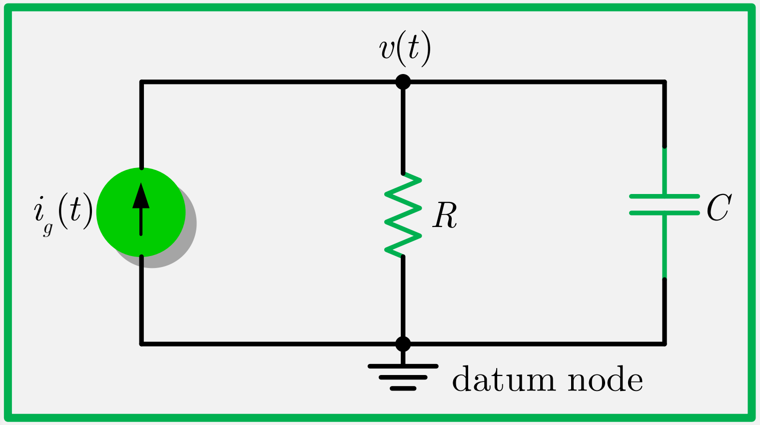
$$i(t) = \left\{\frac{8}{3}e^{-2t} - \frac{2}{3}e^{-0.5t}\right\}u(t) \quad \textbf{(5.50)}$$

- As noted earlier, in the solution of system ODEs, the complementary function is called the free response, where as the forced response is a particular integral, and in the case of constant or periodic excitation, the forced response at $t = \infty$ is the steady-state solution.
- Note, the free response is a function of the system elements alone.

5.5 Solution of System Equations Cont'd

- Whereas, the forced response depends on both the system and excitation.
- If the roots of the characteristic equation $H(p)$ all have negative or zero real parts, the free response is made up of only damped exponential and/or sinusoids. That is, given $p_1 = \sigma \pm j\omega$ the root of $H(p)$, then $\mathrm{Re}(p_1) = \sigma \leq 0$.
- Note that if a characteristic eq. contains only roots whose real parts are zero or negative, and if the $j\omega$ axis roots are simple, then the network it describes is said to be stable; otherwise, it is unstable.

[Example 5.8] Solution of System Equations



For the R-C network in Fig. 5.19 with the excitation given by Eq. 5.25, i.e., $i_g(t) = (I_0 \sin \omega_0 t)u(t)$, find the voltage $v(t)$ across the capacitor; given that $v_C(0+) = v_C(0-) = 0$.

Fig. 5.19.

[Example 5.8] Solution of System Equations Cont'd

[Solution]

We have already obtained the particular integral in Eq. 5.28. Now let us find the complementary function. The ODE on a node basis is

$$C\frac{dv}{dt} + Gv = I_0 \sin \omega t \, u(t) \tag{5.51}$$

from which we obtain the characteristic equation as

$$H(p) = Cp + G \tag{5.52}$$

so that

$$v_C(t) = Ke^{-Gt/C} \tag{5.53}$$

and the incomplete solution is

$$v(t) = Ke^{-Gt/C}u(t) + \frac{I_0}{(G^2 + \omega^2 C^2)^{1/2}} \sin\left\{\omega t - \tan^{-1}\frac{\omega C}{G}\right\} u(t) \tag{5.54}$$

From initial condition $v(0-) = 0$, we obtain

[Example 5.8] Solution of System Equations Cont'd

$$v(0+) = v(0-) = K - \frac{I_0}{(G^2+\omega^2C^2)^{1/2}}\sin\left\{\tan^{-1}\frac{\omega C}{G}\right\} = 0 \qquad \textbf{(5.55)}$$

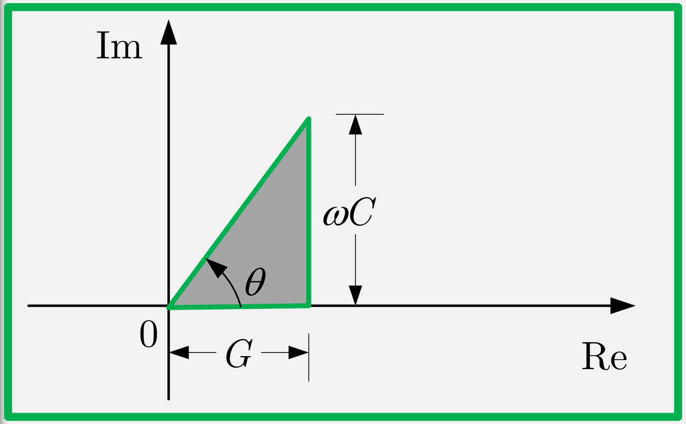


Fig. 5.20.

