2. SOLUTIONS OF EQUATIONS IN ONE VARIABLE

In scientific and engineering work, a frequently occurring problem is to find the solution of equations of the form

$$f(x) = 0. \qquad (2.1)$$

Solutions of (2.1) are called roots or zeros of the function f. If f is a quadratic, cubic or biquadratic expression, then algebraic formulae are available for expressing the roots in terms of the coefficients. On the other hand, when f is a polynomial of higher degree or an expression involving transcendental functions, algebraic methods are not available. In this chapter, we will describe some numerical methods for finding the roots of (2.1) where f is algebraic, transcendental or a combination of both.

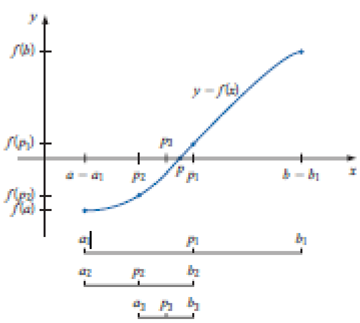
2.1 Bisection Method

Suppose that f is a continuous function defined on the interval $[a, b]$, with $f(a)$ and $f(b)$ of opposite signs. By the Intermediate Value Theorem, there exists at least one number $p \in (a, b)$ such that $f(p) = 0$. The Bisection method calls for a repeated halving (or bisection) of an interval $[a, b]$, at each step, locating the half containing p.

To begin, set $a_1 = a$ and $b_1 = b$, and let p_1 be the midpoint of $[a, b]$. Then,

$$p_1 = \frac{a_1 + b_1}{2}.$$

If $f(p_1) = 0$, then $p = p_1$ is the root of (2.1). If $f(p_1) \neq o$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$. If $f(p_1)$ has the same sign as $f(a_1)$, then $p \in (p_1, b_1)$ and set $a_2 = p_1$ and $b_2 = b_1$. If $f(p_1)$ has the same sign as $f(b_1)$, then $p \in (a_1, p_1)$ and set $a_2 = a_1$ and $b_2 = p_1$. Then, reapply the process to the interval $[a_2, b_2]$.



NOTE: If we select a tolerance $\varepsilon > 0$, then

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon, \quad p_N \neq 0,$$

may be used as the stopping criterion.

Example 2. 1.

1. *Find a real root of the equation*

   $$f(x) = x^3 - x - 1 = 0$$

   in the interval $[1, 2]$.

2. *Use the Bisection method to solve*

   $$f(x) = x^3 + 4x^2 - 10 = 0$$

   which has a root in the interval $[1, 2]$.

Solutions:

1. *Clearly,* $f(1) = -1$ *and* $f(2) = 5$ *so that* $p_1 = \frac{1+2}{2} = 1.5$.

   Since $f(1.5) = 0.875$, *we have that the root lies in the interval* $[1, 1.5]$.

   $p_2 = \frac{1+1.5}{2} = 1.25$ *and* $f(1.25) = -0.296875$ *implying that the root lies in the interval* $[1.25, 1.5]$

   $p_3 = \frac{1.25+1.5}{2} = 1.375$ *and* $f(1.375) = 0.224609375$

   $\therefore$ *A root lies in* $[1.25, 1.375] \Rightarrow p_4 = 1.3125$

   The procedure is repeated in the same manner to approximate a root.

2. *For* $f(x) = x^3 + 4x^2 - 10 = 0$, *we have that* $f(1) = -5$ *and* $f(2) = 14$ *implying that there is a root in the interval* $[1, 2]$.

   The table below shows other results for $n = 1, 2, \ldots 13$*:*

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	2.375
2	1	1.5	1.25	-1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	-0.84839
5	1.3125	1.375	1.34375	-0.35098
6	1.34375	1.375	1.359375	-0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	-0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	-0.01605
11	1.364257813	1.365234375	1.364746094	-0.00799
12	1.364746094	1.365234375	1.364990235	-0.00396
13	1.364990235	1.365234375	1.365112305	-0.00194

The correct value is $p = 1.365230013$*, which shows that* p_9 *is the most accurate approximation and*

$$\frac{|p_9 - p_8|}{|p_9|} = 0.001430615165 < 0.005.$$

Theorem 2. 1.

