EEE 3121 - Signals & Systems

Lecture 6: The Laplace Transforms

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References

Our main reference text book in this course is

[1] B. P. Lathi and R. A. Green, Linear Systems and Signals, 3rd Ed., 2018, Oxford University Press, New York. ISBN 978-0-19-020017-6

[2] Kuo Franklin, F., Network Analysis and Synthesis, 3rd Ed., 1986, J. Wiley (SE), ISBN 0-471-51118-8.

[3] Sundararanjan, D., A Practical Approach to Signals and Systems, 2008, John Wiley & Sons (Asia) Pte Ltd, ISBN 978-0-470-82353-8.

However, feel free to use pretty much any additional text which you might find relevant to our course.

6.1 The Philosophy of transform methods

- We have discussed classical methods for solving differential equations.
- The solutions were obtained directly in the time domain
- In this chapter, we will use Laplace transforms to transform the differential equation to the frequency domain,
- The independent variable is complex frequency *s*.
- In the frequency domain It will be shown that differentiation and integration in the time domain are transformed into algebraic operations.

Thus, the solution can be obtained by simple algebraic operations

6.1 The Philosophy of transform methods

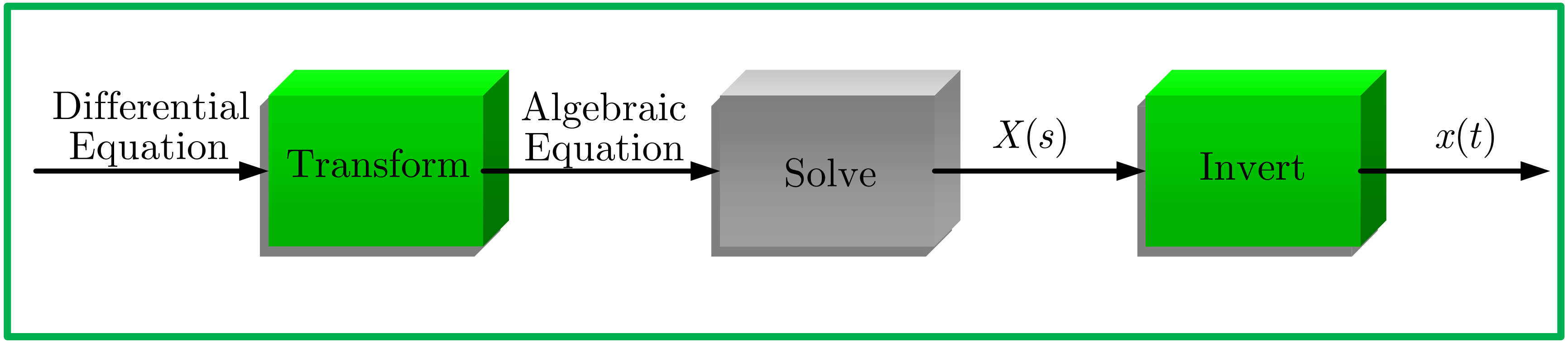


Fig. 6.1

- FIG 6.1 outlines the use of transform methods .
- FIG 6.2 shows the relationship between excitation and response in both the time and frequency domain

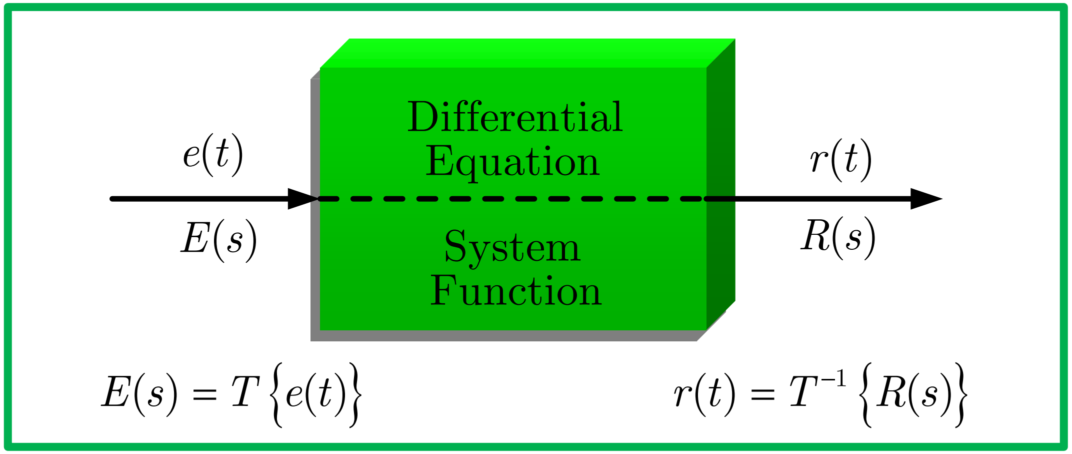


Fig. 6.2

6.2 The Laplace Transform

- The Laplace transform of a function of time $f(t)$ is of the form

$$\mathcal{L}\left\{f(t)\right\} = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt \qquad \textbf{(6.1)}$$

where *s* is the complex frequency variable $s = \sigma + j\omega$

- In order for a function to possess a Laplace transform, it must obey the condition

$$\int_{0-}^{\infty} \left|f(t)\right| e^{-st}dt < \infty \qquad \textbf{(6.2)}$$

- Laplace transform takes directly into account initial conditions at $t = 0-$.
- The inverse transform $\mathcal{L}^{-1}\left\{F(s)\right\}$ is of the form

$$f(t) = \frac{1}{j2\pi}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}ds \qquad \textbf{(6.3)}$$

- Note that the inverse transform, as defined, involves a complex integration. But we will use a partial fraction expansion procedure to obtain the inverse transform.
- To find inverse transforms by recognition, we must remember certain basic transform pairs or use a table of Laplace transforms.

6.2 The Laplace Transform

- Two of the most basic transform pairs are discussed here.

[Example 6.1] $f(t) = u(t)$.

$$\mathcal{L}\left\{f(t)\right\} = F(s) = \int_{0-}^{\infty} u(t)e^{-st}dt = -\frac{e^{-st}}{s}\Bigg|_{0-}^{\infty} = 0 - \left(-\frac{1}{s}\right); \quad \boxed{\mathcal{L}\left\{u(t)\right\} = F(s) = \frac{1}{s}} \quad \textbf{(6.4)}$$

Next, let us find the transform of an exponential function of time.

