EEE 3121 - Signals & Systems

Lecture 7: Transform Methods in System Analysis

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References

Our main reference text book in this course is

[1] B. P. Lathi and R. A. Green, Linear Systems and Signals, 3rd Ed., 2018, Oxford University Press, New York. ISBN 978-0-19-020017-6

[2] Kuo Franklin, F., Network Analysis and Synthesis, 3rd Ed., 1986, J. Wiley (SE), ISBN 0-471-51118-8.

[3] Sundararanjan, D., A Practical Approach to Signals and Systems, 2008, John Wiley & Sons (Asia) Pte Ltd, ISBN 978-0-470-82353-8.

However, feel free to use pretty much any additional text which you might find relevant to our course.

7.1 The Transformed Circuit

The Transformed Circuit

- In lecture 5 we discussed the voltage-current relationships of system elements in the time domain.
- We can also represent these relationships in the complex frequency domain.
- Thus, Ideal energy sources given in the time domain as $v(t)$ and $i(t)$, may now be represented by their transforms $V(s) = \mathcal{L}\left\{v(t)\right\}$ and $I(s) = \mathcal{L}\left\{i(t)\right\}$.

Resistor, the v-i relationship in the time domain is given as

$$\boxed{v(t) = Ri(t)} \tag{7.1}$$

- Eq. (7.1) is transformed to frequency domain to be defined as

$$\boxed{V(s) = RI(s)} \tag{7.2}$$

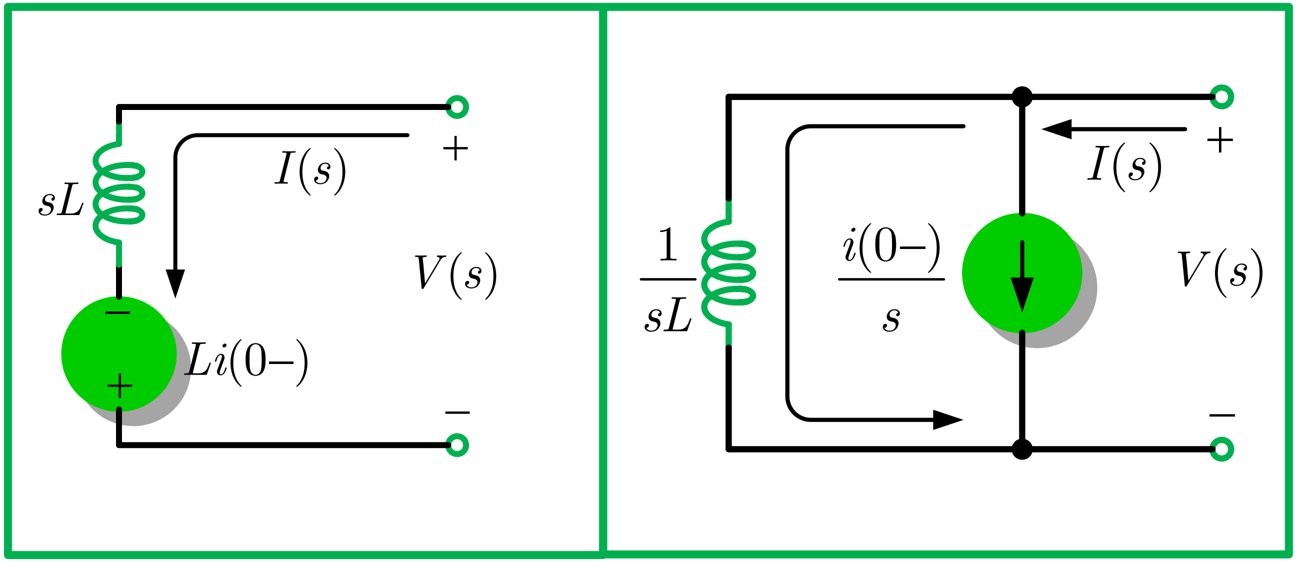
7.1 The Transformed Circuit

Inductor

$$v(t) = L\frac{di(t)}{dt} \xrightarrow{\mathcal{L}\{v(t)\}} V(s) = sLI(s) - Li(0-) \qquad \textbf{(7.3)}$$

$$i(t) = \frac{1}{L}\int_{0-}^{t} v(\tau)d\tau + i(0-) \xrightarrow{\mathcal{L}\{i(t)\}} I(s) = \frac{1}{sL}V(s) + \frac{i(0-)}{s} \qquad \textbf{(7.4)}$$

(Impedance)



(Admittance)

Fig. 7.1: Inductor

Mesh and node equations on impedance and admittance basis can be written directly

7.1 The Transformed Circuit

Capacitor

$$v(t) = \frac{1}{C}\int_{0-}^{t} i(\tau)d\tau + v(0-) \xrightarrow{\mathcal{L}\{v(t)\}} V(s) = \frac{1}{sC}I(s) + \frac{v(0-)}{s} \quad (7.5)$$

$$i(t) = C\frac{dv(t)}{dt} \xrightarrow{\mathcal{L}\{i(t)\}} I(s) = sCV(s) - Cv(0-) \quad (7.6)$$

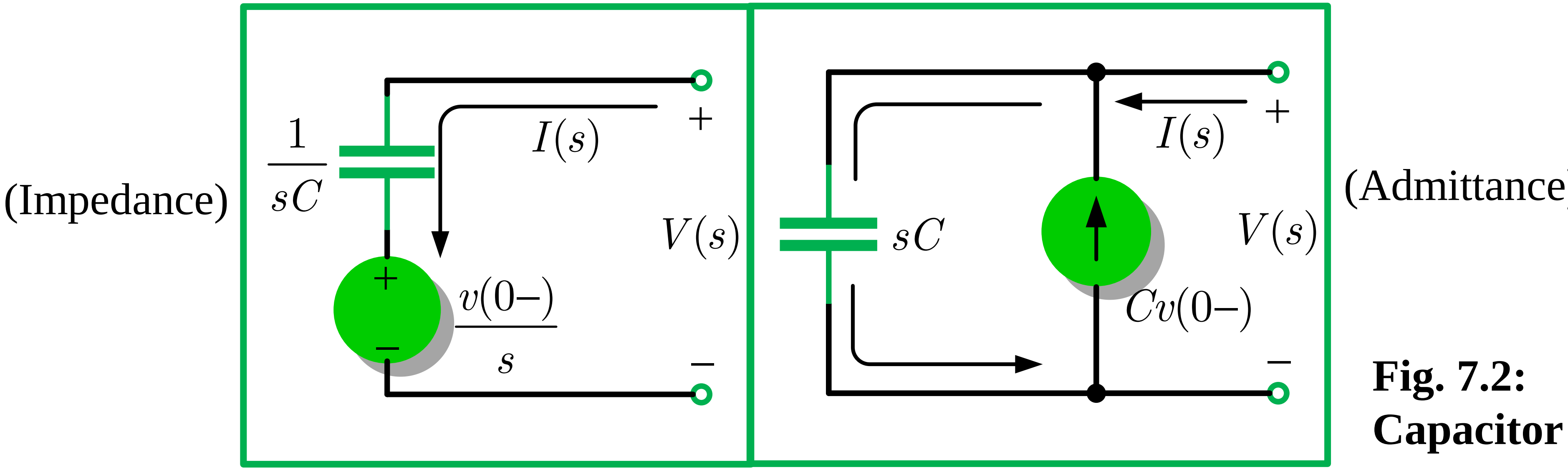


Fig. 7.2: Capacitor

Mesh and node equations on impedance and admittance basis can be written directly

7.1 The Transformed Circuit

Transformer

$$v_1(t) = L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt}$$

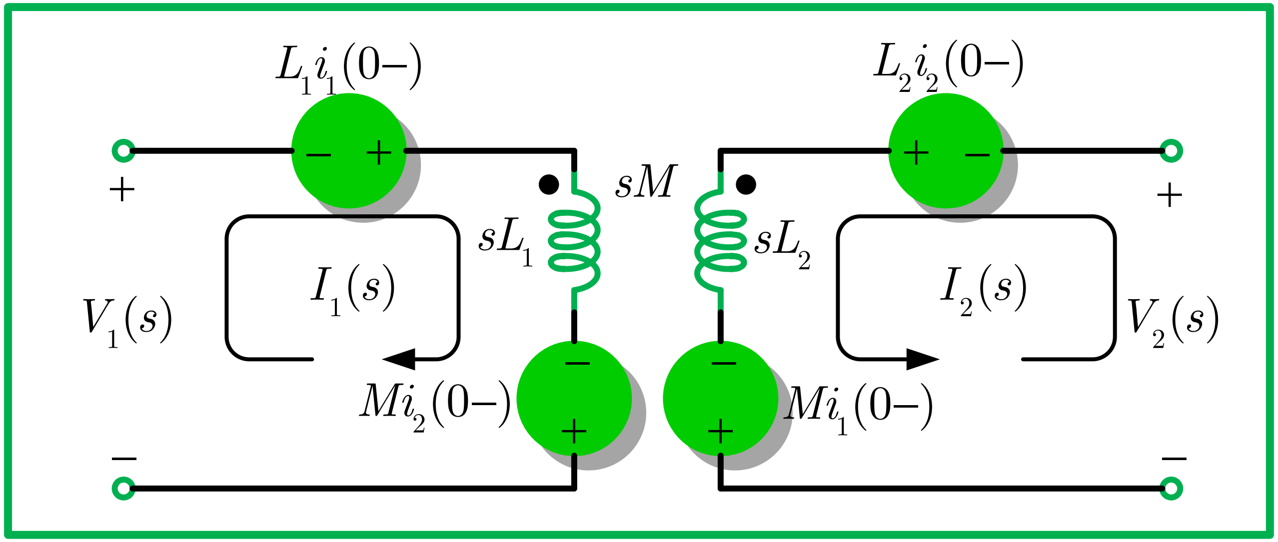
$$v_2(t) = M \frac{di_1(t)}{dt} + L_2 \frac{di_2(t)}{dt}$$