Let $f \in C[a, b]$ *and suppose that* $f(a)f(b) < 0$*. The bisection method generates a sequence* $\{p_n\}_{n=1}^{\infty}$ *approximating* p *with the property*

$$|p - p_n| \leq \frac{b-a}{2^n}, \quad n \geq 1$$

<u>NOTE</u>: Theorem 2.1 gives only a bound for approximation error and that this bound might be quite conservative. For example, from Example 2.1, part (2)

$$|p - p_9| \leq \frac{2-1}{2^9} \approx 0.2 \times 10^{-2}.$$

But the actual absolute error is much smaller

$$|p - p_9| = |1.365230013 - 1.365234375| = 0.000004362$$

Example 2. 2. *Determine the number of iterations necessary to solve*

$$f(x) = x^3 + 4x^2 - 10 = 0$$

with accuracy within 10^{-3} *in the interval* $[1, 2]$.

Solution:

We need to find n *that satisfies*

$$|p - p_n| \leq \frac{b - a}{2^n} = 2^{-n} < 10^{-3}$$

$$\Rightarrow -n \log 2 < -3 \log 10$$

$$\Rightarrow n > \frac{3}{\log 2} \approx 9.96.$$

Hence, about 10 *iterations are needed to obtain the required accuracy.*

Although the bisection method always converges, one disadvantage is that it converges slowly, i.e. n *may become quite large before* $|p - p_n|$ *is sufficiently small.*

2.2 Fixed-Point Iteration

To describe this method, we notice that (2.1) can be written in the form

$$x = g(x).$$

For example,

$$x^3 + x^2 - 1 = 0$$

can be written as

- $x = (1 + x)^{-\frac{1}{2}}$
- $x = (1 - x^3)^{\frac{1}{2}}$
- $x = (1 - x^2)^{\frac{1}{3}}$
- $x = (x + x^2)^{-1}$

and so on.

Definition 2. 1.

A fixed point for a function is a number at which the value of the function does not change when the function is applied.

For example, p is a fixed point for a function g if

$$g(p) = p.$$

Example 2. 3.

The function

$$g(x) = x - \sin \pi x$$

has two fixed points $x = 0$ *and* $x = 1$

A fixed point problem can be used to solve root problem in that if g has a fixed point at p, then f defined as

$$f(x) = x - g(x)$$

has a zero at p. Conversely, if $f(p) = 0$, then g can be defined as

$$g(x) = x - f(x) \text{ or } g(x) = x + 10f(x), \ etc$$

has a fixed point p.

To begin, let p_0 be an approximate value of the desired root p. Then, the first approximation is

$$p_1 = g(p_0).$$

Successive approximations are then given by

$$p_2 = g(p_1)$$

$$p_3 = g(p_2)$$

$$\vdots$$

$$p_n = g(p_{n-1}) \ n \geq 1.$$

Thus, a sequence $\{p_n\}_{n=0}^{\infty}$ is formed. We may ask the following questions:

- Does the sequence $\{p_n\}_{n=0}^{\infty}$ always converge?
- How should we choose g so that $\{p_n\}_{n=0}^{\infty}$ converges to p?

To answer the first question, we give a counter example. If

$$x = 10^x + 1, \text{ i.e. } g(x) = 10^x + 1,$$

then taking $p_0 = 0$ gives

$$p_1 = 2, \;\; p_2 = 101, \;\; p_3 = 10^{101} + 1, \text{ etc.}$$

Clearly, as n increases, $\{p_n\}_{n=0}^{\infty}$ diverges.

The second question is answered by the following theorem, which gives sufficient conditions for the existence and uniqueness of a fixed point:

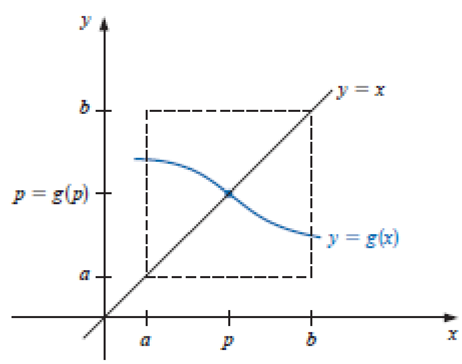
Theorem 2. 2.