[Example 6.2] $f(t) = e^{at}u(t)$.

$$\mathcal{L}\left\{f(t)\right\} = F(s) = \int_{0-}^{\infty} e^{at}e^{-st}dt = -\frac{e^{-(s-a)t}}{s-a}\Bigg|_{0-}^{\infty} = \frac{1}{s-a}; \quad \boxed{\mathcal{L}\left\{e^{at}u(t)\right\} = F(s) = \frac{1}{s-a}} \quad \textbf{(6.5)}$$

- With these two transform pairs, and with the use of the properties of Laplace transforms, which we discuss in Section 6.3, we can build up an extensive table of transform pairs.

6.3 Properties of Laplace Transforms

Linearity

❑The transform of a finite sum of time functions is the sum of the transforms of the individual functions, that is

$$\mathcal{L}\left\{\sum_i f_i(t)\right\} = \sum_i \mathcal{L}\{f_i(t)\} \tag{6.4}$$

[Example 6.3]

$f(t) = \sin \omega t$. Expanding $\sin \omega t$ by Euler's identity, we have

$$f(t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t}) \tag{6.5}$$

The Laplace transform of $f(t)$ is the sum of the transforms of the individual cisoidal $e^{\pm j\omega t}$ terms. Thus

$$\mathcal{L}\{\sin \omega t\} = \frac{1}{2j}\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right) = \frac{\omega}{s^2 + \omega^2}\ ; \Rightarrow \quad \mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2} \tag{6.6}$$

6.3 Properties of Laplace Transforms

Real Differentiation

- Given that $\mathcal{L}\{f(t)\} = F(s)$, then $$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0-) \qquad \textbf{(6.7)}$$
- where $f(0-)$ is the value of $f(t)$ at $t = 0-$.

Proof. By definition, $$\mathcal{L}\{f'(t)\} = \int_{0-}^{\infty} e^{-st} f'(t)dt \qquad \textbf{(6.8)}$$

Integrating Eq. 6.8 by parts yields,

$$\mathcal{L}\{f'(t)\} = e^{-st} f(t)\Big|_{0-}^{\infty} + s\int_{0-}^{\infty} f(t)e^{-st}dt \qquad \textbf{(6.9)}$$

Since $e^{-st} \to 0$ as $t \to \infty$, and because the integral on the right-hand side is $\mathcal{L}\{f(t)\} = F(s)$, we have

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0-) \qquad \textbf{(6.10)}$$

- Similarly, we can show for the *n*th derivative that

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0-) - s^{n-2} f'(0-) - \cdots - f^{(n-1)}(0-) \qquad \textbf{(6.11)}$$

- where $f^{(n-1)}(0-)$ is the $(n-1)$st derivative of $f(t)$ at $t = 0-$.

6.3 Properties of Laplace Transforms

Real integration

- Given that $\mathcal{L}\left\{f(t)\right\} = F(s)$ then the Laplace transform of the integral of $f(t)$ is

$$\mathcal{L}\left\{\int_{0-}^{t} f(\tau)d\tau\right\} = \frac{F(s)}{s} \qquad \textbf{(6.12)}$$

[Example 6.4]

Let us find the transform of the unit ramp function, $\rho(t) = tu(t)$.

[Solution]

We know that $$\int_{0-}^{t} u(\tau)d\tau = \rho(t) \qquad \textbf{(6.13)}$$

Since $$\mathcal{L}\left\{u(t)\right\} = \frac{1}{s} \qquad \textbf{(6.14)}$$

It follows that, $$\mathcal{L}\left\{\rho(t)\right\} = \frac{\mathcal{L}\left\{u(t)\right\}}{s} = \frac{1}{s^2}$$

6.3 Properties of Laplace Transforms

Differentiation by *s*

- Differentiation by s in the complex frequency domain corresponds to multiplication by *t* in the time domain, that is,

$$\mathcal{L}\left\{tf(t)\right\} = -\frac{dF(s)}{ds} \tag{6.15}$$

Complex translation

- By the complex translation property, if $\mathcal{L}\left\{f(t)\right\} = F(s)$ then,

$$F(s-a) = \mathcal{L}\left\{e^{at} f(t)\right\} \tag{6.16}$$

[Example 6.5]

[Solution]

Given $f(t) = \sin \omega t$, find $\mathcal{L}\left\{e^{-at}\sin \omega t\right\}$.

Since $\mathcal{L}\left\{\sin \omega t\right\} = \dfrac{\omega}{s^2+\omega^2}$ **(6.17)**

Thus, by the preceding theorem,

$$\mathcal{L}\left\{e^{-at}\sin \omega t\right\} = \frac{\omega}{(s+a)^2+\omega^2} \tag{6.18}$$

Similarly, we can show that

$$\mathcal{L}\left\{e^{-at}\cos \omega t\right\} = \frac{s+a}{(s+a)^2+\omega^2} \tag{6.19}$$

6.3 Properties of Laplace Transforms

Real Translation (Shifting theorem)

- Here we consider the very important concept of the transform of a shifted or delayed function of time. If $\mathcal{L}\{f(t)\} = F(s)$, then the transform of the function delayed by time *a* is,

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \quad \textbf{(6.20)}$$

[Example 6.5]

Given the square pulse $f(t)$ in Fig. 6.3, let us first find its transform $F(s)$. Then let us determine the inverse transform of the square of $F(s)$, i.e., let us find

[Solution]

$$f_1(t) = \mathcal{L}^{-1}\{F^2(s)\} \quad \textbf{(6.21)}$$

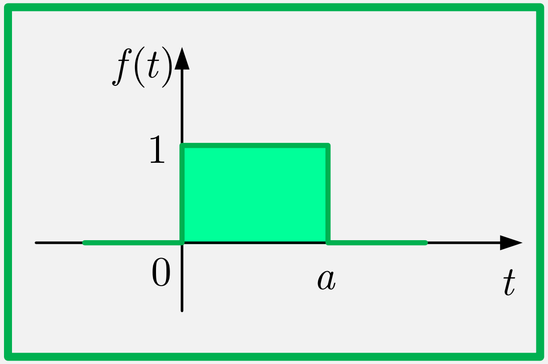
The square pulse is given in terms of step functions as

$$f(t) = u(t) - u(t-a) \quad \textbf{(6.22)}$$

Its Laplace transform is then

$$\mathcal{L}\{f(t)\} = F(s) = \frac{1}{s}(1 - e^{-as}) \quad \textbf{(6.23)}$$

Fig. 6.3



6.3 Properties of Laplace Transforms

[Example 6.5] Cont'd

- Squaring $F(s)$, yields

$$F^2(s) = \frac{1}{s^2}(1 - 2e^{-as} + e^{-2as}) \tag{6.24}$$

- Thus, to find the inverse transform of $F^2(s)$, we need only to determine the inverse transform of the term with zero delay, which is

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = tu(t) \tag{6.25}$$

- Thus,

$$\mathcal{L}^{-1}\left\{F^2(s)\right\} = tu(t) - 2(t-a)u(t-a) + (t-2a)u(t-2a) \tag{6.26}$$

- The resulting waveform is shown in Fig. 6.4.