$\mathcal{L}\{v_1(t)\}$

$\mathcal{L}\{v_2(t)\}$

$$V_1(s) = sL_1 I_1(s) - L_1 i_1(0-) + sM\, I_2(s) - M i_2(0-) \tag{7.7}$$

$$V_2(s) = sM\, I_1(s) - M i_1(0-) + sL_2 I_2(s) - L_2 i_2(0-) \tag{7.8}$$



In general the use of transformed equivalent Circuits is considered an easier way to solve the problem

Fig. 7.3

7.1 The Transformed Circuit

[Example 7.1]

In Fig. 7.4, the switch is thrown from position 1 to 2 at time $t = 0$. Just before the switch is thrown, the initial conditions are $i_L(0-) = 2\,\mathrm{A}$, $v_C(0-) = 2\,\mathrm{V}$.

Let us find the current $i_1(t)$ after the switching action.

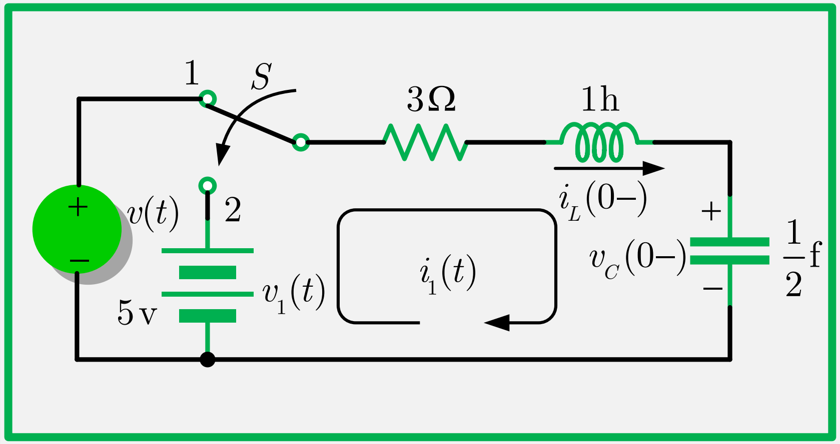


Fig. 7.4

[Solution]

- Apply KVL to the transformed circuit.

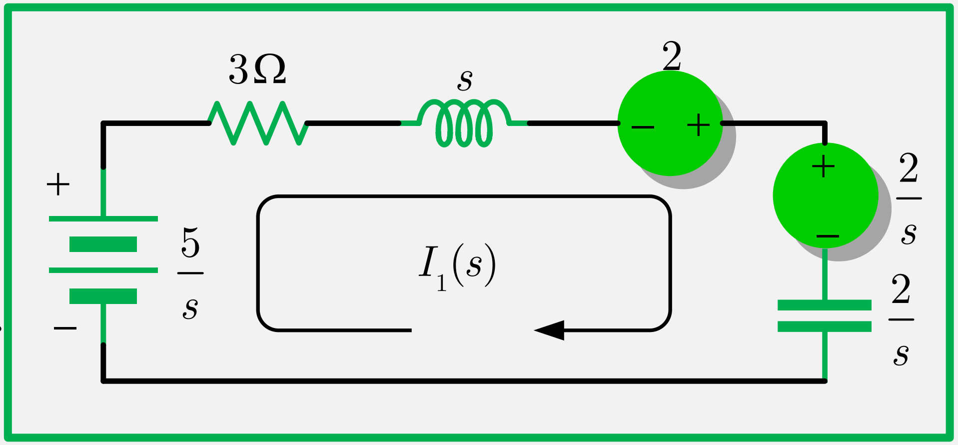


Fig. 7.5

$$\frac{5}{s} - \frac{2}{s} + 2 = \left(3 + s + \frac{2}{s}\right) I_1(s);$$

$$\Rightarrow I_1(s) = \frac{2s+3}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{1}{s+2};$$

$$i_1(t) = \mathcal{L}^{-1}\left\{I_1(s)\right\} = e^{-t} + e^{-2t}$$

7.1 The Transformed Circuit

[Example 7.2]

- Consider the network in Fig. 7.6. At $t=0$, the switch is opened.
- Let us find the current $v_1(t)$ and $v_2(t)$ for the circuit. It is given that

$$L=\frac{1}{2}\text{h} \qquad C=1\text{f} \qquad G=1\text{mho} \qquad V=1\text{v}$$

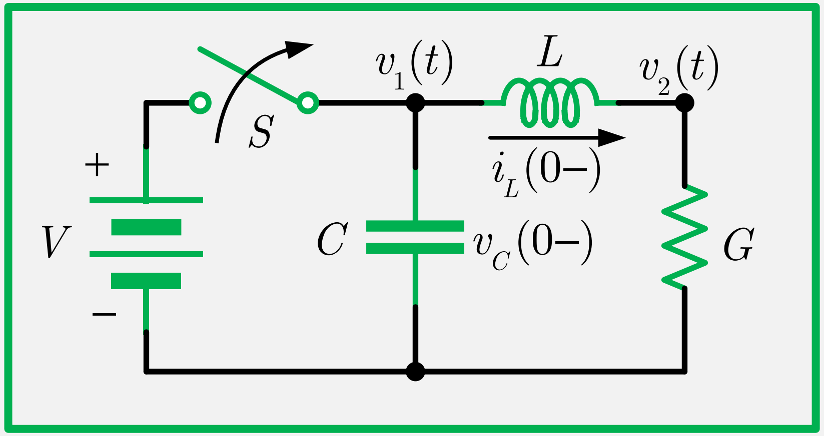


Fig. 7.6

$$1-\frac{1}{s}=\left(s+\frac{2}{s}\right)V_1(s)-\frac{2}{s}V_2(s)\cdots(i)$$

$$\frac{1}{s}=-\frac{2}{s}V_1(s)+\left(\frac{2}{s}+1\right)V_2(s)\cdots(ii)$$

[Solution]

- Apply KCL to the transformed circuit.

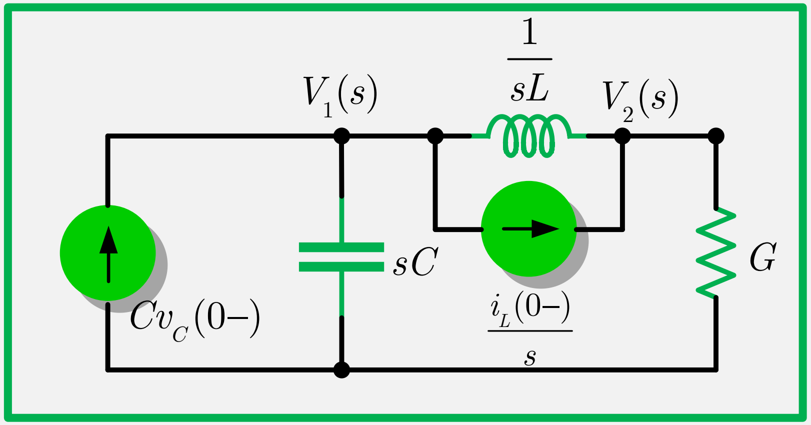


Fig. 7.7

$$s-1=(s^2+2)V_1(s)-2V_2(s)\cdots(i)$$

$$1=-2V_1(s)+(s+2)V_2(s)\cdots(ii)$$

7.1 The Transformed Circuit

[Example 7.2] [Solution] Cont'd

- Solving the simultaneous equations yields,

$$V_1(s) = \frac{s+1}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1} \;; \qquad V_2(s) = \frac{s+2}{s^2+2s+2} = \frac{s+2}{(s+1)^2+1}$$

- Thus, the inverse transforms are of the form,

$$v_1(t) = e^{-t}\cos t \qquad v_2(t) = e^{-t}(\cos t + \sin t)$$

7.2 Thevenin's and Norton's Theorems

- In system analysis, the objective of a problem is often to determine a single branch current through a given element or a single voltage across an element.
- It is generally not practical to write a complete set of mesh and node equations and to solve them for this one current or voltage.
- It is then convenient to use two very important theorems on equivalent circuits, known as *Thevenin's and Norton's theorems*.