(a) *Let $g \in C[a,b]$ and $g(x) \in [a,b]$ for all x, then g has a fixed point in $[a,b]$.*

(b) *If, in addition, a positive constant $k < 1$ exists with*

$$|g'(x)| \le k < 1, \;\; \textit{for all } x \in (a,b),$$

then g has a unique fixed point on $[a,b]$.



Example 2. 4.

1. *Show that*

$$g(x) = \frac{x^2 - 1}{3}$$

has a fixed point on $[-1,1]$

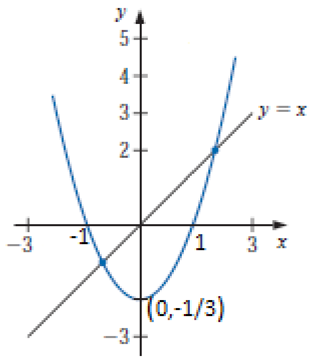
2. *Show that Theorem 2.2 does not ensure unique fixed point of*

$$g(x) = 3^{-x}$$

on the interval $[0,1]$, even though a unique fixed point on this interval does exist.

Solutions:

1 (a) *Clearly, $g(x)$ is differentiable and continuous on $[-1, 1]$ as the graph shows*



Also, $g(x) \in [-1, 1]$. For example,

$$g(-1) = 0 = g(1) \in [-1, 1],\ g\left(\frac{1}{2}\right) = -0.25 \in [-1, 1]$$

$\therefore$ *g has a fixed point.*

(b) $|g'(x)| = \left|\frac{2x}{3}\right| \leq \frac{2}{3} < 1$ *for all $x \in (-1, 1)$*

$\therefore$ *g has a unique fixed point.*

This value can be determined explicitly.

$$g(p) = p \Rightarrow p = \frac{p^2 - 1}{3}$$

$\Rightarrow p^2 - 3p - 1 = 0$

$\Rightarrow p = \frac{3-\sqrt{13}}{2}$ or $p = \frac{3+\sqrt{13}}{2}$.

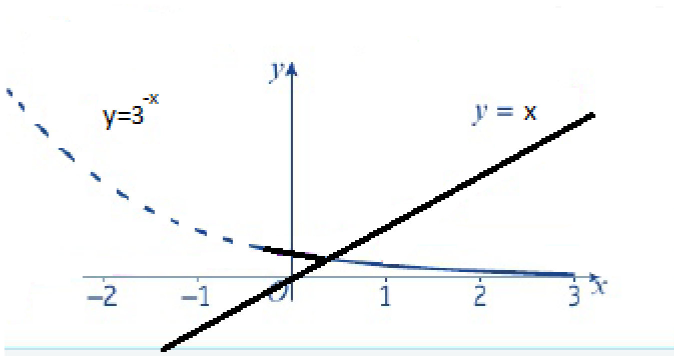
Clearly, only $p = \frac{3-\sqrt{13}}{2}$ lies in $[-1, 1]$.

NOTE: $p = \frac{3+\sqrt{13}}{2}$ *is a fixed point of g on the interval $[3, 4]$, but*

- $g(4) = 5 \notin [3, 4]$
- $|g'(4)| = \left|\frac{2(4)}{3}\right| = \frac{8}{3} > 1$

Therefore, the conditions of Theorem 2.2 are satisfied to guarantee a unique fixed point but are not necessary.

2 (a) *Clearly,* $g(x)$ *is differentiable on* $[0,1]$ *and* $g \in [0,1]$.



Therefore, g *has a fixed point.*

(b) $g'(x) = -3^{-x}\ln 3 \Rightarrow |g'(0)| = |-\ln 3| = 1.098612289 > 1.$

Since $|g'(x)| \not< 1$ *on* $(0,1)$*, Theorem 2.2 cannot be used to determine uniqueness.*

Example 2. 4 part (2) shows that we cannot always determine the fixed point explicitly. We can, however, choose an initial approximation p_0 in the given interval and generate the sequence $\{P_n\}_{n=0}^{\infty}$ using

$$p_n = g(p_{n-1}), \quad n \geq 1.$$

If the sufficient conditions for existence and uniqueness of $g(x)$ on $[a,b]$ are met, then

$$p = \lim_{n\to\infty} p_n$$

is the solution of (2.1).