- From the argand diagram in Fig. 5.20 we see that,

$$\sin\left\{\tan^{-1}\frac{\omega C}{G}\right\} = \frac{\omega C}{(G^2+\omega^2C^2)^{1/2}} \qquad \textbf{(5.56)}$$

- Consequently,

$$v(t) = \frac{I_0 u(t)}{(G^2+\omega^2C^2)^{1/2}}\left[\frac{\omega C e^{-Gt/C}}{(G^2+\omega^2C^2)^{1/2}} + \sin\left\{\omega t - \tan^{-1}\frac{\omega C}{G}\right\}\right] \qquad \textbf{(5.57)}$$

From the complementary function in Eq. 5.53, we see that the time constant of the circuit is $\tau = C/G = RC$.

5.5 Solution of System Equations Cont'd

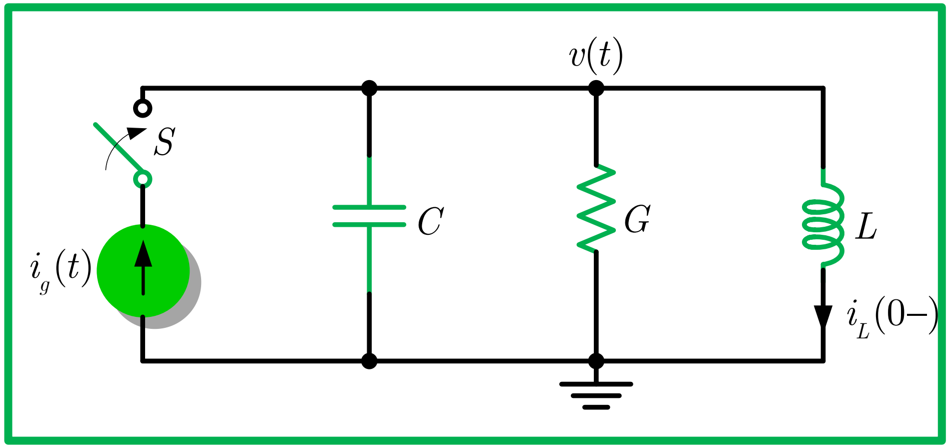


Fig. 5.21.

- Next, we examine an example of different kinds of free responses of a second-order system equation that depend on relative values of the system elements.
- Suppose we are given the system in Fig. 5.21; let us find the free response $v_C(t)$ for the ODE

$$i_g(t) = C\frac{dv}{dt} + Gv + \frac{1}{L}\int_{0-}^{t} v(\tau)\,d\tau + i_L(0-) \tag{5.58}$$

- Differentiating both sides of Eq. 5.58, we have

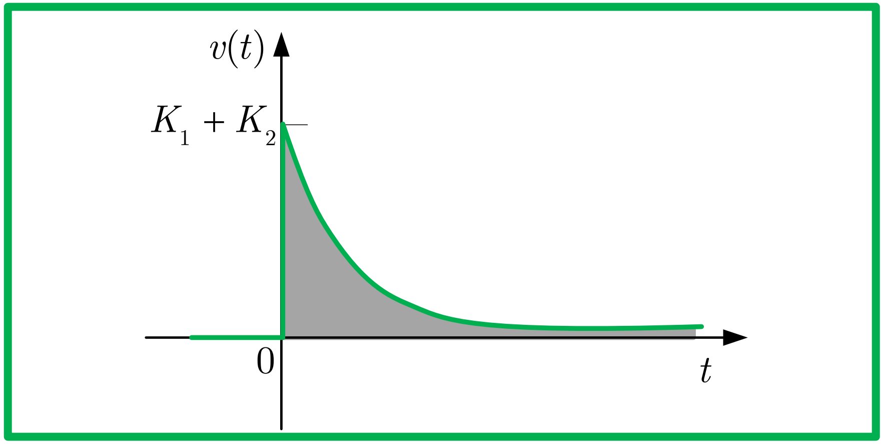
$$i_g'(t) = Cv''(t) + Gv'(t) + \frac{1}{L}v(t) \tag{5.59}$$