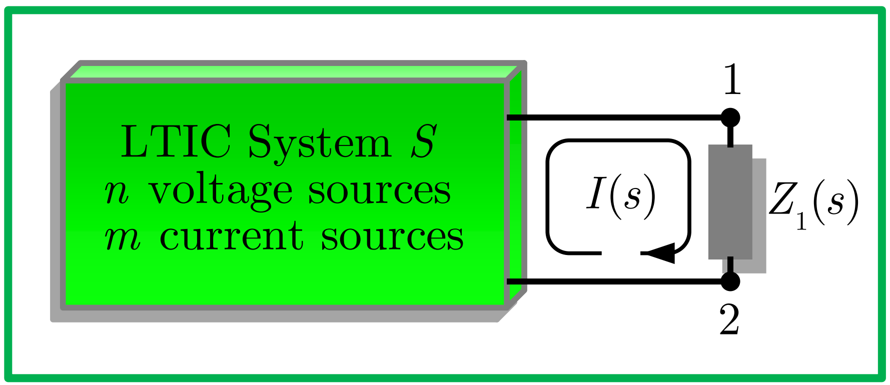


Fig. 7.8

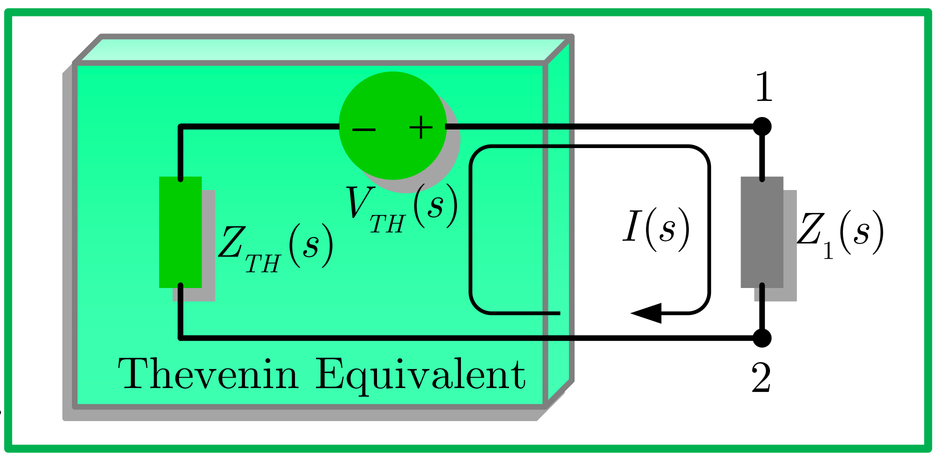


Fig. 7.9: Thevenin's equivalent circuit.

Thevenin's theorem

- From the standpoint of determining the current $I(s)$ through an element of impedance $Z_1(s)$, the rest of the system S can be replaced by an equivalent impedance $Z_{TH}(s)$ in series with an equivalent voltage source $V_{TH}(s)$.

7.2 Thevenin's and Norton's Theorems

- ❑ $\boldsymbol{V_{TH}(s)}$ is the voltage across 1-2 when Z is removed or open circuited
- ❑ $\boldsymbol{Z_{TH}(s)}$ is equivalent impedance "looking into" S from terminals of $Z_1(s)$ when all voltage sources are short circuited and current sources are open circuited.
- ❑ **Note:** $Z_1(s)$ must note be magnetically coupled to an element in S.

[Example 7.3]

✍Let us determine by Thevenin's theorem the current $I_1(s)$ flowing through the capacitor in the system shown in Fig. 7.10.

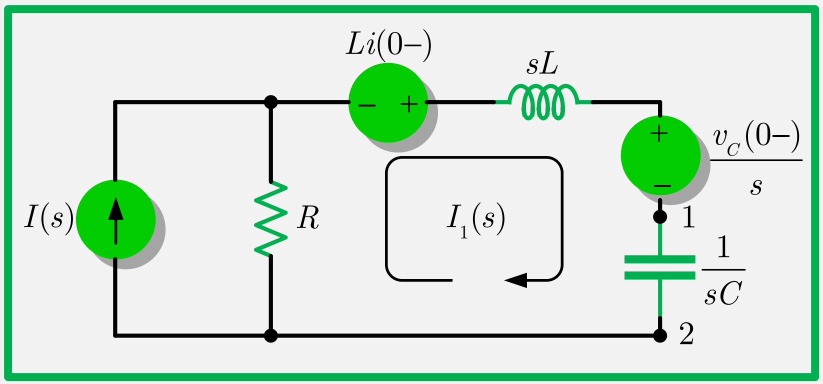
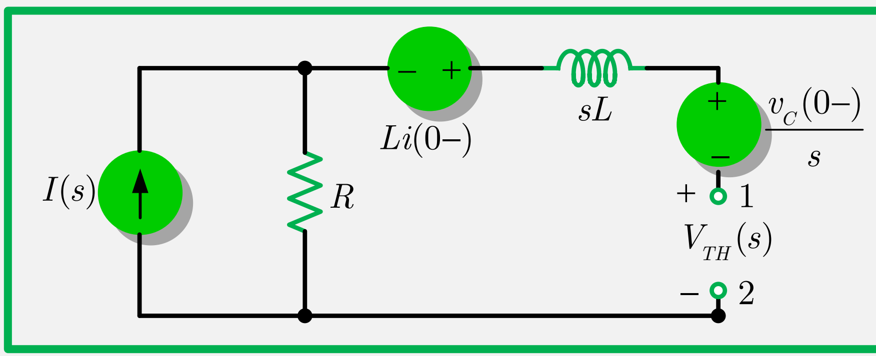


Fig. 7.10

[Solution]

- ❑ Capacitor removed to find $Z_{TH}(s)$ and $V_{TH}(\mathrm{s})$.

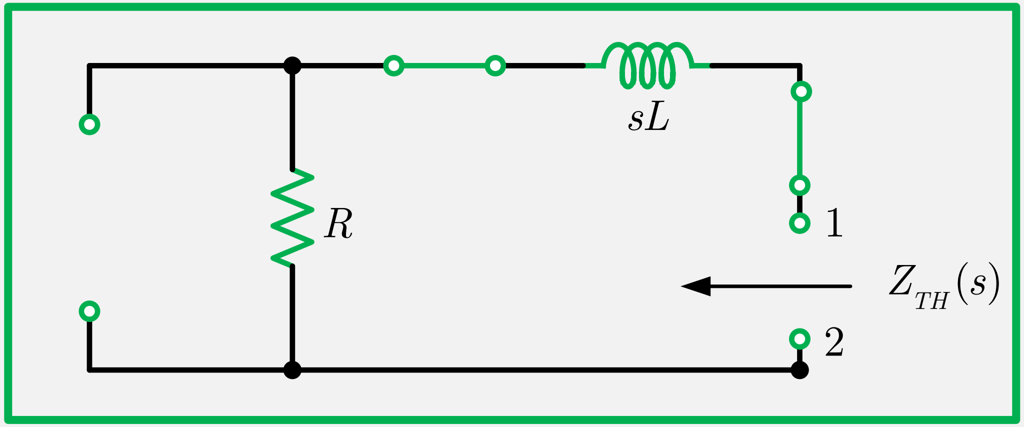


$$V_{TH}(s) = I(s)R + Li(0-) - \frac{v_C(0-)}{s}$$

Fig. 7.11

7.2 Thevenin's and Norton's Theorems
[Example 7.3] [Solution]

By the Thevenin's theorem we have



$$\Rightarrow \quad Z_{TH}(s) = R + sL\ ;$$

Fig. 7.12

$$I_1(s) = \frac{V_{TH}(s)}{Z_{TH}(s) + Z(s)} = \frac{I(s)R + Li(0-) - v_C(0-)/s}{R + sL + 1/sC}$$

7.2 Thevenin's and Norton's Theorems

Norton's theorem

- ❑ When it is required to find the voltage across an element whose admittance is $Y_1(s)$, the rest of the LTIC System S can be represented by an equivalent admittance $Y_e(s)$ in parallel with an equivalent current source $I_e(s)$.