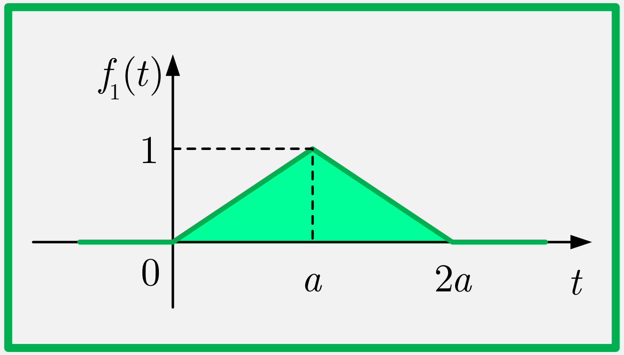


Fig. 6.4

6.3 Properties of Laplace Transforms

TABLE 6.1

Laplace Transforms

	$f(t)$	$F(s)$
1.	$f(t)$	$F(s) = \int_{0-}^{\infty} f(t) e^{-st}\, dt$
2.	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
3.	$\frac{d}{dt} f(t)$	$s\,F(s) - f(0-)$
4.	$\frac{d^n}{dt^n} f(t)$	$s^n F(s) - \sum_{j=1}^{n} s^{n-j} f^{j-1}(0-)$
5.	$\int_{0-}^{t} f(\tau)\, d\tau$	$\frac{1}{s} F(s)$
6.	$\int_{0-}^{t} \int_{0-}^{t} f(\tau)\, d\tau\, d\sigma$	$\frac{1}{s^2} F(s)$
7.	$(-t)^n f(t)$	$\frac{d^n}{ds^n} F(s)$
8.	$f(t-a)\, u(t-a)$	$e^{-as} F(s)$
9.	$e^{at} f(t)$	$F(s-a)$
10.	$\delta(t)$	1
11.	$\frac{d^n}{dt^n} \delta(t)$	s^n
12.	$u(t)$	$\frac{1}{s}$
13.	t	$\frac{1}{s^2}$
14.	$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
15.	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
16.	$\frac{1}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})$	$\frac{1}{(s+\alpha)(s+\beta)}$
17.	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
18.	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
19.	$\sinh at$	$\frac{a}{s^2 - a^2}$
20.	$\cosh at$	$\frac{s}{s^2 - a^2}$
21.	$e^{-\alpha t} \sin \omega t$	$\frac{\omega}{(s + \alpha^2) + \omega^2}$
22.	$e^{-\alpha t} \cos \omega t$	$\frac{(s+\alpha)}{(s+\alpha)^2 + \omega^2}$
23.	$\frac{e^{-\alpha t} t^n}{n!}$	$\frac{1}{(s+\alpha)^{n+1}}$
24.	$\frac{t}{2\omega} \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}$
25.	$\frac{1}{\alpha^n} J_n(\alpha t);\ n = 0, 1, 2, 3, \ldots$ (Bessel function of first kind, nth order)	$\frac{1}{(s^2+\alpha^2)^{1/2}[(s^2+\alpha^2)^{1/2} - s]^{-n}}$
26.	$(\pi t)^{-1/2}$	$s^{-1/2}$
27.	t^k (k need not be an integer)	$\frac{\Gamma(k+1)}{s^{k+1}}$

6.4 Uses of Laplace Transforms

Solution of integrodifferential equations

- Let us consider an example using Laplace transforms in solving ODEs.

[Example 6.6]

Let us solve the differential equation (given that $x(0-)=1$ and $x'(0-)=-1$)

$$x''(t)+3x'(t)+2x(t)=4e^{t} \tag{6.27}$$

[Solution]

By taking the Laplace transform of the differential equation, which then becomes

$$\left[s^{2}X(s)-s\,x(0-)-x'(0-)\right]+3\left[s\,X(s)-x(0-)\right]+2X(s)=\frac{4}{s-1} \tag{6.28}$$

Substituting the initial conditions and simplifying, we have

$$(s^{2}+3s+2)X(s)=\frac{4}{s-1}+s+2 \tag{6.29}$$

Expanding $X(s)$ in partial fractions, we have

$$X(s)=\frac{s^{2}+s+2}{(s-1)(s+1)(s+2)} \tag{6.30}$$

6.4 Uses of Laplace Transforms

[Example 6.6] Cont'd

- To find the inverse transform $x(t) = \mathcal{L}^{-1}\left\{X(s)\right\}$, we expand $X(s)$ into partial fractions

$$X(s) = \frac{K_{-1}}{(s-1)} + \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)} \qquad \textbf{(6.31)}$$

- Solving for K_{-1}, K_1 and K_2 algebraically, we obtain

$$K_{-1} = \frac{2}{3}\;;\quad K_1 = -1\;;\quad K_2 = \frac{4}{3}$$

- Thus, the final solution is the inverse transform of $X(s)$, i.e.,

$$x(t) = \mathcal{L}^{-1}\left\{X(s)\right\} = \frac{2}{3}e^{t} - e^{-t} + \frac{4}{3}e^{-2t} \qquad \textbf{(6.32)}$$

- **The ease with which we use transform methods depends upon how quickly we are able to obtain the partial fraction expansion of a given transform function.**

6.5 Partial-Fraction Expansion

- The ease with which we use transform methods depends upon how quickly we are able to obtain the partial-fraction expansion of a given transform function
- Two methods that may be used to determine the partial-fractions will be introduced

Real Roots

- Method for simple real roots, consider the function

$$F(s) = \frac{N(s)}{(s-s_0)(s-s_1)(s-s_2)} \tag{6.33}$$

- where s_0, s_1 and s_2 are distinct, real roots, and the degree of $N(s) < 3$. Expanding $F(s)$ we have

$$F(s) = \frac{K_0}{(s-s_0)} + \frac{K_1}{(s-s_1)} + \frac{K_2}{(s-s_2)} \tag{6.34}$$