- The characteristic equation is then

$$H(p) = Cp^2 + Gp + \frac{1}{L} = C\left\{p^2 + \frac{G}{C}p + \frac{1}{LC}\right\} \tag{5.60}$$

5.5 Solution of System Equations Cont'd

- In factored form, $H(p)$ is $$H(p) = C(p-p_1)(p-p_2)$$ **(5.61)**
- where $$\left.\begin{matrix} p_1 \\ p_2 \end{matrix}\right\} = -\frac{G}{2C} \pm \frac{1}{2}\left[\left(\frac{G}{C}\right)^2 - \frac{4}{LC}\right]^{1/2} = -A \pm B$$ **(5.62)**
- There are three different kinds of responses depending upon whether B is real, zero, or imaginary.
- **CASE 1**. B is real, that is, $$\left(\frac{G}{C}\right)^2 > \frac{4}{LC}$$
- Then the free response is $$v_C(t) = K_1 e^{-(A-B)t} + K_2 e^{-(A+B)t}$$ **(5.63)**



- which is a sum of damped exponentials. Here the response is said to be overdamped, see Fig. 5.22.

Fig. 5.22. Overdamped response.

5.5 Solution of System Equations Cont'd

- **CASE 2**. $B = 0$, that is, $$\left(\frac{G}{C}\right)^2 = \frac{4}{LC}$$
- Then $$p_1 = p_2 = -A$$
- so that $$v_C(t) = (K_1 + K_2 t)e^{-At} \qquad \textbf{(5.64)}$$
- When $B = 0$, the response is critically damped, as shown in Fig. 5.23.

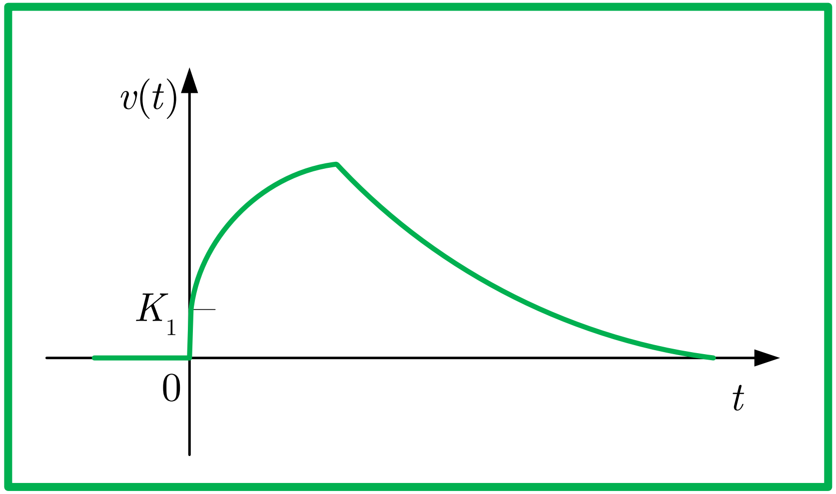


Fig. 5.23. Critically damped response.

5.5 Solution of System Equations Cont'd

- **CASE 3**. B is imaginary, that is, $\left(\frac{G}{C}\right)^2 < \frac{4}{LC}$

- Letting $B = j\beta$, we have

$$v_C(t) = e^{-At}(K_1 \sin \beta t + K_2 \cos \beta t) \qquad \textbf{(5.63)}$$

- Here, the response is said to be underdamped, and is shown by the damped oscillatory curve in Fig. 5.24.