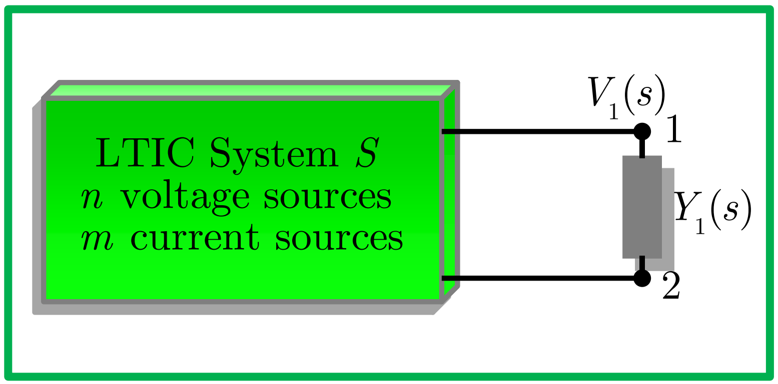
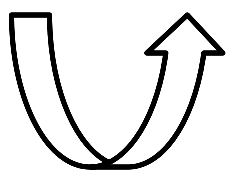


Fig. 7.13



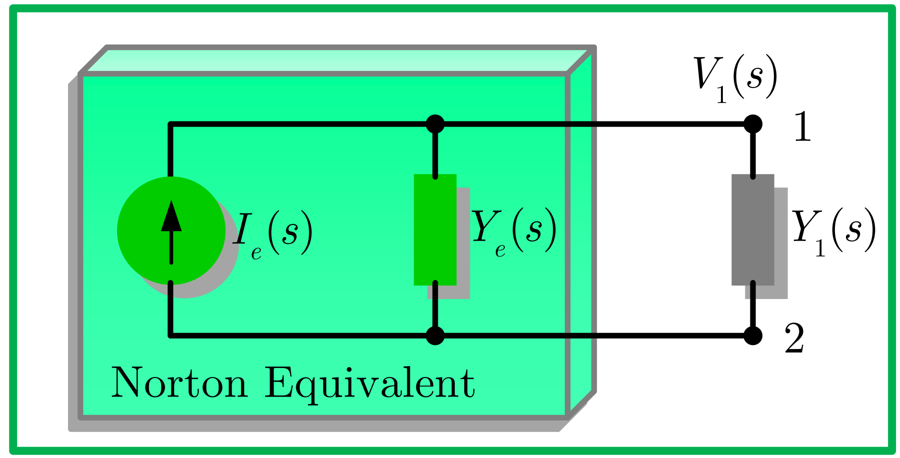


Fig. 7.14

- ❑ $I_e(s)$ is current which flows through a short circuit across $Y_1(s)$.
- ❑ $Y_e(s)$ is the reciprocal of the Thevenin impedance.

Norton's theorem

[Example7.4]

Consider the LTIC network in FIG. 7.15. Let us find the voltage across the capacitor by Norton's theorem.

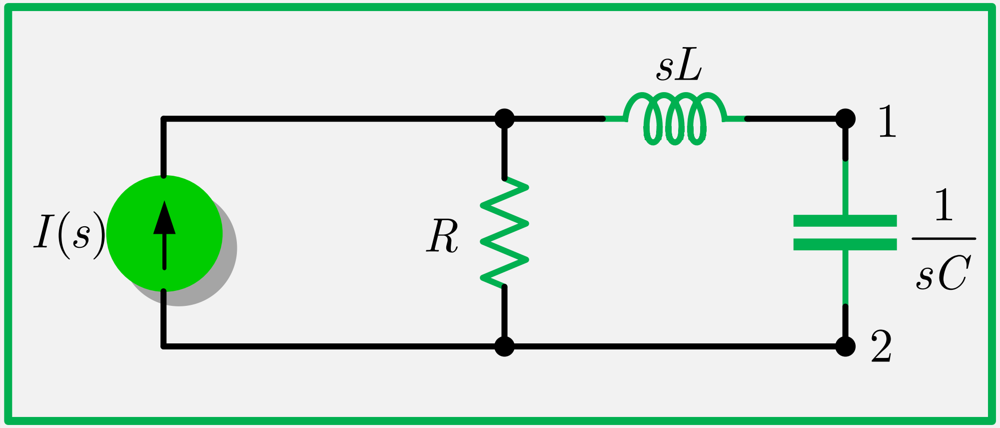


FIG. 7.15

[SOLUTION]

By current divider rule, the current through the capacitor $I_C(s)$ is of the form

$$I_C(s) = \frac{R}{R + sL + 1/sC} I(s)$$

$$\Rightarrow V_C(s) = \left[\frac{1}{sC}\right] I_C(s) = \frac{R}{sC[R + sL + 1/sC]} I(s)$$

$$\therefore V_C(s) = \frac{I(s)R}{(sL + R)sC + 1}$$

7.3 System Analysis

- In earlier discussions a linear system with excitation $e(t)$ was related to the response $r(t)$ by a differential equation.
- When the Laplace transform is used in describing the system, the relationship between the excitation $E(s)$ and the response $R(s)$ is an algebraic one. In particular, when we discuss initially inert systems, the excitation and response are related by the system function $H(s)$ by the given relation

$$R(s)=E(s)H(s) \tag{7.9}$$

7.3 System Analysis

Forms of System Functions

- It was mentioned in lecture 1 that the system function may assume many forms and may have special names driving-point admittance, transfer impedance, voltage or current ratio transfer function.
- This is because the form of the system function depends on whether the excitation is a voltage or current source and whether the response is a specified current or voltage.

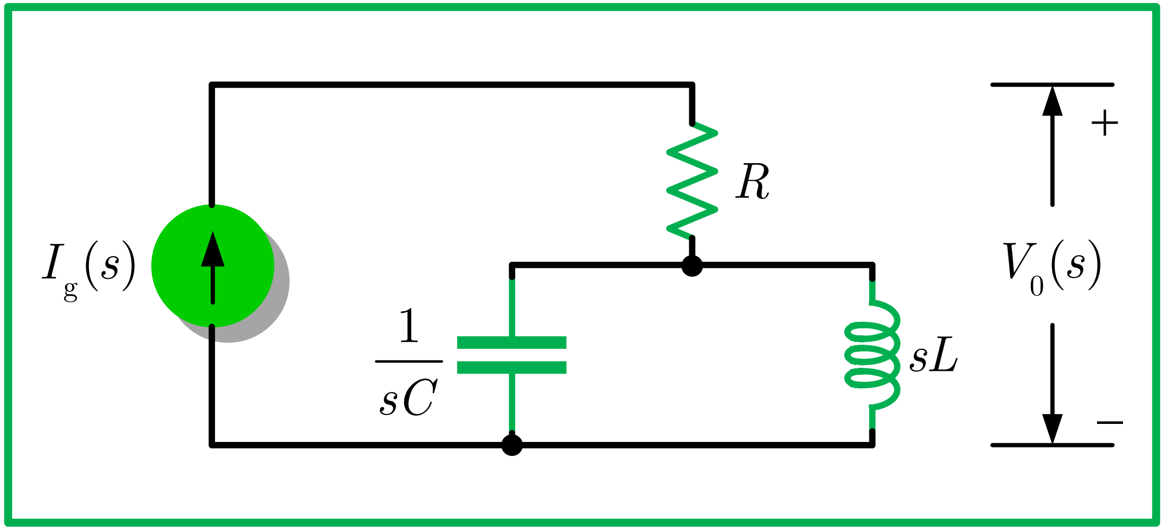


FIG. 7.16

7.3 System Analysis

Forms of System Functions: Impedance

- If the excitation is a current source and the response is a voltage then the system function is an impedance.
- When both excitation and response are measured between the same pair of terminals, we have a driving point impedance.

For example a driving-point impedance is given in FIG. 7.17, here $H(s)$ is of the form

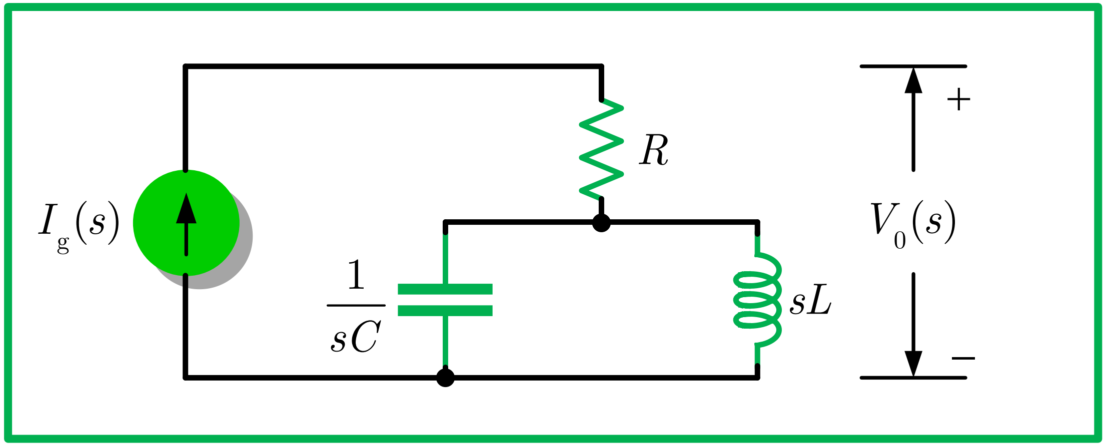


FIG. 7.17

$$H(s) = \frac{V_0(s)}{I_g(s)} = R + \frac{[1/sC]sL}{sL + 1/sC} \tag{7.10}$$

7.3 System Analysis

Forms of System Functions: Admittance

- When excitation is a voltage source and the response is a current, $H(s)$ is an admittance.
- In Fig. 7.18, the transfer admittance $I_2(s)/V_g(s)$ is obtained from the network as

