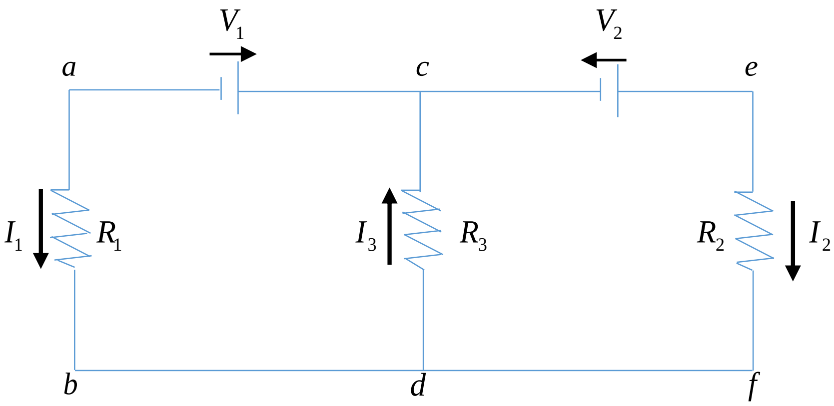
## 3. SYSTEMS OF EQUATIONS

Many engineering and scientific problems can be modelled in terms of systems of simultaneous linear and non-linear equations.

### 3.1. LINEAR SYSTEMS OF EQUATIONS

Consider the electric circuit below with two separate networks, loop $acdb$ and loop $aefb$:




Applying Kirchhoff's first law at junction $c$ yields the equation

$$I_1 + I_2 - I_3 = 0,$$

applying Kirchhoff's second law at network loop $acdb$ yields the equation

$$V_1 = R_1 I_1 + R_3 I_3$$

and applying Kirchhoff's second law at network loop $aefb$ yields the equation

$$V_1 - V_2 = R_1 I_1 - R_2 I_2.$$

Assuming that $R_1 = 2,\ R_2 = 4,\ R_3 = 5,\ V_1 = 6$ and $V_2 = 2,$ we get the following system of three linear simultaneous equations:

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ 2I_1 \quad + 5I_3 &= 6 \\ 2I_1 - 4I_2 \quad &= 4. \end{aligned}$$

The solution to these three equations produces the current flow in the network.

In general,

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n
\end{aligned} \qquad \textbf{(3.1)}$$

We will use two methods to solve linear systems of equations. These are direct and iterative methods.

### 3.1.1. <u>Direct Methods</u>

Notice that system (3.1) can be written as

$$AX = B, \qquad \textbf{(3.2)}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then we can write the augmented matrix for system (3.1) as

$$A \,|\, B = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right)$$

and apply row operations to it to find the solution. This procedure is called Gaussian elimination with backward substitution. The three row operations we use are:

1) Multiply (or divide) the sides of an equation by a non-zero number
2) Add a multiple of one equation to another equation in the system
3) Interchange two equations in the system.

When row operations are applied, the augmented matrix is left in the form with the following characteristics:

(a) The first no-zero number (starting from the left) in any row is called the pivot

(b) Any row (if any) consisting of zero entries appears at the bottom of the matrix

(c) If two successive rows do not consist of entirely zeros, then the pivot in the lower row occurs farther to the right than the first pivot in the highest row.

***Example 3.1.1***

Represent the linear system

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= -8 \\ 2x_1 - 2x_2 + 3x_3 - 3x_4 &= -20 \\ x_1 + x_2 + x_3 &= -2 \\ x_1 - x_2 + 4x_3 + 3x_4 &= 4 \end{aligned}$$

as an augmented matrix and use Gaussian elimination with backward substitution to find its solution.