- Let us first obtain the constant K_0,. We proceed by multiplying both sides of the equation by $(s-s_0)$ to give

$$(s-s_0)F(s) = K_0 + \frac{(s-s_0)K_1}{(s-s_1)} + \frac{(s-s_0)K_2}{(s-s_2)} \tag{6.35}$$

6.5 Partial-Fraction Expansion

- If we let $s = s_0$ in Eq. 6.26, we obtain

$$K_0 = (s - s_0)F(s)\Big|_{s=s_0} \tag{6.36}$$

- Similarly, we see that the other constants can be evaluated through the general relation

$$K_i = (s - s_i)F(s)\Big|_{s=s_i} \tag{6.37}$$

[Example 6.7]

Let us find the partial-fraction expansion for

$$F(s) = \frac{s^2 + 2s - 2}{s(s+2)(s-3)} = \frac{K_0}{s} + \frac{K_1}{(s+2)} + \frac{K_2}{(s-3)} \tag{6.38}$$

[Solution]

Clearly,

$$K_0 = sF(s)\Big|_{s=0} = \frac{1}{3}; \qquad K_1 = (s+2)F(s)\Big|_{s=-2} = -\frac{1}{5};$$

$$K_2 = (s-3)F(s)\Big|_{s=3} = \frac{13}{15} \tag{6.39}$$

6.5 Partial-Fraction Expansion

Complex roots

Self study (You are implored to study this on your own)

Multiple roots: Method A

- For the case in which the partial fraction involves repeated or multiple roots there are two methods that can be used. Method A we will be discussed next whilst method B is Self study
- Suppose we are given the function

$$F(s) = \frac{N(s)}{(s-s_0)^n D_1(s)} \tag{6.40}$$

- With multiple roots of degree *n* at $s = s_0$. The partial fraction expansion is

$$F(s) = \frac{K_0}{(s-s_0)^n} + \frac{K_1}{(s-s_0)^{n-1}} + \frac{K_2}{(s-s_0)^{n-2}} + \cdots + \frac{K_{n-1}}{s-s_0} + \frac{N_1(s)}{D_1(s)} \tag{6.41}$$

- where $N_1(s)/D_1(s)$ represents the remaining terms of the expansion. The problem is to obtain $K_0, K_1, \ldots, K_{n-1}$. For the K_0 term, we use the method cited earlier for simple roots, that is

$$K_0 = (s-s_0)^n F(s)\Big|_{s=s_0} \tag{6.42}$$

6.5 Partial-Fraction Expansion

Multiple roots: Method A

- However, if we were to use the same formula to obtain factors $K_0, K_1, \ldots, K_{n-1}$, we would invariably arrive at the indeterminate $0/0$ condition. Instead, let us multiply both sides of Eq. 6.41 by $(s-s_0)^n$ and define

$$F_1(s) \overset{def}{=} (s-s_0)^n F(s) \tag{6.43}$$

- Thus,

$$F_1(s) = K_0 + K_1(s-s_0) + \cdots + K_{n-1}(s-s_0)^{n-1} + R(s)(s-s_0)^n \tag{6.42}$$

- where $R(s)$ indicates the remaining terms. If we differentiate Eq. 6.42 by *s*, we obtain

$$\frac{d}{ds}F_1(s) = K_1 + 2K_2(s-s_0) + \cdots + K_{n-1}(n-1)(s-s_0)^{n-2} + \cdots \tag{6.43}$$

- It is evident that

$$K_1 = \frac{d}{ds}F_1(s)\bigg|_{s=s_0} \tag{6.44}$$

- On the same basis

$$K_2 = \frac{1}{2}\frac{d^2}{ds^2}F_1(s)\bigg|_{s=s_0} \tag{6.45}$$

- Thus, in general

$$K_j = \frac{1}{j!}\frac{d^j}{ds^j}F_1(s)\bigg|_{s=s_0}\ ; \ \ j = 0, 1, 2, \ldots, n-1 \tag{6.46}$$

6.5 Partial-Fraction Expansion

[Example 6.8]

- Consider the function

$$F(s)=\frac{s-2}{s(s+1)^3} \tag{6.47}$$

- Which we represent in expanded form as

$$F(s)=\frac{K_0}{(s+1)^3}+\frac{K_1}{(s+1)^2}+\frac{K_2}{s+1}+\frac{A}{s} \tag{6.48}$$

- The constant A for the simple root at $s=0$ is

$$A=sF(s)\Big|_{s=0}=-2 \tag{6.49}$$

- To obtain the constants for the multiple roots we first find $F_1(s)$.

$$F_1(s)=(s+1)^3F(s)=\frac{s-2}{s} \tag{6.50}$$

- Using the general formula for the multiple root expansion, we obtain

$$K_0=\frac{1}{0!}\frac{d^0}{ds^0}\left(\frac{s-2}{s}\right)\Bigg|_{s=-1}=3 \tag{6.51}$$

6.5 Partial-Fraction Expansion

[Example 6.8] Cont'd

$$K_1 = \frac{1}{1!}\frac{d}{ds}\left(\frac{s-2}{s}\right)\bigg|_{s=-1} = \frac{2}{s^2}\bigg|_{s=-1} = 2 \qquad \textbf{(6.52)}$$

$$K_2 = \frac{1}{2!}\frac{d}{ds}\left(\frac{2}{s^2}\right)\bigg|_{s=-1} = \left(-\frac{2}{s^2}\right)\bigg|_{s=-1} = 2 \qquad \textbf{(6.53)}$$

Thus we have,

$$F(s) = \frac{3}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{2}{s+1} - \frac{2}{s} \qquad \textbf{(6.54)}$$

- **Once again to reiterate the aforesaid statement do a self study on Method B.**

6.6 Poles and Zeros

- In this section we shall discuss the implications of a pole-zero description of a given rational function with real coefficients $F(s)$.
- We define the poles of $F(s)$ to be the roots of the denominator of $F(s)$.
- And the zeros of $F(s)$ as the roots of the numerator.
- At a pole the system will produce an infinite output.
- At a zero the system will produce a zero output.

$$F(s) = \frac{s(s-1+j1)(s-1-j1)}{(s+1)^2(s+j2)(s-j2)} \qquad \textbf{(6.55)}$$

- Thus, the poles for the system in Eq. 6.55 are at

$$s=-1 \quad (\text{double});$$
$$s=-j2\ ; \quad s=+j2$$

- Fig. 6.5 depicts crosses for poles and zero by a small circle.