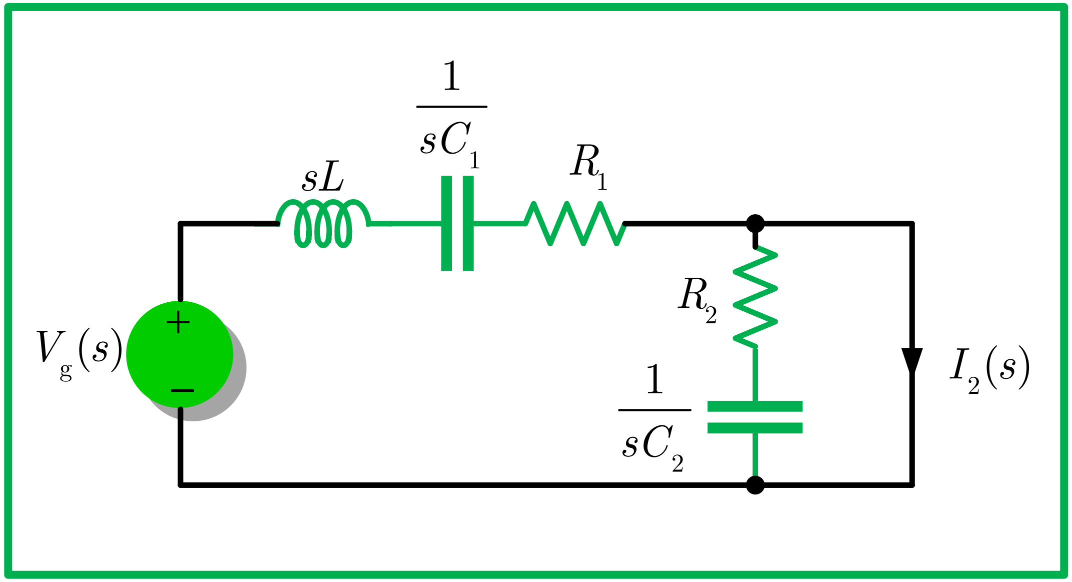


FIG. 7.18

$$H(s) = \frac{I_2(s)}{V_g(s)} = \frac{1}{R_1 + sL + 1/sC_1} \tag{7.11}$$

7.3 System Analysis

Forms of System Functions: Voltage-ratio transfer function

- When excitation is a voltage source and the response is also a voltage, $H(s)$ is a voltage-ratio transfer function .
- In Fig. 7.19, the Voltage-ratio transfer function $V_0(s)/V_g(s)$ is obtained as follows:

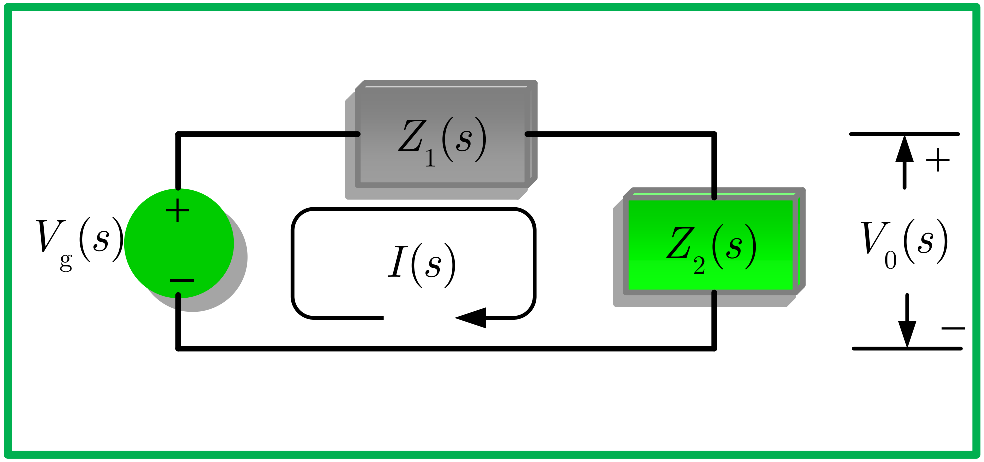


FIG. 7.18

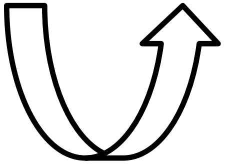
- First find the current

$$I(s) = \frac{V_g(s)}{Z_1(s) + Z_2(s)}$$

- Since

$$V_0(s) = Z_2(s)I(s)$$

- Vividly,



$$\boxed{\frac{V_0(s)}{V_g(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}} \qquad \textbf{(7.12)}$$

7.3 System Analysis

Forms of System Functions: Current-ratio transfer function

- When excitation is a current source and the response is another current, $H(s)$ is called a current-ratio transfer function .

- Let us find the ratio $I_0(\mathrm{s})/I_g(\mathrm{s})$ for the LTIC system in Fig. 7.20. From the LTIC system we know that

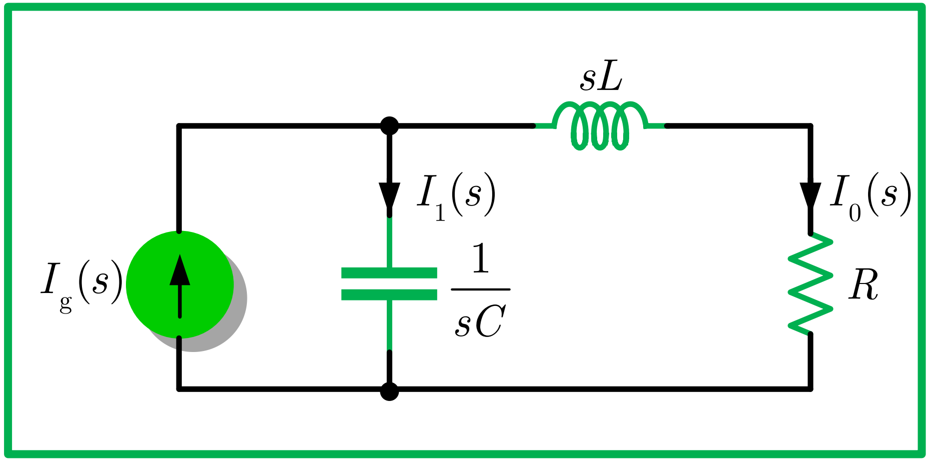


FIG. 7.20

$$I_g(s) = I_1(s) + I_0(s);$$

$$I_1(s)\frac{1}{sC} = I_0(s)[R + sL]$$

- Eliminating the variable I_1, we find

$$I_g(s) = I_0(s)\left[1 + \frac{R + sL}{1/sC}\right]$$

- So that the current-ratio transfer function is

$$\boxed{\frac{I_0(s)}{I_g(s)} = \frac{1/sC}{R + sL + 1/sC}} \qquad \mathbf{(7.13)}$$

7.3 System Analysis

Obtaining $R(s)$ given $E(s)$ and $H(s)$

- From the preceding examples, we have seen that the system function is a function of the elements of the network alone, and is obtained from the network alone, and is obtained by applying Kirchhoff's laws.

- Now let us obtain $R(s)$, given the excitation and the system function. Consider Fig. 7.21, assume that the network is initially inert when the switch is closed at $t = 0$.

- Determine response $V(s)$ for:

1. $i_g(t) = (\sin\omega_0 t)u(t)$.
2. $i_g(t)$ is the square pulse in Fig. 7.22.
3. $i_g(t)$ has the waveform in FIG. 7.23.