Solution:

The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 2 & -2 & 3 & -3 & -20 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 4 & 3 & 4 \end{array}\right)$$

Applying row operations, we get

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 2 & -2 & 3 & -3 & -20 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 4 & 3 & 4 \end{array}\right) \xrightarrow[\substack{r_3 \to r_3 - r_1 \\ r_4 \to r_4 - r_1}]{r_2 \to r_2 - 2r_1} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right) \xrightarrow[r_4 \to \frac{1}{2} r_4]{r_2 \leftrightarrow r_3} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 1 & 2 & 6 \end{array}\right)$$

$$\xrightarrow{r_4 \to r_3 + r_4} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$$\begin{aligned} \therefore\ x_4 = 2 \ &\Rightarrow -x_3 - x_4 = -4 \\ &\Rightarrow x_3 = 2 \\ 2x_2 - x_3 + x_4 = 6 \quad &\Rightarrow x_2 = 3 \\ x_1 - x_2 + 2x_3 - x_4 = -8 \quad &\Rightarrow x_1 = -7 \end{aligned}$$

$\therefore\ X = (x_1, x_2, x_3, x_4)^t = (-7, 3, 2, 2)^t$.

△

*Pivoting Strategies*

Note that applying Gaussian elimination method to the augmented matrix involves multiplying a number

$$m_{jk} = \frac{a_{jk}^{(k)}}{a_{kk}^{(k)}}$$

To one row to "reduce" it. For example, $m_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} = \frac{a_{21}}{a_{11}}$ so that $r_2 \to r_2 - (\frac{a_{21}}{a_{11}})r_1$. If $a_{kk}^{(k)}$ is small in magnitude, any round-off error in the numerator will be increased. The next example illustrates how a wrong choice of the pivot can affect the final answer.

***Example 3.1.2***

The exact solution to the system

$$\begin{aligned} 0.003000x_1 + 59.14x_2 &= 59.17 \\ 5.291x_1 - 6.130x_2 &= 46.78 \end{aligned}$$

using four-digit rounding-off is $x_1 = 10.00$ and $x_2 = 1.000$. Apply Gaussian elimination method to find the solution.

Solution:

$$m_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} = \frac{5.291}{0.003000} = 1764.$$

$$\left(\begin{array}{cc|c} 0.003000 & 59.14 & 59.17 \\ 5.291 & -6.130 & 46.78 \end{array}\right) \xrightarrow{r_2 \to r_2 - m_{21}r_1} \left(\begin{array}{cc|c} 0.003000 & 59.14 & 59.17 \\ 0 & -104300 & -104400 \end{array}\right)$$

$$\therefore x_2 \approx \frac{-104300}{-104400} \approx 1.001$$

$$\Rightarrow x_1 = \frac{59.17 - 59.14(1.001)}{0.003000} = -10.00.$$

Comparing with the exact solution, we notice serious errors. △

Example 3.1.2 shows that serious errors may arise when pivoting element $a_{kk}^{(k)}$ is small relative to the entries $a_{ij}^{(k)}$, for $k \le i \le n$ and $k \le j \le n$. To avoid this problem, we use scaled partial pivoting. The first step in this procedure is to define a scale factor $s_i$ for each row as

$$s_i = \max_{1 \le j \le n} |a_{ij}|.$$

Assuming that $s_i \neq 0,$ the appropriate row interchange to place zeros in the first column is determined by choosing the least integer $p$ with

$$\frac{|a_{p1}|}{s_p} = \underset{1\le j\le n}{Max}\left(\frac{|a_{k1}|}{s_k}\right)$$

and then interchange row 1 and row $p.$ In a similar manner, before eliminating the variable $x_i,$ we select the smallest integer $p \geq i$ with

$$\frac{|a_{pi}|}{s_p} = \underset{1\le j\le n}{Max}\left(\frac{|a_{ki}|}{s_k}\right)$$

and perform row interchange. The scale factors $s_1, s_{2,}, ..., s_n$ are computed only once at the start of the procedure. They are row dependent, so they must be interchanged when row interchanges are performed.

***<u>Example 3.1.3</u>***

Apply Gaussian elimination with scaled partial pivoting to find the solution to the system

$$\begin{aligned}
2.11x_1 - 4.21x_2 + 0.921x_3 &= 2.01 \\
4.01x_1 + 10.2x_2 - 1.12x_3 &= -3.09 \\
1.09x_1 + 0.987x_2 + 0.832x_3 &= 4.21
\end{aligned}$$

using three-digit rounding.