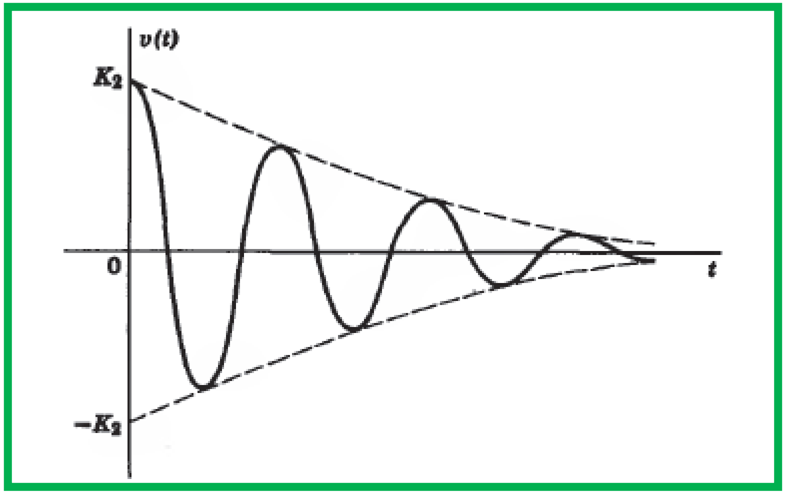
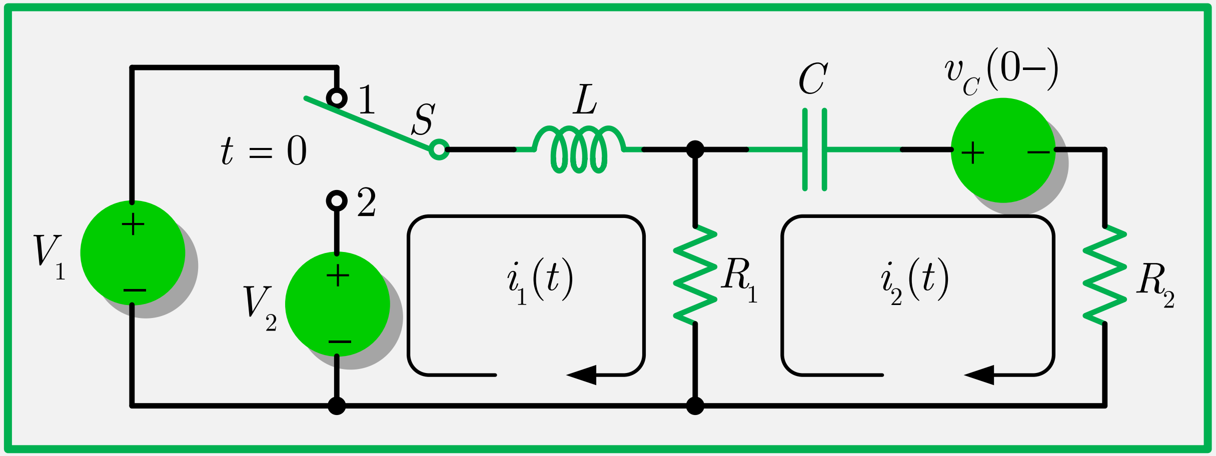


Fig. 5.24. underdamped response.

[Example 5.8] Solution of System Equations

- A look at simultaneous system equations. We once again rely upon physical reasoning rather than formal mathematical operations to obtain the initial currents and voltages as well as the steady-state solutions. In the system of Fig. 5.25, the switch S is thrown from position 1 to position 2 at $t = 0$. it is known that prior to $t = 0$, the circuit had been in steady state. We make the idealized assumption that the switch closes instantaneously at $t = 0$.
- Our task is to find $i_1(t)$ and $i_2(t)$ after the switch position changes. The values of the batteries V_1 and V_2 are $V_1 = 2\text{v}$, $V_2 = 3\text{v}$; and the element values are given as



$$L = 1\,\text{h}, \quad R_1 = 0.5\,\Omega$$

$$C = \frac{1}{3}\,\text{f}, \quad R_2 = 2.0\,\Omega$$

Fig. 5.25.

[Example 5.8] Solution of System Equations Cont'd

The mesh equations for $i_1(t)$ and $i_2(t)$ after $t = 0$ are

$$V_2 = Li_1'(t) + R_1 i_1(t) - R_1 i_2(t) \tag{5.64}$$

$$-v_C(0-) = -R_1 i_1(t) + \frac{1}{C}\int_{0-}^{t} i_2(\tau)d\tau + (R_1 + R_2) i_2(t) \tag{5.65}$$

Since Eq. 5.65 contains an integral, we differentiate it to give

$$0 = -R_1 i_1'(t) + \frac{1}{C} i_2(t) + (R_1 + R_2)\, i_2'(t) \tag{5.66}$$

Using Eqs. 5.64 and 5.66 as our system of equations, we obtain the characteristic equation

$$H(p) = \begin{vmatrix} Lp + R_1 & -R_1 \\ -R_1 p & \frac{1}{C} + (R_1 + R_2)p \end{vmatrix} = L(R_1 + R_2)p^2 + \left(\frac{L}{C} + R_1 R_2\right)p + \frac{R_1}{C} \tag{5.67}$$