Example 2. 5.

Find the unique root of the equation

$$x^3 + 4x^2 - 10 = 0$$

in the interval $[1,2]$ *with* $p_0 = 1.5$ *as the initial approximation.*

Solution:

$f(x) = x^3 + 4x^2 - 10 = 0$ *can be written as* $x = g(x)$ *in many ways:*

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$

(b) $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{\frac{1}{2}}$

(c) $x = g_3(x) = \frac{1}{2}(10 - x^3)^{\frac{1}{2}}$

(d) $x = g_4(x) = \left(\frac{10}{4+x}\right)^{\frac{1}{2}}$

(e) $x = g_5(x) = x - \frac{x^3+4x-10}{3x^2+8x}$

The table below shows the results for all $g_i(x),\ i = 1, 2, 3, 4, 5$:

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768	1.348399725	1.373333333
2	-469.7	2.9969	1.402540804	1.367376372	1.365262015
3	1.08×10^8	$-(8.65)^{\frac{1}{2}}$	1.345458374	1.364957015	1.365230014
4	6.732		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.3652300013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

This method converges faster than Bisection method which required more iterations to get the same accuracy. However, sequences for $g_1(x)$ and $g_2(x)$ diverge. Theorem 2.2 can be used to determine the correct choice of $g(x)$. The following Corollary also helps to determine how fast the sequence will converge:

Corollary 2. 1.

If g satisfies hypotheses for Theorem 2.2, then the bound for the error involved in using p_n to approximate p is given by

$$|p - p_n| \leq \frac{k^n}{1-k}|p_1 - p_0|, \text{ for all } n \geq 1.$$

<u>NOTE</u>: The convergence of $\{p_n\}_{n=0}^{\infty}$ to p depends on the factor k^n. The smaller the value of k, the faster the convergence implying that values of k closer to 1 would converge slower.

Example 2. 6.

Apply Theorem 2.2 and Corollary 2.1 to example 2.5 to determine which $g_i(x)$'s converge and compare how fast they converge.

<u>Solutions</u>:

(a) $g_1(x) = x - x^3 - 4x^2 + 10$

- $g_1(x)$ *is differentiable on* $[1,2]$.
- $g_1(x) \notin [1,2]$, *since* $g_1(1) = 6$ *and* $g_1(2) = -12$
- $g_1'(x) = 1 - 3x^2 - 8x$

$\Rightarrow\ |g_1'| > 1, \forall x \in (1,2)$

$\therefore$ *There is no reason to expect the sequence to converge.*

(b) $g_2(x) = \left(\frac{10}{x} - 4x\right)^{\frac{1}{2}}$

- $g_2(x) \notin [1,2]$ *because* $g_2(1) = 2.4495$ *and* $g_2(2)$ *gives a complex number*

$\therefore$ *There is no reason to expect the sequence to converge.*

(c) $g_3(x) = \frac{1}{2}\left(10 - x^3\right)^{\frac{1}{2}}$

- $g_3(x)$ *is differentiable on* $[1,2]$.
- $g_3(x) \in [1,2]$
- $g_3'(x) = -\frac{3}{4}x^2\left(10 - x^2\right)^{-\frac{1}{2}}$

$\Rightarrow\ |g_3'(2)| \approx 2.12.$

But it can be shown that $g_3(x)$ *satisfies conditions of Theorem 2.2 in the interval* $[1,1.5]$ *and that*

$|g_3'(1.5) \approx 0.66 < 1$

Therefore, the sequence will converge but will converge slowly because $k = 0.66$ is closer to 1.

(d) *Check that other conditions are satisfied and that* $|g_4(x)| = \left|\frac{1}{\sqrt{10}(4+x)^{\frac{3}{2}}}\right| \leq \frac{5}{\sqrt{10}(5)^{\frac{3}{2}}} < 0.15, \forall x \in [1, 2].$

Therefore, we expect the sequence to converge faster than $g_3(x)$ since $k = 0.15$.