<u>Solution:</u>

$$\begin{aligned}
s_1 &= Max\{|\,2.11\,|, |-4.21|, |\,0.921\,|\} = 4.21 \\
s_2 &= Max\{|\,4.01\,|, |\,10.2\,|, |-1.12\,|\} = 10.2 \\
s_3 &= Max\{|\,1.09\,|, |\,0.987\,|, |\,0.832\,|\} = 1.09
\end{aligned}$$

$$\therefore \frac{|a_{11}|}{s_1} = \frac{2.11}{4.21} \approx 0.501$$

$$\frac{|a_{21}|}{s_2} = \frac{4.01}{10.2} \approx 0.393$$

$$\frac{|a_{31}|}{s_3} = 1.09 = 1.$$

Since $Max\left\{\frac{|a_{11}|}{s_1}, \frac{|a_{21}|}{s_2}, \frac{|a_{31}|}{s_3}\right\} = Max\{0.501,\ 0.393, 1\} = 1 = \frac{|a_{31}|}{s_3}$, we interchange row 1 and row 3. Thus, we have the augmented matrix

$$\left(\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 4.01 & 10.2 & -1.12 & -3.09 \\ 2.11 & -4.21 & 0.921 & 2.01 \end{array}\right) \quad m_{21} = \frac{4.01}{1.09} \approx 3.68, \quad m_{31} = \frac{2.11}{1.09} \approx 1.94$$

$$\left(\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 4.01 & 10.2 & -1.12 & -3.09 \\ 2.11 & -4.21 & 0.921 & 2.01 \end{array}\right) \xrightarrow[r_3 \to r_3 - m_{31}r_1]{r_2 \to r_2 - m_{21}r_1} \left(\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & 6.57 & -4.18 & -18.6 \\ 0 & -6.21 & -0.693 & -6.16 \end{array}\right)$$

$$\frac{|a_{22}|}{s_2} = \frac{6.57}{10.2} \approx 0.644$$
$$\frac{|a_{32}|}{s_1} = \frac{6.12}{4.21} \approx 1.45$$

Since $Max\left\{\frac{|a_{22}|}{s_2}, \frac{|a_{32}|}{s_1}\right\} = Max\{0.644,\ 1.45\} = 1.45 = \frac{|a_{32}|}{s_1}$, we interchange row 2 and row 3.

$$\left(\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & -6.12 & -0.693 & -6.16 \\ 0 & 6.57 & -4.18 & -18.6 \end{array}\right) \quad m_{32} = \frac{6.57}{-6.12} \approx -1.07$$

$$\left(\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & -6.12 & -0.693 & -6.16 \\ 0 & 6.57 & -4.18 & -18.6 \end{array}\right) \xrightarrow{r_3 \to r_3 - m_{32}r_2} \left(\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & -6.12 & -0.693 & -6.16 \\ 0 & 0 & -4.92 & -25.2 \end{array}\right)$$

$$\begin{aligned} \therefore -4.92x_3 &= -25.2 & &\Rightarrow x_3 \approx 5.12 \\ -6.12x_2 - 0.693x_3 &= -6.16 & &\Rightarrow x_2 \approx 0.426 \\ 1.09x_1 + 0.987x_2 + 0.832x_3 &= 4.21 & &\Rightarrow x_1 \approx -0.431 \end{aligned}$$

$$\therefore X = (x_1,\ x_2,\ x_3)^t = (-0.431,\ 0.426,\ 5.12)^t$$

△

<u>Exercise:</u> Use scaled partial pivoting to solve the system in Example 3.1.2.

### 3.1.2. Iterative Methods

Iterative or indirect methods use trial-and-error procedure to solve large systems of linear equations with high percentage of zero entries. An iterative method to solve the $(n \times n)$ linear system (3.2) starts with an initial approximation $X^{(0)}$ to the solution $X$ and generate a sequence of vectors $\left\{X^{(k)}\right\}_{k=0}^{\infty}$ that converges to $X$. We will discuss two of the common methods, the Jacobi and Gauss-Siedel iteration techniques. To determine the difference between the approximations and the exact solution, we will define the norm of a vector which will help us to determine the distance between $n$ – dimensional column vectors.