[Example 5.8] Solution of System Equations Cont'd

- Substituting the element values into $H(p)$, we have

$$H(p) = 2.5p^2 + 4p + 1.5 = 2.5(p+1)(p+0.6) \quad \textbf{(5.68)}$$

- The free response are then

$$\begin{aligned} i_{1C}(t) &= (K_1 e^{-0.6t} + K_2 e^{-t})u(t) \\ i_{2C}(t) &= (K_3 e^{-0.6t} + K_4 e^{-t})u(t) \end{aligned} \quad \textbf{(5.69)}$$

- The steady-state solutions for the mesh currents are obtained at $t = \infty$ by considering the circuit from a dc viewpoint. The inductor is then a short circuit and the capacitor is an open circuit; thus we have

$$\begin{aligned} i_{1p}(t) &= \frac{V_2}{R_1} = 6\,\mathrm{amp} \\ i_{2p}(t) &= 0 \end{aligned} \quad \textbf{(5.70)}$$

[Example 5.8] Solution of System Equations Cont'd

- Now let us determine the initial currents and voltages, which incidentally, have the same values at $t = 0-$ and $t = 0+$ because the voltage sources are not impulses.
- Before the switch is thrown at $t = 0$, the circuit with V_1 as the voltage source was at steady state. Consequently,

$$v_C(0-) = V_1 = 2\,\text{v}; \quad i_1(0-) = \frac{V_1}{R_1} = 4 \text{ amp}; \quad i_2(0-) = 0 \tag{5.71}$$

- We next find $i_1'(0+)$ from Eq. 5.64 at $t = 0+$.

$$V_2 = Li_1'(0+) + R_1 i_1(0+) - R_1 i_2(0+) \tag{5.72}$$

- Substituting numerical values into Eq. 5.74, we find

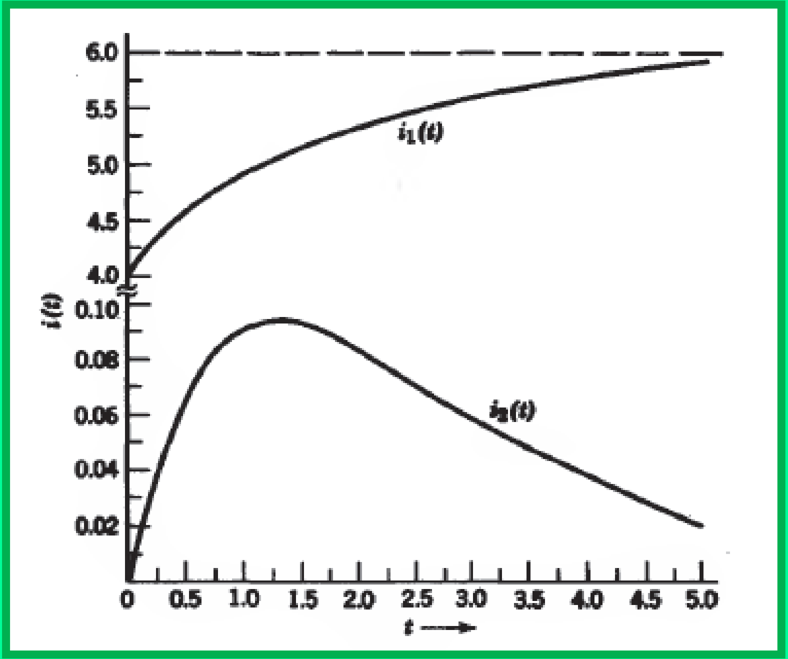
$$i_1'(0+) = 1 \text{ amp}/\text{sec}$$

[Example 5.8] Solution of System Equations Cont'd

- From Eq. 5.68 at $t = 0+$, we obtain similarly

$$i_2'(0+) = \frac{R_1}{R_1 + R_2} i_1'(0+) = 0.2 \text{ amp/sec} \qquad \textbf{(5.73)}$$

- With these initial values of $i_1(t)$ and $i_2(t)$, we can quickly arrive at the final solutions



$$\begin{aligned} i_1(t) &= \left[0.5e^{-t} - 2.5e^{-0.6t} + 6\right]u(t) \\ i_2(t) &= \left[-0.5e^{-t} + 0.5e^{-0.6t}\right]u(t) \end{aligned} \qquad \textbf{(5.74)}$$

- The two functions are plotted in Fig. 5.26.

Fig. 5.26.