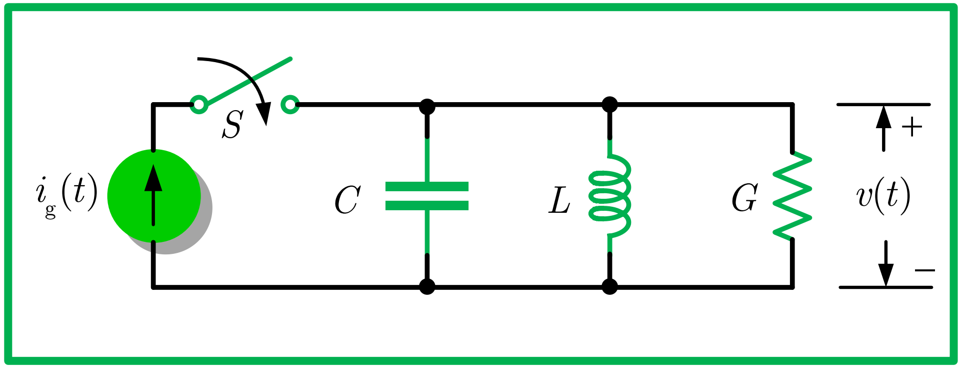


FIG. 7.21

First, we obtain the system function as

$$H(s) = \frac{1}{sC + 1/sL + G} = \frac{s}{C[s^2 + s(G/C) + 1/LC]} \qquad (7.14)$$

7.3 System Analysis

Obtaining $R(s)$ given $E(s)$ and $H(s)$

1. For $i_g(t) = [\sin \omega_0 t]u(t)$ the transform is of the form

$$I_g(s) = \frac{\omega_0}{s^2 + \omega_0^2} \; ; \quad \text{so that} \qquad \textbf{(7.15)}$$

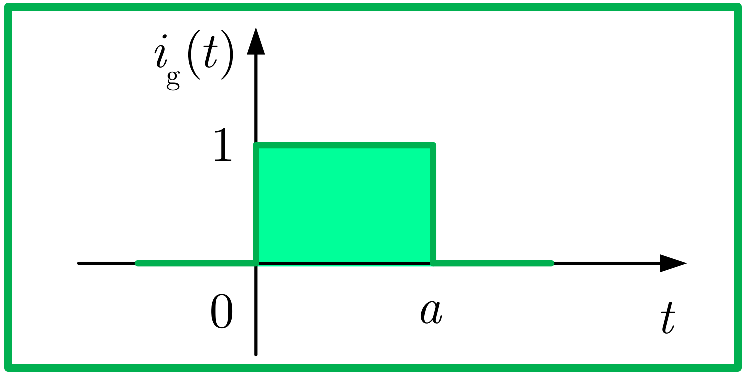


FIG. 7.22

$$V(s) = I_g(s)H(s) = \frac{\omega_0}{s^2 + \omega_0^2} \cdot \frac{s}{C[s^2 + s(G/C) + 1/LC]} \qquad \textbf{(7.16)}$$

2. For the square pulse in FIG. 7.22, $i_g(t)$ can be written as

$i_g(t) = u(t) - u(t-a)$. Its transform is of the form $I_g(s) = \frac{1}{s}[1 - e^{-as}]$; **(7.17)**

- Thus, the response $V(s)$ is given by
- Notice that $V(s)$ is of the form

$$V(s) = \frac{1 - e^{-as}}{C[s^2 + s(G/C) + 1/LC]} \qquad \textbf{(7.18)}$$

$$V(s) = \frac{H(s)}{s}[1 - e^{-as}] \qquad \textbf{(7.19)}$$

Corresponds to delay in time domain

7.3 System Analysis

Obtaining $R(s)$ given $E(s)$ and $H(s)$

- Let the inverse transform of $V_1(s)$ be $v_1(t)$. It follows that $v_1(t)$ is the step response of the LTIC system $H(s)$. That is,

$$v_1(t) = \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\} \quad \textbf{(7.20)}$$

- Vividly, using the above concept we obtain inverse transform of $V(s)$, that is the time response of the LTIC system to the excitation $i_g(t)$ to be of the form,

$$v(t) = v_1(t)u(t) - v_1(t-a)u(t-a) \quad \textbf{(7.21)}$$

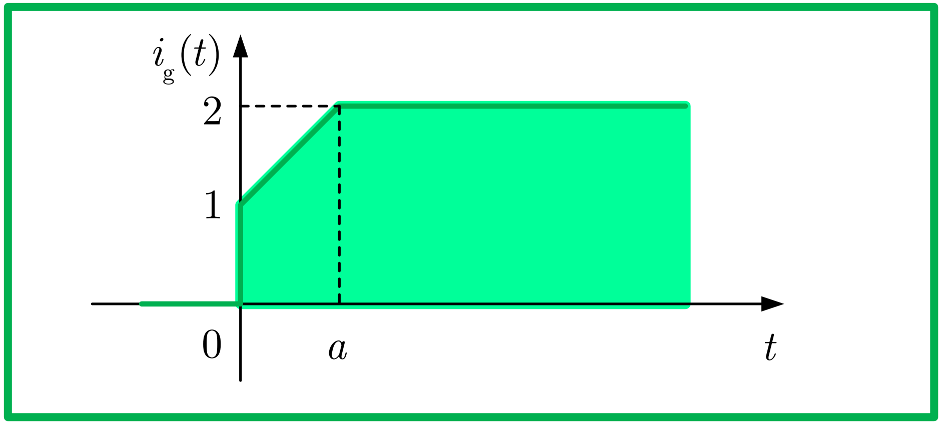


FIG. 7.23

7.3 System Analysis

Obtaining $R(s)$ given $E(s)$ and $H(s)$

3. The waveform depicted in FIG. 7.24 can be represented as

$$i_g(t) = u(t) + \frac{t}{a}u(t) - \frac{(t-a)}{a}u(t-a) \tag{7.22}$$

- Thus, its transform is of the form $I_g(s) = \frac{1}{s}\left[1 + \frac{1}{as} - \frac{e^{-as}}{as}\right];$ **(7.23)**
- It follows that $V(s)$ is of the form

$$V(s) = \left[1 + \frac{1}{as} - \frac{e^{-as}}{as}\right] \cdot \frac{1}{C[s^s + s(G/C) + 1/LC]} \tag{7.24}$$

- If we let $v_2(t)$ be the response of the system to a ramp excitation, we see that

$$v(t) = \mathcal{L}^{-1}\left\{V(s)\right\} = v_1(t) + \frac{1}{a}v_2(t) - \frac{1}{a}v_2(t-a)u(t-a) \tag{7.25}$$

- where is the step response of the LTIC system given in FIG. 7.22.

7.3 System Analysis

Obtaining $R(s)$ given $E(s)$ and $H(s)$

- Consider the partial-fraction expansion of $R(s)$:

$$R(s) = \sum_i \frac{A_i}{s - s_i} + \sum_k \frac{A_k}{s - s_k} \tag{7.26}$$

- Where s_i depict poles of the LTIC system $H(s)$, and s_k are poles of the excitation $E(s)$.

- The inverse transform of R(s) is $$r(t) = \underbrace{\sum_i A_i e^{s_i t}}_{} + \underbrace{\sum_k A_k e^{s_k t}}_{} \tag{7.27}$$

- Associated with the system $H(s)$ and are called the free response terms.
- s_i are the natural frequencies of the system

- These terms are due to the excitation and are known as the forced response terms.
- s_k are the forced frequencies

7.3 System Analysis

R-C differentiator and integrator

- Fourier analysis of the LTIC system in FIG. 7.24 (a) yields the system function $H(j\omega)$ of the form

$$\frac{V_0(j\omega)}{V_g(j\omega)} = \frac{R}{R + 1/j\omega C} = \frac{j\omega RC}{j\omega RC + 1} \qquad \textbf{(7.28)}$$

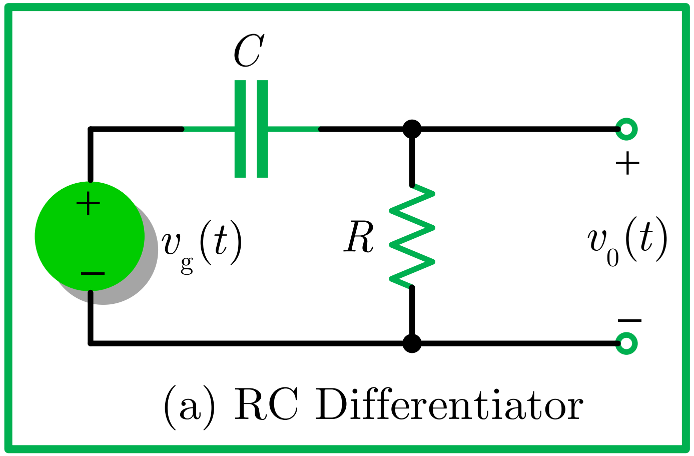
- If we let $\omega RC << 1$, this yields an approximation,

$$\frac{V_0(j\omega)}{V_g(j\omega)} \approx j\omega RC; \qquad \textbf{(7.29)}$$

$$\boxed{\Rightarrow V_0(j\omega) \approx RC[j\omega V_g(j\omega)]} \qquad \textbf{(7.30)}$$