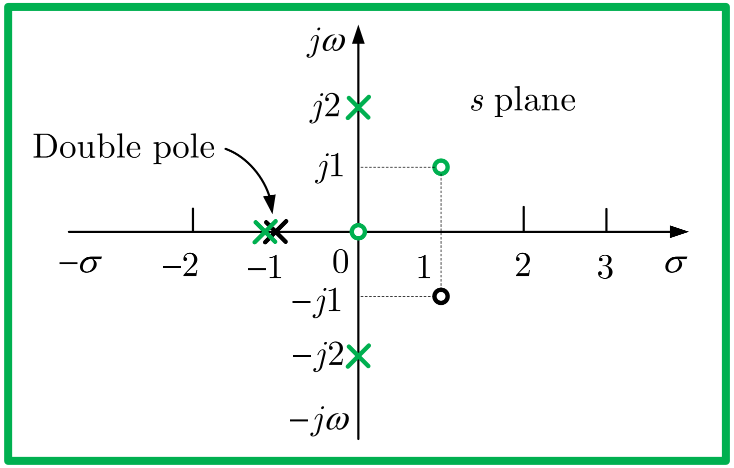


Fig. 6.5 Pole-zero diagram of $F(s)$.

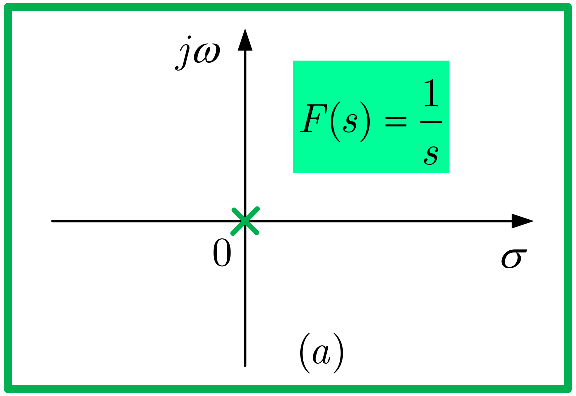
- Zeros are at

$$s=0\ ;$$
$$s=1+j1\ ;$$
$$s=1-j1\ ;$$
$$s=\infty$$

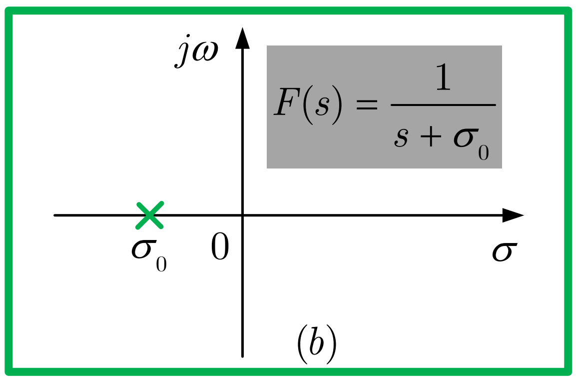
6.6 Poles and Zeros

Pole-zero diagrams for standard signals

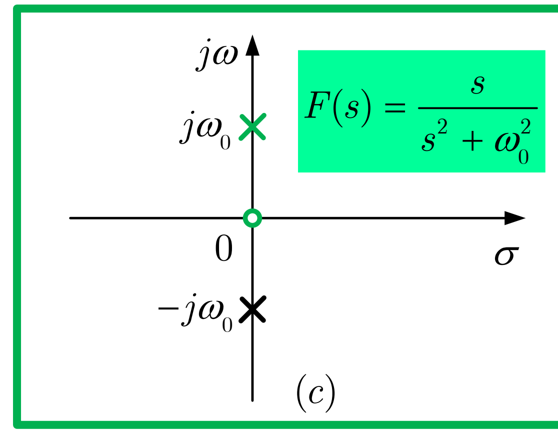
$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$



$$\mathcal{L}\{e^{-\sigma_0 t}\} = \frac{1}{s+\sigma_0}$$



$$\mathcal{L}\{\cos \omega_0 t\} = \frac{s}{s^2+\omega_0^2}$$



- ❑ **we note that the poles corresponding to decaying exponential waves are on the -σ axis and have zero imaginary parts.**

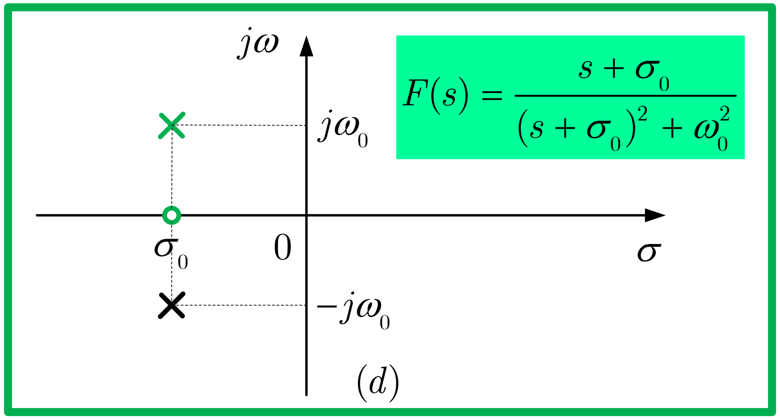
- ❑ **The poles and zeros corresponding to undamped sinusoids are on the $j\omega$ axis, and have zero real parts.**

Fig. 6.6

6.6 Poles and Zeros

Pole-zero diagrams for standard signals

$$\mathcal{L}\left\{e^{-\sigma_0 t}\cos\omega_0 t\right\} = \frac{s+\sigma_0}{(s+\sigma_0)^2+\omega_0^2}$$



- The poles and zeros for damped sinusoids must have real and imaginary parts that are both nonzero.