5.6 Analysis of Transformers

- By Faraday's law of induction, a current i_1 flowing in a coil L_1 may induce a current i_2 in a closed loop containing a second coil L_2.

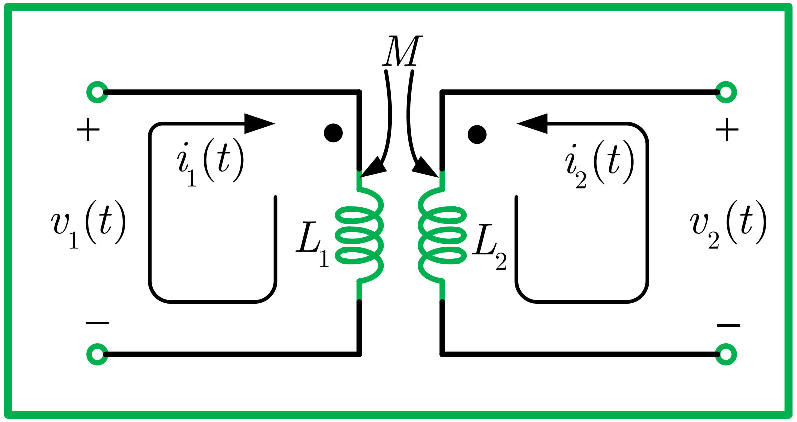


Fig. 5.27. Transformer.

- To induce the current i_2, (a) part of the flux Φ_1 in the coil L_1 must be coupled magnetically to the coil L_2, (b) the flux Φ_1 must be changing with time.
- Here we will analyze circuits having a transformer. A schematic of a transformer is shown in Fig. 5.27.
- The L_1 side is usually referred to as the primary coil and L_2 side called as the secondary coil. The energy source is generally at the primary. Mathematically the transformer in Fig. 5.27 is described as

$$v_1(t) = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}; \qquad v_2(t) = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \tag{5.75}$$

5.6 Analysis of Transformers Cont'd

- Where M is the mutual inductance associated with the flux linking L_1 to L_2, and is related to L_1 and L_2 by

$$M = K\sqrt{L_1 L_2} \tag{5.76}$$

- Here K is called the coefficient of coupling. It is bounded by the limits $0 \le |K| \le 1$. If $|K| = 1$, then all of flux Φ_1 in coil L_1 is linked magnetically to L_2.
- If $K = 0$, the coils may be regarded as two separate coils having no effect upon one another.
- For circuits with transformers, reference polarities for mutually induced voltages $M\, di/dt$, are given by small dots painted on the input and output leads of a transformer.
- If both currents $i_1(t)$ and $i_2(t)$ flow into the dots or away from the dots, $M\, di/dt$ is positive. If one current flows into the dot and the other away from the second dot, the sign of $M\, di/dt$ is negative.

5.6 Analysis of Transformers Cont'd

- If N_1 and N_2 are the number of turns of coils L_1 and L_2, then the flux linkages of L_1 and L_2 are given by $N_1\Phi_1$ and $N_2\Phi_2$, respectively.
- If both currents flow into the dots, sum of flux linkages is

$$\sum \Phi \text{ linkages} = N_1\Phi_1 + N_2\Phi_2 \quad \textbf{(5.77)}$$

- If, however, one of the currents, for example, i_1, flows into the dot, and the other i_2 flows out of the other dot, then

$$\sum \Phi \text{ linkages} = N_1\Phi_1 - N_2\Phi_2 \quad \textbf{(5.78)}$$

- Note, the sum of flux linkages is continuous with time.
- The differential equations for the transformer in Fig. 5.28 are

$$\begin{aligned} Vu(t) &= L_1 i_1'(t) + R_1 i_1(t) + M i_2'(t) \\ 0 &= M i_1'(t) + R_2 i_2(t) + L_2 i_2'(t) \end{aligned} \quad \textbf{(5.79)}$$

5.6 Analysis of Transformers Cont'd

- Integrating this set of equations between $t = 0-$ and $t = 0+$ results in the determinant

$$\begin{vmatrix} L_1\left[i_1(0+) - i_1(0-)\right] & M\left[i_2(0+) - i_2(0-)\right] \\ M\left[i_1(0+) - i_1(0-)\right] & L_2\left[i_2(0+) - i_2(0-)\right] \end{vmatrix} \quad \textbf{(5.80)}$$