- Thus, the inverse transform of $V_0(j\omega)$ yields

$$\boxed{v_0(t) \approx RC\frac{d}{dt}v_g(t)} \qquad \textbf{(7.31)}$$



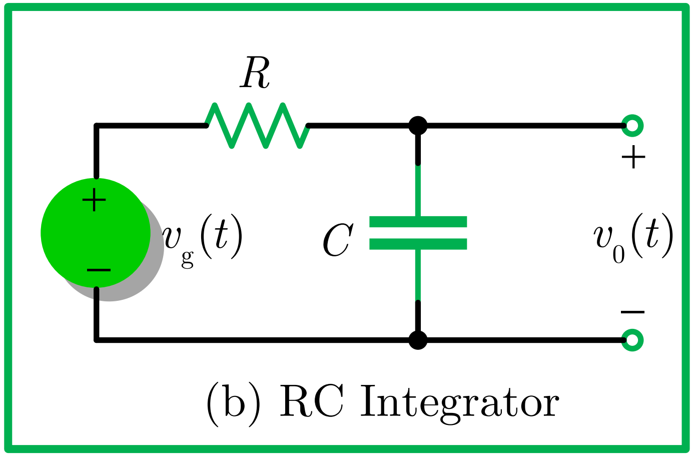


FIG. 7.24

7.3 System Analysis

R-C differentiator and integrator

- Similarly, Fourier analysis of FIG. 7.24(b) yields,

$$\frac{V_0(j\omega)}{V_g(j\omega)} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1} \qquad \textbf{(7.32)}$$

- Assuming $\omega RC >> 1$ yields

$$\boxed{V_0(j\omega) \approx \frac{1}{j\omega RC} V_g(j\omega)} \qquad \textbf{(7.33)}$$

- Under the above stated conditions it follows that the inverse transform of $V_0(j\omega)$

$$\boxed{V_0(j\omega) \approx \frac{1}{j\omega RC} V_g(j\omega)}$$

- yields,

$$\boxed{v_0(t) \approx \frac{1}{RC}\int_0^t v_g(\tau)d\tau} \qquad \textbf{(7.34)}$$

7.4 The Step and Impulse Response

- In this section we will show that the impulse response $h(t)$ and the system function $H(s)$ constitute a transform pair. So that we can obtain step and impulse responses directly from the system function.

- We know that the transform for unit impulse $\delta(t)$ is unity. Suppose that the system excitation were a unit impulse, then the response $R(s)$ would be

$$R(\mathrm{s}) = E(\mathrm{s})H(\mathrm{s}) = 1 \bullet H(\mathrm{s}) = H(\mathrm{s}) \qquad \textbf{(7.35)}$$

- We see that the impulse response $h(t)$ and the system function $H(s)$ constitute a transform pair

$$\mathcal{L}\left\{h(t)\right\} = H(s)$$
$$\mathcal{L}^{-1}\left\{H(s)\right\} = h(t) \qquad \textbf{(7.36)}$$

Since the system function is usually easy to obtain, it is apparent that we can find the impulse response of a system by taking the inverse transform of $H(s)$.

[Example7.5]

- Let us find the impulse response of the current $i(t)$ in the *RC* LTIC circuit of FIG. 7.25.

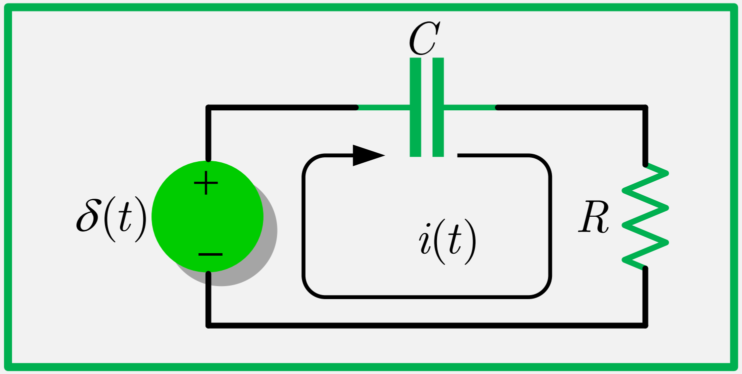


FIG. 7.25

[SOLUTION]

- Vividly, the system function is of the form

$$H(s)=\frac{I(s)}{V_g(s)}=\frac{1}{R+1/sC}=\frac{s}{R[s+1/RC]}$$

- Simplifying $H(s)$ further yields,

$$H(s)=\frac{1}{R}\left[1-\frac{1/RC}{s+1/RC}\right]\quad ; \quad \text{so that,}$$

$$h(t)=\mathcal{L}^{-1}\{H(s)\}=\frac{1}{R}\left[\delta(t)-\frac{1}{RC}e^{-t/RC}\right]u(t)$$

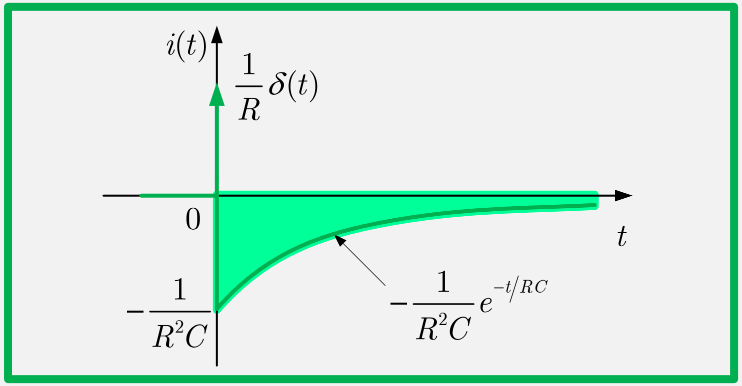


FIG. 7.26

7.4 The Step and Impulse Response

- Since the step response is the integral of the impulse response, we can use the integral property of the Laplace transforms to obtain the step response as

$$\alpha(t) = \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\} \tag{7.37}$$

- Similarly, we obtain the unit ramp response from the equation

$$\gamma(t) = \mathcal{L}^{-1}\left\{\frac{H(s)}{s^2}\right\} \tag{7.38}$$

1. If we know the impulse response of an <u>initially inert linear system</u>, we can obtain all transient response data that are needed to characterize the system.
2. the Impulse response alone is sufficient to characterize the system from the standpoint of excitation and response.

[Example7.6]

Let us find the current step response of the current $I(s)$ in the RL LTIC circuit of FIG. 7.27. **[SOLUTION]**

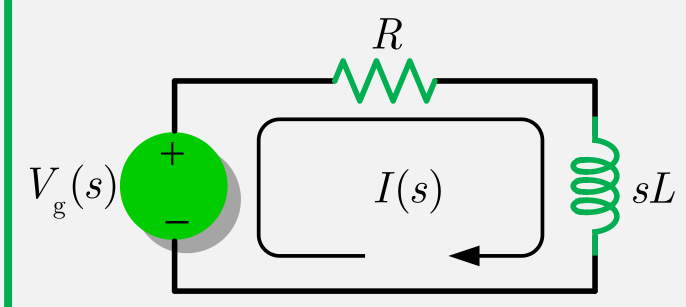


FIG. 7.27

Vividly, the system function $H(s)$ is of the form

$$H(s)=\frac{I(s)}{V_g(s)}=\frac{1}{R+sL};$$

$$\Rightarrow\frac{H(s)}{s}=\frac{1}{s(R+sL)}=\frac{1}{R}\left[\frac{1}{s}-\frac{1}{s+R/L}\right];$$

Thus, the step response is of the form

$$\alpha(t)=\mathcal{L}^{-1}\left\{H(s)/s\right\}=\frac{1}{R}\left[1-e^{-(R/L)t}\right]u(t)$$

To prove this result, recall that impulse of the RL circuit of FIG. 7.27 is

$$h(t)=\frac{1}{L}e^{-(R/L)t}u(t);\quad\Rightarrow\quad \alpha(t)=\int_0^t h(\tau)d\tau=\frac{1}{R}\left[1-e^{-(R/L)t}\right]u(t)$$

- Hence the proof for the step response of the LTIC given.

7.5 The Convolution Integral

- Given two functions $f_1(t)$ and $f_2(t)$, which are zero for $t < 0$, the convolution theory states that if the transform of $f_1(t)$ is $F_1(s)$, for $f_2(t)$ is $F_2(s)$, the transform of the convolution of $f_1(t)$ and $f_2(t)$ is the product of the individual transforms $F_1(s)F_2(s)$, that is

$$\mathcal{L}\left\{\int_{0-}^{t} f_1(t-\tau)f_2(\tau)d\tau\right\} = F_1(s)F_2(s) \tag{7.39}$$

- where the integral $\int_{0-}^{t} f_1(t-\tau)f_2(\tau)d\tau$ is the convolutional integral or folding integral, and is denoted operationally as

$$\int_{0-}^{t} f_1(t-\tau)f_2(\tau)d\tau = f_1(t) * f_2(t) \tag{7.40}$$

- Proof:

$\mathcal{L}\left\{f_1(t) * f_2(t)\right\} = \int_0^{\infty} e^{-st}\int_0^{t}[f_1(t-\tau)f_2(\tau)d\tau]dt;$ by shifted the step function we have

$\mathcal{L}\left\{f_1(t) * f_2(t)\right\} = \int_0^{\infty} e^{-st}\int_0^{\infty}[f_1(t-\tau)u(t-\tau)f_2(\tau)d\tau]\,dt;$ let $x = t-\tau$, so that

$$\mathcal{L}\left\{f_1(t) * f_2(t)\right\} = \int_0^{\infty}\int_0^{\infty} f_1(x)u(x)f_2(\tau)e^{-s\tau}e^{-sx}d\tau dx = \int_0^{\infty} f_1(x)u(x)e^{-sx}dx\int_0^{\infty} f_2(\tau)e^{-s\tau}d\tau$$

$$\therefore\ \mathcal{L}\left\{f_1(t) * f_2(t)\right\} = F_1(s)F_2(s)$$

7.5 The Convolution Integral

[Example7.7]

Let us evaluate the convolution of the functions $f_1(t)=e^{-2t}u(t)$ and $f_2(t)=tu(t)$, and then compare the results with the inverse transform of $F_1(s)F_2(s)$.