6.6 Poles And Zeros

Effect of pole location upon exponential decay

- Now let us consider two exponential waves $f_1(t) = e^{-\sigma_1 t}$ and $f_2(t) = e^{-\sigma_2 t}$, where $\sigma_2 > \sigma_1 > 0$, so the $f_2(t)$ decays faster than $f_1(t)$, as shown in Fig. 6.7a. The transforms of the two functions are

$$F_1(s) = \frac{1}{s+\sigma_1} \qquad F_2(s) = \frac{1}{s+\sigma_2} \tag{6.56}$$

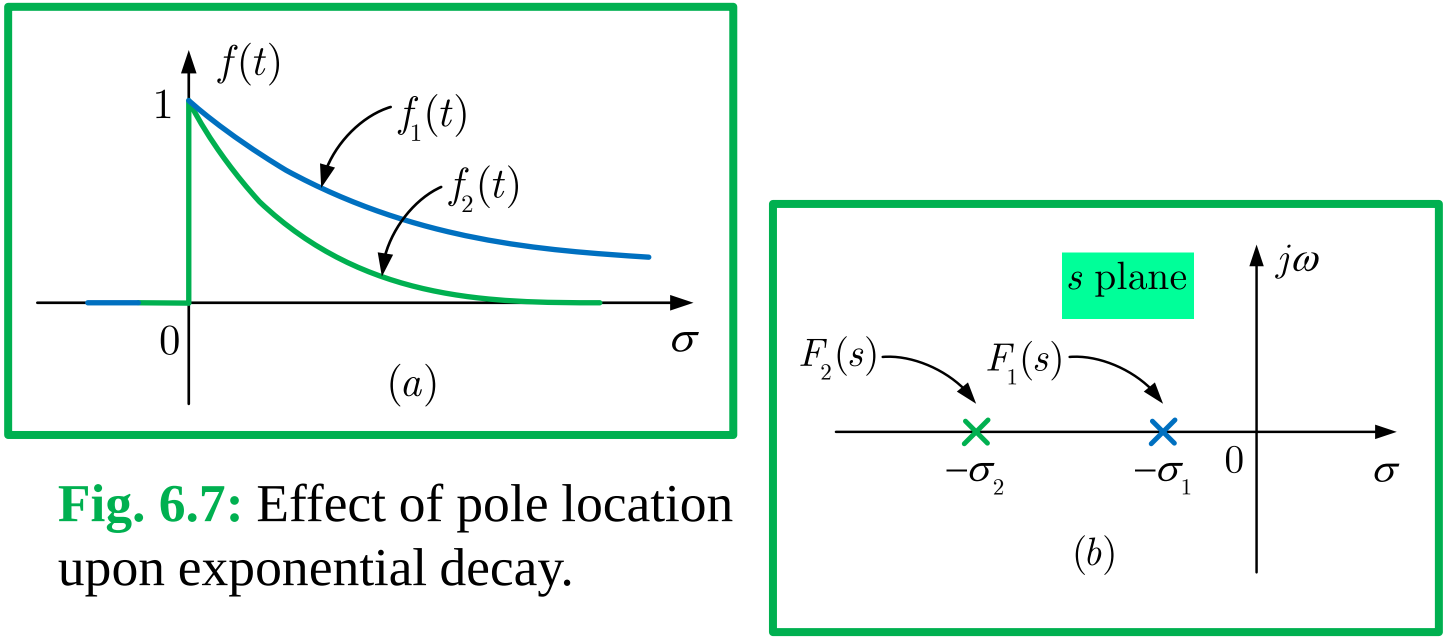


Fig. 6.7: Effect of pole location upon exponential decay.

6.6 Poles And Zeros

Effect of pole location: Complex Poles

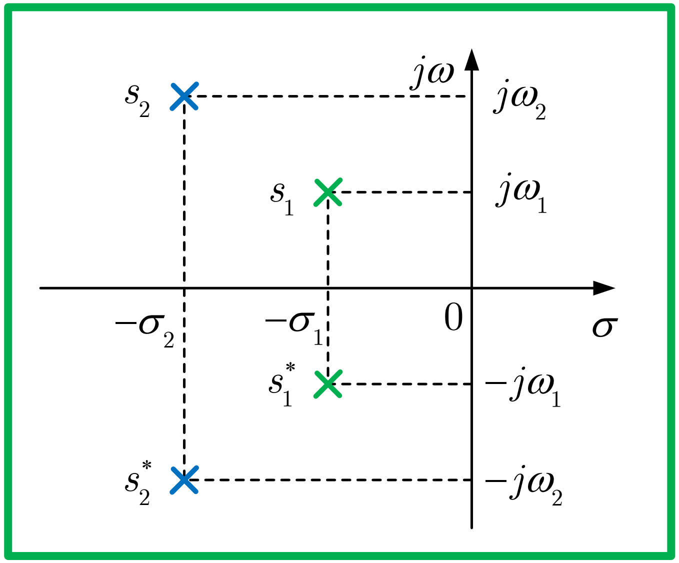
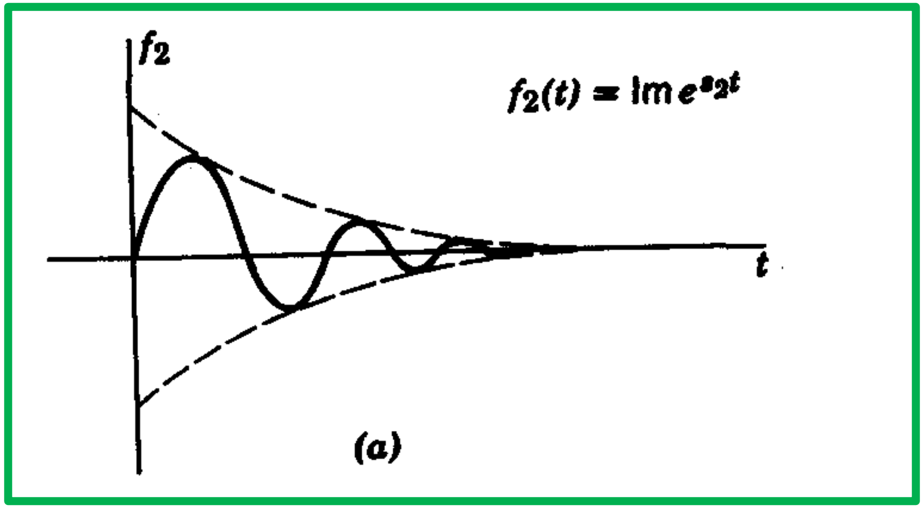


Fig. 6.8.