- Evaluating the determinant we get

$$(L_1L_2 - M^2)\left[i_1(0+) - i_1(0-)\right]\left[i_2(0+) - i_2(0-)\right] = 0 \quad \textbf{(5.81)}$$

- If $L_1L_2 > M^2$, that is, $K < 1$, then the currents must be continuous at $t = 0$ in order for the determinant in Eq. 5.82 to be equal to zero. Thus

$$i_1(0+) = i_1(0-), \qquad K < 1 \quad \textbf{(5.82)}$$

and

$$i_2(0+) = i_2(0-), \qquad K < 1 \quad \textbf{(5.83)}$$

[Example 5.9] Analysis of Transformers

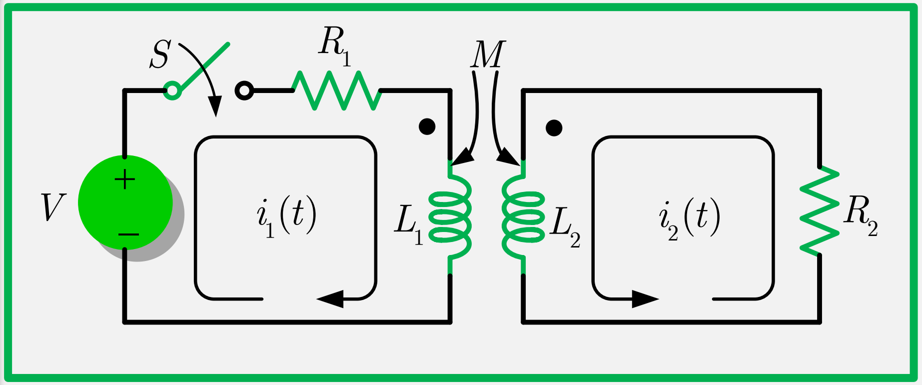


Fig. 5.28.

For the transformer circuit in Fig. 5.28, $L_1 = 1h$, $L_2 = 2h$, $R_1 = 3\,\Omega$, $R_2 = 8\,\Omega$ and $M = 1h$. The excitation is $V = 6u(t)$. Let us find $i_1(t)$ and $i_2(t)$, assuming that $i_1(0-) = i_2(0-) = 0$.

[Solution]

The differential equations for the circuit are

$$\begin{aligned} i_1'(t) + 3i_1(t) + i_2'(t) &= 6u(t) \\ i_1'(t) + 2i_2'(t) + 8i_2(t) &= 0 \end{aligned} \qquad \textbf{(5.84)}$$

The characteristic equation is given by the determinant

$$\begin{aligned} H(p) &= \begin{vmatrix} p+3 & p \\ p & 2p+8 \end{vmatrix} = (p+3)(2p+8) - p^2 = (p^2 + 14p + 24) \\ &= (p+2)(p+12) \end{aligned} \qquad \textbf{(5.85)}$$

[Example 5.9] Analysis of Transformers Cont'd

- Thus the complementary functions are

$$\begin{aligned} i_{1c}(t) &= K_1 e^{-2t} + K_2 e^{-12t} \\ i_{2c}(t) &= K_3 e^{-2t} + K_4 e^{-12t} \end{aligned} \tag{5.86}$$

- To obtain the particular integral, or steady-state solution, we rely upon physical reasoning to arrive at

$$\begin{aligned} i_{1p}(t) &= \frac{V}{R_1} = 2\,\mathrm{amp} \\ i_{2p}(t) &= 0\,\mathrm{amp} \end{aligned} \tag{5.87}$$

- Since the excitation does not contain an impulse (and since $L_1L_2 - M^2 \neq 0$), we can assume that $i_1(0+)$ and $i_2(0+)$ are also zero. Then using Eq. 5.86 we can find $i_1'(0+)$ and $i_2'(0+)$.

$$\begin{aligned} i_1'(0+) + i_2'(0+) &= 6 \\ i_1'(0+) + 2i_2'(0+) &= 0 \end{aligned} \tag{5.88}$$

[Example 5.9] Analysis of Transformers Cont'd

Solving, we find $i_1'(0+) = 12$ and $i_2'(0+) = 6$. With these initial conditions, we obtain the final solutions of $i_1(t)$ and $i_2(t)$.

$$i_1(t) = \left(2 - \frac{6}{5}e^{-2t} - \frac{4}{5}e^{-12t}\right)u(t); \quad i_2(t) = \left(-\frac{3}{5}e^{-2t} + \frac{3}{5}e^{-12t}\right)u(t) \qquad \textbf{(5.89)}$$