2.3 Newton-Raphson Method

One way of deriving the Newton-Raphson method is by using the Taylor polynomial. Suppose that $f \in C^2[a, b]$. Let $p_0 \in [a, b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is "small". Then, expanding f about p_0 and evaluating it at $x = p$ yields

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2!}f''(\xi(p)),$$

where $p_0 < \xi(p) < p$. Since $f(p) = 0$, we have that

$$0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2!}f''(\xi(p)).$$

Since we are assuming that $|p - p_0|$ is small, the term involving $(p - p_0)^2$ is much smaller, so that

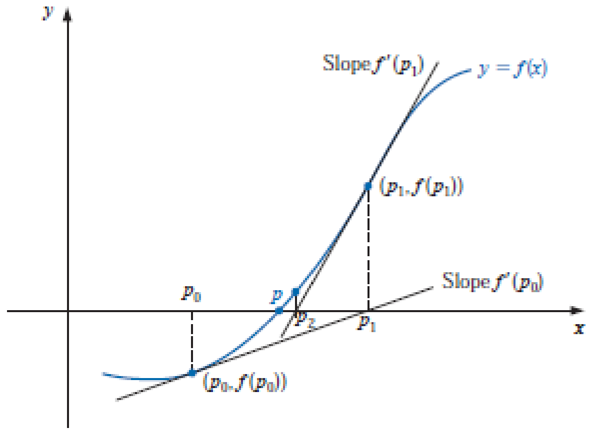
$$0 \approx f(p_0) + (p - p_0)f'(p_0)$$

and making p the subject gives

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

This generates a sequence $\{p_n\}_{n=0}^{\infty}$ given by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \ n \geq 1$$



NOTE: 1. Newton-Raphson's method cannot be applied if $f'(p) = 0$ for some n. Thus, we require that f' be bounded bounded away from zero near p.

2. Newton-Raphson's method can be considered as the fixed-point iteration method with $p_n = g(p_{n-1})$, for which

$$g(p_{n-1}) = p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \ n \geq 1$$

(See Example 2.2.3 (e))

3. The method always converges provided that a sufficiently accurate initial approximation is chosen.

Example 2. 7.

Use the Newton-Raphson's method to approximate a root of $f(x) = \cos x - x = 0$ *starting with* $p_0 = \frac{\pi}{4}$.

Solution:

$f'(x) = -\sin x - 1$. *Starting with* $p_0 = \frac{\pi}{4} = 0.7853981635$,

$$p_1 = \frac{\pi}{4} - \frac{\cos\left(\frac{\pi}{4}\right) - \frac{\pi}{4}}{-\sin\left(\frac{\pi}{4}\right) - 1} \approx 0.7395361337.$$

The table below shows the other results:

n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

Exercise: Use fixed- point iteration to solve the same problem and compare the rate of convergence.

2.4 Secant Method

This is an extension of Newton-Raphson's method to avoid the use of the derivative of the function.

By definition of the derivative,

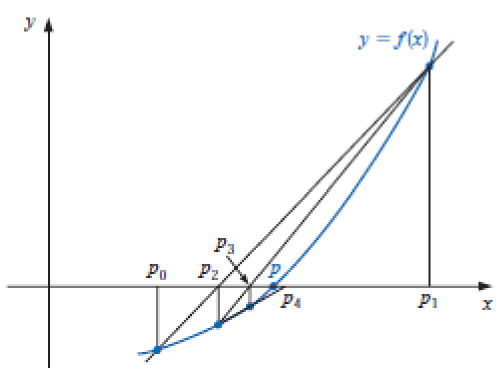
$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}.$$

If p_{n-2} is close to p_{n-1}, then

$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}} = \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}.$$

Using this approximation for $f'(p_{n-1})$ in Newton-Raphson's method, we get

$$p_n = p_{n-1} - \frac{f(p_{n-1})\,(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$



NOTE: Secant method requires two approximations p_0 and p_1. Then, p_2 is the $x-$intercept of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$, i.e.

$$\frac{0 - f(p_0)}{x - p_0} = \frac{f(p_1) - f(p_0)}{p_1 - p_0}$$

$$\Rightarrow x\,(f(p_1) - f(p_0)) - p_0\,(f(p_1) - f(p_0)) = -f(p_0)(p_1 - p_0)$$

$$\Rightarrow x = \frac{p_0\,[f(p_1) - f(p_0)] - f(p_0)(p_1 - p_0)}{f(p_1) - f(p_0)} = p_0 - \frac{f(p_0)\,(p_1 - p_0)}{f(p_1) - f(p_0)}.$$

Similarly, p_3 is the $x-$intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$, and so on.