- **Match the poles to the time response**



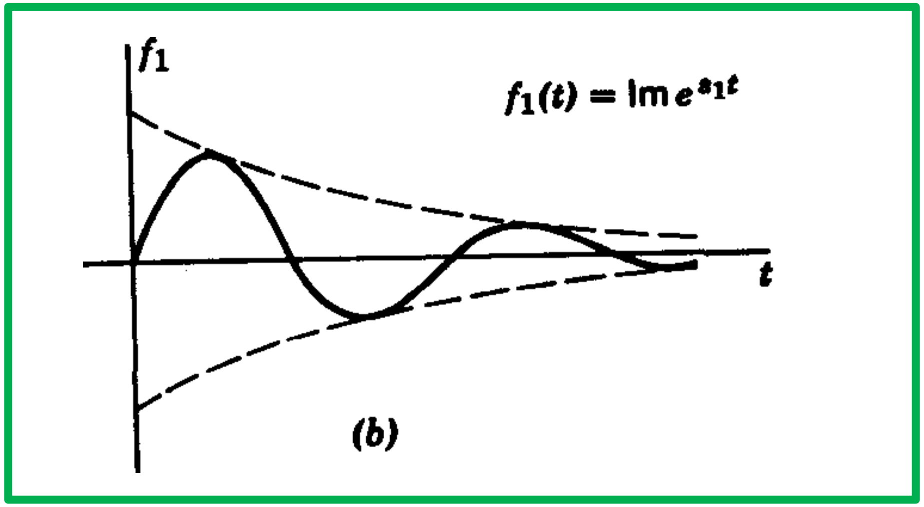


Fig. 6.9. Responses for poles in Fig. 6.8.

6.6 Poles And Zeros

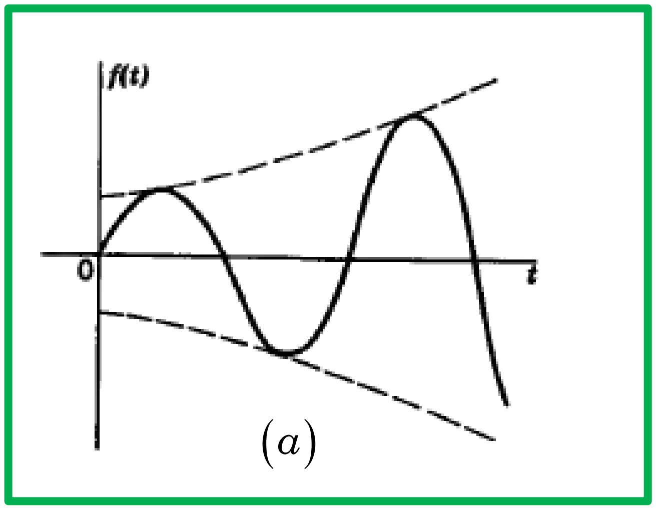
Effect of pole location: Complex Poles with positive real part

- For a given function
$$F(s)=\frac{K_0}{s-p_0}+\frac{K_1}{s-p_1}+\cdots+\frac{K_n}{s-p_n} \tag{6.57}$$

- The inverse transform is
$$\begin{aligned} f(t) &= K_0 e^{p_0 t}+K_1 e^{p_1 t}+\cdots+K_n e^{p_n t} \\ &= K_0 e^{\sigma_0 t} e^{\omega_0 t}+K_1 e^{\sigma_1 t} e^{\omega_1 t}+\cdots+K_n e^{\sigma_n t} e^{\omega_n t} \end{aligned} \tag{6.58}$$

- If the real part of any of the terms is positive, this will result in an exponentially increasing sinusoid.

A System function that has poles in the right-hand plane is therefore, unstable.



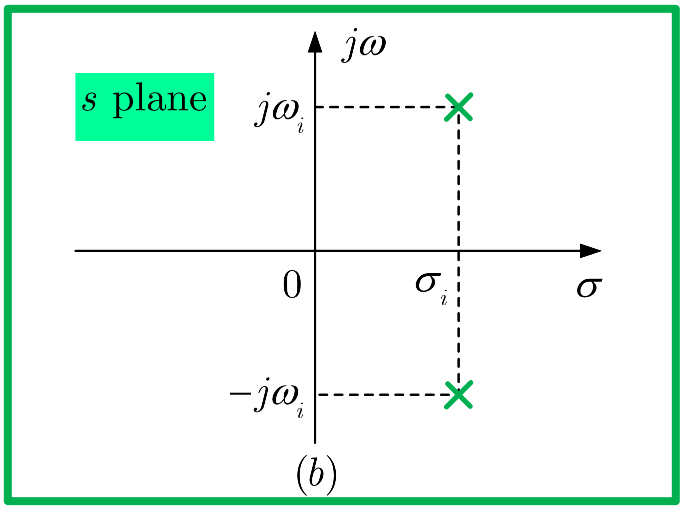
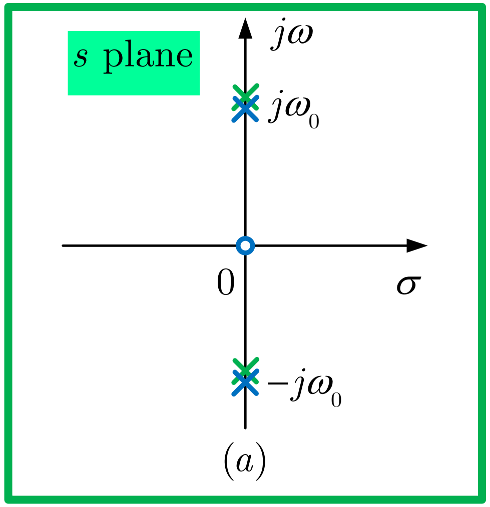


Fig. 6.10. Responses for poles in Fig. 6.8.

6.6 Poles and Zeros

Effect of pole location: Complex Poles with double poles on the $j\omega$ axis



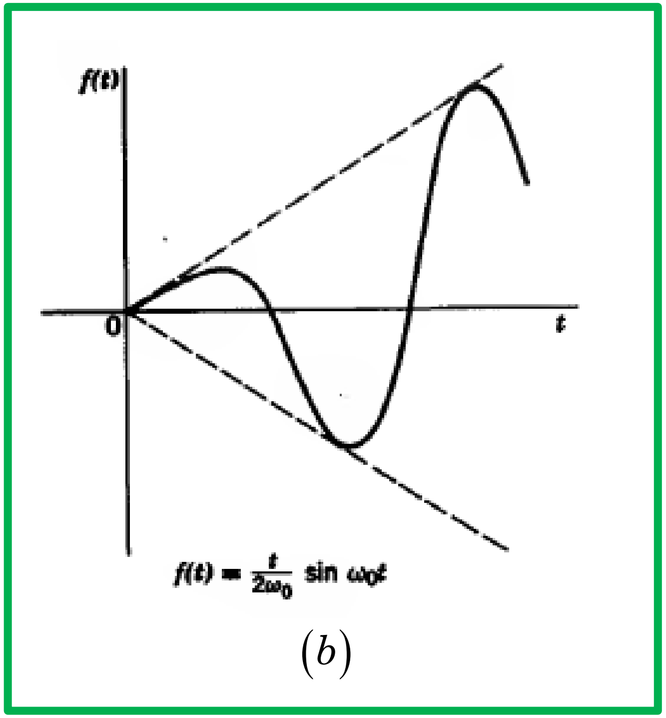


Fig. 6.11. Effect of double poles.