5.6 Analysis of Transformers Cont'd

- Suppose, now, $L_1 L_2 = M^2$; that is, $K = 1$, then $i_1(t)$ and $i_2(t)$ need not be continuous at $t = 0$. In fact, we will show that the currents are discontinuous at $t = 0$ for a unity-coupled transformer.
- Assuming that $K = 1$, consider the mesh eq for the secondary at $t = 0+$

$$R_2 i_2(0+) = -M i_1'(0+) - L_2 i_2'(0+) \qquad \textbf{(5.90)}$$

$$= -\frac{M}{L_1}\left[L_1 i_1'(0+) + M i_2'(0+)\right] \qquad \textbf{(5.91)}$$

5.6 Analysis of Transformers Cont'd

- The mesh equation of the primary side then becomes

$$V = R_1 i_1(0+) + \left[L_1 i_1'(0+) + M i_2'(0+)\right] = R_1 i_1(0+) - \frac{L_1}{M} R_2 i_2(0+) \tag{5.92}$$

- We need an additional equation to solve for $i_1(0+)$ and $i_2(0+)$. This is provided by the equation

$$L_1\left[i_1(0+) - i_1(0-)\right] - M\left[i_2(0+) - i_2(0-)\right] = 0 \tag{5.93}$$

- Which is obtained from Eq. 5.80. Since $i_1(0-) = i_2(0-) = 0$, we solve Eqs. 5.92 and 5.93 directly to give

$$i_1(0+) = \frac{VL_2}{R_1L_2 + R_2L_1}$$
$$i_2(0+) = \frac{-VM}{R_1L_2 + R_2L_1} \tag{5.94}$$

5.6 Analysis of Transformers Cont'd

- Consider the following example. For the transformer in Fig. 5.28 the element values are

$$
\begin{aligned}
L_1 &= 4\,h, & L_2 &= 1\,h \\
R_1 &= 8\,\Omega, & R_2 &= 3\,\Omega \\
M &= 2\,h, & V &= 10\,\text{v}
\end{aligned}
$$

- Assuming that the circuit is at steady-state before the switch is closed at $t = 0$, let us find $i_1(t)$ and $i_2(t)$. The DEs written on mesh basis are

$$
\begin{aligned}
4\frac{di_1}{dt} + 8i_1 + 2\frac{di_2}{dt} &= 10 \\
2\frac{di_1}{dt} + \frac{di_2}{dt} + 3i_2 &= 0
\end{aligned}
\qquad \textbf{(5.95)}
$$

- The characteristic equation is

$$
H(p) = \begin{vmatrix} 4(p+2) & 2p \\ 2p & (p+3) \end{vmatrix} = 0 \qquad \textbf{(5.96)}
$$

5.6 Analysis of Transformers Cont'd

- which yields

$$H(p) = 20p + 24 = 20\left(p + \frac{6}{5}\right) \qquad (5.97)$$

- The complementary functions are

$$\begin{aligned} i_{1C}(t) &= K_1 e^{-1.2t} \\ i_{2C}(t) &= K_2 e^{-1.2t} \end{aligned} \qquad (5.98)$$

- The particular integrals that we obtain by inspection are

$$i_{1p}(t) = \frac{V}{R_1} = \frac{10}{8} = \frac{5}{4}; \qquad i_{2p}(t) = 0 \qquad (5.99)$$

- The initial conditions are

$$\begin{aligned} i_1(0+) &= \frac{VL_2}{R_1L_2 + R_2L_1} = \frac{10}{20} = 0.5 \\ i_2(0+) &= \frac{-VM}{R_1L_2 + R_2L_1} = -1.0 \end{aligned} \qquad (5.100)$$

5.6 Analysis of Transformers Cont'd

- We thus find $K_1 = -0.75$ and $K_2 = -1.0$, so that

$$\boxed{\begin{aligned} i_1(t) &= (-0.75e^{-1.2t} + 1.25)\,u(t) \\ i_2(t) &= -e^{-1.2t}u(t) \end{aligned}} \qquad \textbf{(5.101)}$$

- Notice that as t approaches infinity $i_2(t) \to 0$, while $i_1(t)$ goes to its steady-state value of 1.25.

End of Lecture 5

Thank you for your attention!