Example 2. 8.

Find a zero of $f(x) = \cos x - x = 0$ *given the initial approximations* $p_0 = 0.5$ *and* $p_1 = \frac{\pi}{4}$.

Solution:

Using

$$p_n = p_{n-1} - \frac{f(p_{n-1})\,(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})},$$

we have that

$$\begin{aligned} p_2 &= p_1 - \frac{f(p_1)\,(p_1 - p_0)}{f(p_1) - f(p_0)} \\ &= \frac{\pi}{4} - \frac{\left(\cos(\frac{\pi}{4}) - \frac{\pi}{4}\right)\left(\frac{\pi}{4} - 0.5\right)}{\left(\cos(\frac{\pi}{4}) - \frac{\pi}{4}\right) - (\cos(0.5) - 0.5)} \approx 0.73638413880 \end{aligned}$$

The table below shows the other results:

n	p_n
0	0.5
1	0.7833981635
2	0.7336841388
3	0.7390581392
4	0.7390851493
5	0.7390851332

NOTE: Generally, secant method converges slightly slower than Newton-Raphson's method.

2.5 Error Analysis for Iterative Methods

We have seen that Newton-Raphson's method converges faster than other functional iteration techniques we have considered. We now consider the order of convergence of functional iteration techniques in a general sense, and deduce why the Newton-Raphson's method converges faster than others.

Definition 2. 2. (Order of convergence)

Suppose that $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda,$$

then $\{p_n\}_{n=0}^{\infty}$ converges to p of order α, with asymptotic error constant λ.

Definition 2.2 implies that an iterative technique of the form $p_n = g(p_{n-1})$ is of order α if $\{p_n\}_{n=0}^{\infty}$ converges to $p = g(p)$. Two cases of order are given special names:

1. If $\alpha = 1$ (and $\lambda < 1$), the sequence is linearly convergent.

2. If $\alpha = 2$, the sequence is quadratically convergent.

The next example uses linear and quadratic convergence to show that higher-order of convergence converges more rapidly than lower-order of convergence.

Example 2. 9.

Suppose that $\{p_n\}_{n=0}^{\infty}$ *is linearly convergent to* 0 *with*

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = 0.75$$

and that $\{\overline{p}_n\}_{n=0}^{\infty}$ *is quadratically convergent to* 0 *with*

$$\lim_{n\to\infty} \frac{|\overline{p}_{n+1} - \overline{p}|}{|\overline{p}_n - \overline{p}|^2} = \lim_{n\to\infty} \frac{|\overline{p}_{n+1}|}{|\overline{p}_n|^2} = 0.75$$

For simplicity, suppose also that $\frac{|p_{n+1}|}{|p_n|} \approx 0.75$ *and* $\frac{|\overline{p}_{n+1}|}{|\overline{p}_n|^2} \approx 0.75$. *This means that*

$$\begin{aligned}
|p_n - p| = |p_n| &\approx (0.75)|p_{n-1}| \approx 0.75(0.75|p_{n-2}|) = (0.75)^2|p_{n-2}| \\
&\approx (0.75)^2(0.75|p_{n-3}|) = (0.75)^3|p_{n-3}| \\
&\vdots \\
&\approx (0.75)^n|p_0|
\end{aligned}$$

and

$$\begin{aligned}
|\overline{p}_n - \overline{p}| = |\overline{p}_n| &\approx (0.75)\,|\overline{p}_{n-1}|^2 \approx 0.75\left[(0.75)\,|\overline{p}_{n-2}|^2\right]^2 = (0.75)^3|\overline{p}_{n-2}|^4 \\
&\approx (0.75)^3\left[0.75|\overline{p}_{n-3}|^2\right]^4 = (0.75)^7|\overline{p}_{n-3}|^8 \\
&\vdots
\end{aligned}$$