- **A System function that has double poles on the $j\omega$ axis is unstable.**

$$F(s) = \frac{s}{(s^2 + \omega_0^2)^2} \quad \textbf{(6.59)}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{t}{2\omega_0}\sin\omega_0 t \quad \textbf{(6.60)}$$

6.6 Poles and Zeros

Significance of poles and zeros at the origin

- Dividing a given system function $H(s)$ by s (i.e., placing a pole at the origin) corresponds to integration in the time domain.

a pole at the origin implies integration in the time domain

$$\frac{1}{s}F(s) \xrightarrow{\mathcal{L}^{-1}\left\{H(s)\right\}} \int_{0-}^{t} f(\tau)d\tau \qquad \textbf{(6.61)}$$

$$sF(s) - f(0-) \xrightarrow{\mathcal{L}^{-1}\left\{H(s)\right\}} \frac{d}{dt}f(t) \qquad \textbf{(6.62)}$$

A zero at the origin corresponds to a differentiation in the time domain.

6.7 Evaluation of Residues Graphically

- We have seen that that poles are complex frequencies of associated times responses.
- But what role do zeros play? Thus, to answer this question, let us look at the partial fraction expansion

$$F(s) = \sum_{i}^{N} \frac{K_i}{s - s_i} \tag{6.61}$$

- Here, $F(s)$ is assumed to have only simple poles and no poles at $s = \infty$.
- The inverse transform is

$$f(t) = \sum_{i}^{N} K_i e^{s_i t} \tag{6.62}$$

- Notice that the response $f(t)$ depends complex frequencies s_i and constant multipliers K_i called residues if associated with first-order poles.
- Residuals can also be obtained directly from a pole-zero diagram.

6.7 Evaluation of Residues Graphically

- The residue K_i of any pole p_i is equal to the ratio of the product of the vectors drawn from the zeros to p_i, to the product of the vectors from the other poles to p_i.

- To illustrate this idea, consider

$$F(s) = \frac{A_0(s-z_0)(s-z_1)}{(s-p_0)(s-p_1)(s-p_1^*)} \qquad \textbf{(6.63)}$$

$$F(s) = \frac{K_0}{s-p_0} + \frac{K_1}{s-p_1} + \frac{K_1^*}{s-p_1^*} \qquad \textbf{(6.64)}$$

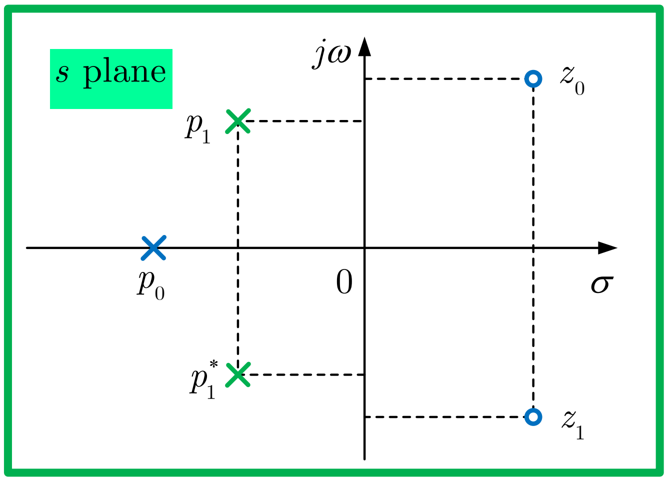
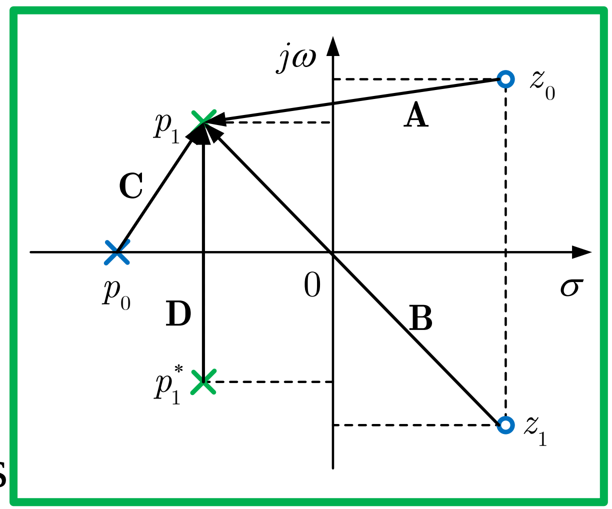
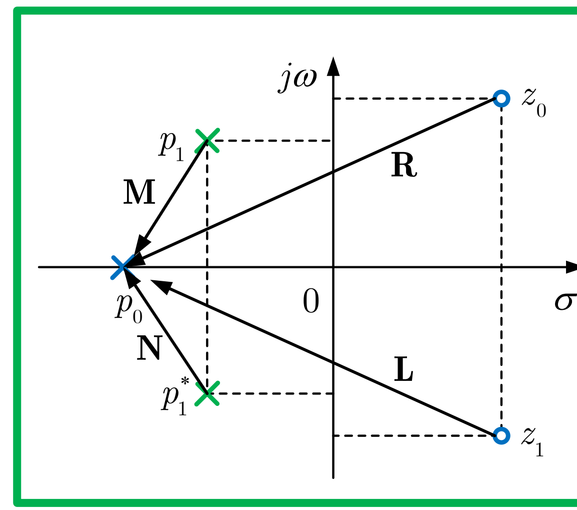


Fig. 6.12. Poles and zeros for Eq. 6.63.





- We evaluate K_i as follows

$$K_1 = \frac{A_0\mathbf{AB}}{\mathbf{CD}} \qquad K_0 = \frac{A_0\mathbf{RL}}{\mathbf{MN}} \qquad \textbf{(6.65)}$$

6.8 The Initial and Final Value Theorems

The Initial Value Theorem

$$\lim_{t \to 0+} f(t) = \lim_{s \to \infty} sF(s) \qquad \textbf{(6.66)}$$

($f(t)$ must be continuous or contain at most a step discontinuity OR $F(s)$ must be a proper function; degree of denominator is greater)

The Final Value Theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \qquad \textbf{(6.67)}$$

(provided the poles of the denominator of $F(s)$ have negative or zero real parts OR must not be in the right half of the complex frequency plane)

End of Lecture 6

Thank you for your attention!