[SOLUTION] Vividly, $F_1(s)=\mathcal{L}\{f_1(t)\}=\dfrac{1}{s+2}$; and $F_2(s)=\dfrac{1}{s^2}$

The convolution $f_1(t)*f_2(t)$ is obtained by first substituting the dummy variable $t-\tau$ for t in $f_1(t)$, so that

$$f_1(t-\tau)=e^{-2(t-\tau)}u(t-\tau);$$

$$f_1(t)*f_2(t)=\int_0^t f_1(t-\tau)f_2(\tau)d\tau=\int_0^t \tau e^{-2(t-\tau)}d\tau=e^{-2t}\int_0^t \tau e^{2\tau}d\tau$$

Integration by parts yields

$$f_1(t)*f_2(t)=\left[\frac{t}{2}-\frac{1}{4}+\frac{1}{4}e^{-2t}\right]u(t)$$

Next we evaluate

$$F_1(s)=\mathcal{L}^{-1}\{F_1(s)F_2(s)\};\quad \text{i.e.,}\quad F_1(s)F_2(s)=\frac{1}{s^2(s+2)}=\frac{1/2}{s^2}-\frac{1/4}{s}+\frac{1/4}{s+2};$$

$$\therefore\ \mathcal{L}^{-1}\{F_1(s)F_2(s)\}=\left[\frac{t}{2}-\frac{1}{4}+\frac{1}{4}e^{-2t}\right]u(t)$$

Hence the proof.

7.5 The Convolution Integral

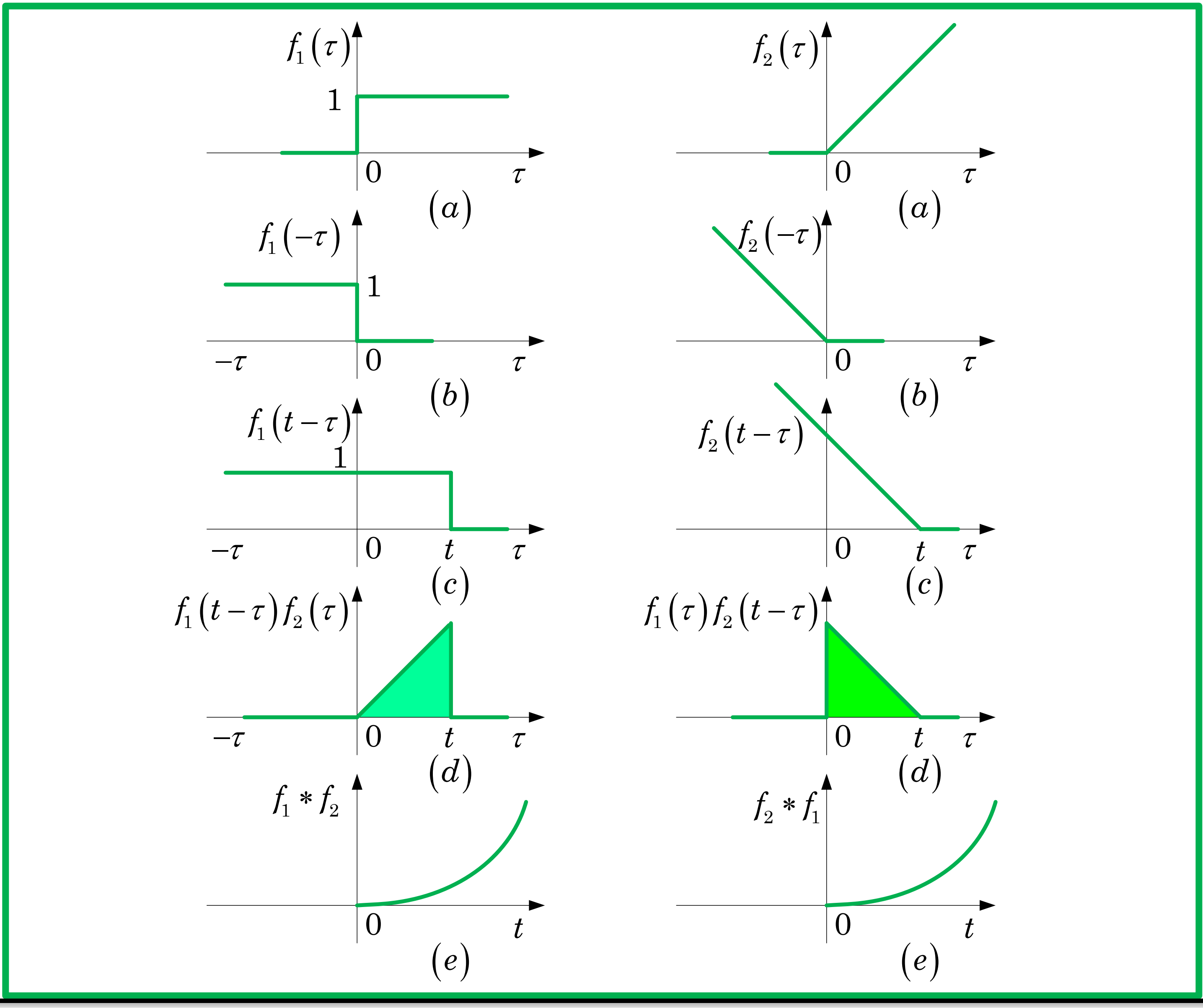


FIG. 7.28

7.5 The Convolution Integral

- Let us now examine the role of the convolution integral in system analysis.
- From the familiar equation $R(s) = E(s)H(s)$
- We obtain the time response as

$$r(t) = \mathcal{L}^{-1}\{E(s)H(s)\} = \int_0^t e(\tau)h(t-\tau)d\tau \qquad \textbf{(7.41)}$$

- where $e(\tau)$ is the excitation and $h(\tau)$ is the impulse response of the system.

- Given that the impulse response of a system is known the convolution integral can be used to obtain the response of the system directly in the time domain.

[Example7.8]

- Let us find the response $i(t)$,of the *RL* LTIC system in FIG. 7.29 due to the excitation $v(t) = 2e^{-t}u(t)$.

[SOLUTION]

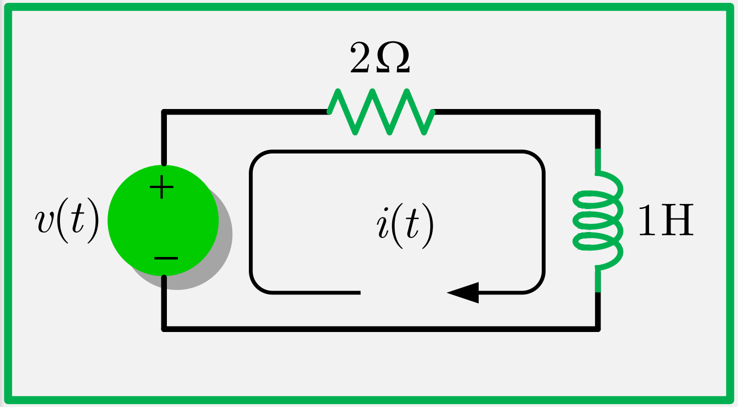


FIG. 7.29

- Vividly, the impulse response for the current is of the form

$$h(t) = \frac{1}{L}e^{-(R/L)t}u(t)$$

- For the *RL* LTIC system under discussion

$$h(t) = e^{-2t}u(t)$$

- Using the convolution integral, we have

$$i(t) = \int_0^t v(t-\tau)h(\tau)d\tau = 2\int_0^t e^{-(t-\tau)}e^{-2\tau}d\tau = 2e^{-t}\int_0^t e^{-\tau}d\tau;$$

$$\boxed{\therefore \; i(t) = 2[e^{-t} - e^{-2t}]u(t)}$$

Which is the free response ?

End of Lecture 7

Thank you for your attention!