$$\approx (0.75)^{2^n-1}|\overline{p}_0|^{2^n}.$$

Now, assume that $|p_0| = |\overline{p}_0| = 0.5$ *and that the error does not exceed* 10^{-8}*. Then, the above estimates imply that*

$$|p_n| \approx (0.75)^n(0.5) \leq 10^{-8}$$
$$\Rightarrow n \geq \frac{(\log 2) - 8}{\log(0.75)} \approx 62.$$

and

$$|\overline{p}_n| \approx (0.75)^{2^n-1}(0.5)^{2^n} \leq 10^{-8}$$

$$\begin{aligned}
&\Rightarrow (0.75)^{-1}(0.75)^{2^n})(0.5)^{2^n} \leq 10^{-8}\\
&\Rightarrow (0.75)^{-1}(0.75 \times 0.5)^{2^n} \leq 10^{-8}\\
&\Rightarrow (0.75)^{-1}(0.375)^{2^n} \leq 10^{-8}\\
&\Rightarrow \log(0.75)^{-1} + \log(0.375)^{2^n} \leq \log 10^{-8}\\
&\Rightarrow 2^n \log(0.375) \leq -8 - (-1)\log(0.75)\\
&\Rightarrow 2^n \geq \frac{-8 + \log(0.75)}{\log(0.375)} \approx 19\\
&\Rightarrow n \geq 5
\end{aligned}$$

Therefore, quadratic convergence requiring only 5 *iterations is vastly superior to the linear convergence requiring* 62 *iterations.*

Theorem 2. 3.

Let $g \in C[a,b]$ *be such that* $g(x) \in [a,b]$*, for all* $x \in [a,b]$*. Suppose, in addition, that* g' *is continuous on* (a,b) *and a positive constant* $k < 1$ *exists with*

$$g'(x) \leq k, \text{ for all } x \in (a,b).$$

If $g'(p) \neq 0$*, then for any number* $p_0 \neq p$ *in* $[a,b]$ *the sequence*

$$p_n = g(p_{n-1}),\ n \geq 1$$

converges only *linearly* *to the fixed point* $p \in [a,b]$*.*

Theorem 2.3 shows that the conditions for existence and uniqueness of the fixed point p imply that the sequence $\{p_n\}_{n=0}^{\infty}$ will converge slowly. We can only ensure rapid convergence if the order is higher, say if it is quadratic, and this can only happen if $g'(p) = 0$ as the next theorem shows:

Theorem 2. 4.

Let p be a solution of the equation $x = g(x)$. Suppose that $g'(p) = 0$ and g'' is continuous with $g''(x) < M$ on an open interval I containing p. Then, there exists $\delta > 0$ such that, for $p_0 \in [p - \delta, p + \delta]$, the sequence defined by

$$p_n = g(p_{n-1}), \ n \geq 1$$

converges at least quadratically to p.

Moreover, for sufficiently large value of n,

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$

Theorem 2.3 and 2.4 show that the functional iteration technique will converge quadratically if $g(p) = p$ and $g'(p) = p$. To achieve that, remember that functional iteration method for solving equation (2.1) will imply writing $g(x)$ in the form

$$g(x) = x - \phi(x)f(x),$$

where ϕ is a differentiable function. Then,

$$g'(x) = 1 - \phi'(x)f(x) - \phi(x)f'(x).$$

Since $f(p) = 0$, we have that

$$g'(p) = 1 - \phi(p)f'(p).$$

Thus, $g'(p) = 0$ if and only if

$$\phi(p) = \frac{1}{f'(p)}.$$

If we let $\phi(x) = \frac{1}{f'(x)}$, then we will ensure that $\phi(p) = \frac{1}{f'(p)}$ and produce the quadratically convergent procedure

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f(p_{n-1})},$$

which is simply the Newton-Raphson's method. Hence, if $f(p) = 0$ and $f'(p) \neq 0$, then for starting values sufficiently close to p, Newton-Raphson's method will converge at least quadratically.

Definition 2. 3.

A solution p of equation (2.1) is a zero of multiplicity m of f if for $x \neq p$, we can write $f(x) = (x-p)^m q(x)$, where $\lim_{x \to p} q(x) \neq 0$.

Theorem 2. 5.

The function $f \in C^1[a,b]$ has a simple zero at p in (a,b) if and only if $f(p) = 0$ and $f'(p) \neq 0$.

Theorem 2.5 implies that Newton-Raphson's method will only converge quadratically if f has a simple zero.

Theorem 2. 6.

The function $f \in C^m[a,b]$ has a zero of multiplicity m at p in (a,b) if and only if

$$0 = f(p) = f'(p) = f''(p) = \cdots = f^{(m-1)}(p),$$

but $f^{(m)}(p) \neq 0$.

Newton-Raphson's method may not converge quadratically if f has a zero of multiplicity m, as the next example shows:

Example 2. 10.

Show that $f(x) = e^x - x - 1$ has a zero of multiplicity 2 at $x = 0$ but Newton-Raphson's method with $p_0 = 1$ does not converge quadratically to this zero.

<u>Solution</u>:

We expect

$0 = f(0) = f'(0)$, *but* $f''(0) \neq 0$.

$f(x) = e^x - x - 1 \;\Rightarrow f'(x) = e^x - 1$ *and* $f''(x) = e^x$

$\Rightarrow f(0) = e^0 - 0 - 1 = 0,\; f'(0) = e^0 - 1 = 0$ *and* $f''(0) = e^0 = 1 \neq 0$.

Therefore, f has a zero of multiplicity 2 at $x = 0$.

With $p_0 = 1$,

$$p_1 = 1 - \frac{e-2}{e-1} \approx 0.58198$$

$$p_2 = 0.58198 - \frac{e^{0.58198} - 0.58198 - 1}{e^{0.58198} - 1} \approx 0.31906$$

The table below shows other values:

n	p_n
0	1.0
1	0.5819
2	0.31906
3	0.16800
4	0.08635
5	0.04380
6	0.02206
7	0.01107
8	0.005545
9	2.7750×10^{-3}
10	1.3881×10^{-3}
11	6.9411×10^{-4}
12	3.4703×10^{-4}
13	1.7416×10^{-4}
14	8.8041×10^{-5}
15	4.2610×10^{-5}
16	1.9142×10^{-5}

Clearly, the sequence $\{p_n\}_{n=0}^{\infty}$ does not converge quadratically (it converges slowly).

One way of accelerating convergence of Newton-Raphson's method when the zero of f is not simple is to define a function μ as $\mu(x) = \dfrac{f(x)}{f'(x)}$. If p is a zero of multiplicity m, then $f(x) = (x-p)^m q(x)$ and

$$\mu(x) = \frac{(x-p)^m q(x)}{m(x-p)^{m-1}q(x) + (x-p)^m q'(x)} = (x-p)T(x),$$

where

$$T(x) = \frac{q(x)}{m\,q(x) + (x-p)q'(x)}.$$

If $T(p) \neq 0$, then we conclude that $\mu(x)$ has a simple zero at p. Since $q(x) \neq 0$, we have that

$$T(p) = \frac{q(p)}{m\,q(p) + (p - p)q'(p)} = \frac{1}{m}.$$

Thus, μ has a simple zero at p and Newton-Raphson's method can be applied giving

$$\begin{aligned} g(x) &= x - \frac{\mu(x)}{\mu'(x)} \\ &= x - \frac{f(x)}{f'(x)} \div \left[\frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}\right] \\ &= x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)} \end{aligned}$$

Example 2. 11.

Repeat Example 2.10 *using the modified Newton-Raphson's method.*

Solution:

With $p_0 = 1$*,*

$$p_1 = p_0 - \frac{f(p_0)f'(p_0)}{[f'(p_0)]^2 - f(p_0)f''(p_0)} = 1 - \frac{(e-2)(e-1)}{(e-1)^2 - (e-2)e} \approx -2.3421061 \times 10^{-1}.$$

More results are shown in the table below:

n	p_n
0	1.0
1	$-2.3421061 \times 10^{-1}$
2	$-8.4582788 \times 10^{-3}$
3	$-1.1889524 \times 10^{-5}$
4	$-6.8638230 \times 10^{-6}$
5	$-2.8085217 \times 10^{-7}$

THE END!