*Definition 3.1.1*

A vector norm on $\mathbb{R}^n$ is a function $\|.\|$, from $\mathbb{R}^n$ to $\mathbb{R}$ with the following properties:

(i) $\|X\| \geq 0 \ \ \forall X \in \mathbb{R}^n$

(ii) $\|X\| = 0 \Leftrightarrow X = 0$

(iii) $\|\alpha X\| = |\alpha| \, \|X\|, \ \forall \alpha \in \mathbb{R}$ and $X \in \mathbb{R}^n$

(iv) $\|X + Y\| \leq \|X\| + \|Y\|, \ \forall X, Y \in \mathbb{R}^n$

◇

*Definition 3.1.2*

The $l_2$ and $l_\infty$ norms for the vector $X = \left(x_1, x_2, ..., x_n\right)^t$ are defined by

$$\|X\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \quad \text{and} \quad \|X\|_\infty = \underset{1 \leq i \leq n}{Max} |x_i|.$$

◇

***Example 3.1.4***

The $l_2$ and $l_\infty$ norms for the vector $X = \left(-1, 1, 5, -20\right)^t$ are

$$\|X\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = \sqrt{(-1)^2 + 1^2 + 5^2 + (-20)^2} = \sqrt{427}$$

$$\|X\|_\infty = \underset{1 \leq i \leq n}{Max} |x_i| = Max\left\{|-1|, |1|, |5|, |-20|\right\} = 20$$

△

*Definition 3.1.3*

If $X = (x_1, x_2, ..., x_n)^t$ and $Y = (y_1, y_2, ..., y_n)^t$ are vectors in $\mathbb{R}^n$, then $l_2$ and $l_\infty$ distances between $X$ and $Y$ are defined by

$$\|X - Y\|_2 = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} \quad \text{and} \quad \|X - Y\|_\infty = \underset{1\le i\le n}{Max}|x_i - y_i|.$$

◇

*Jacobi's Method*

The Jacobi's iterative method is obtained by solving the $i^{th}$ equation in the system (3.2) for $x_i$ to obtain

$$x_i = \sum_{\substack{j=1 \\ j\neq i}}^{n}\left(\frac{-a_{ij}x_j}{a_{ii}}\right) + \frac{b_i}{a_{ii}}, \quad \text{for } i = 1, 2, ..., n$$

provided that $a_{ii} \neq 0$. For each $k \geq 1$, generate the components $x_i^{(k-1)}$ of $X^{(k)}$ from the components of $X^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1 \\ j\neq i}}^{n}\left(-a_{ij}x_j^{(k-1)}\right) + b_i\right], \quad \text{for } i = 1, 2, ..., n.$$

***Example 3.1.5***

The following system

$$\begin{aligned} 10x_1 - x_2 + 2x_3 \qquad &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11 \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}$$

has the unique solution $X = (1, 2, -1, 1)^t$. Use Jacobi's iterative method with four-decimal places rounding-off to find approximations $X^{(k)}$ to $X$ with $X^{(0)} = (0, 0, 0, 0)^t$ until

$$\frac{\|X^{(k)} - X^{(k-1)}\|_\infty}{\|X^{(k)}\|_\infty} < 10^{-3}.$$

Solution:

We first check that $a_{ii} \neq 0$ for $i = 1,2,3,4.$ $a_{11} = 10,\ a_{22} = 11,\ a_{33} = 10,\ a_{44} = 8.$ Thus,

$$x_1^{(k)} = \frac{1}{10}(x_2^{(k-1)} - 2x_3^{(k-1)} + 6)$$
$$x_2^{(k)} = \frac{1}{11}(x_1^{(k-1)} + x_3^{(k-1)} - 3x_4^{(k-1)} + 25)$$
$$x_3^{(k)} = \frac{1}{10}(-2x_1^{(k-1)} + x_2^{(k-1)} + x_4^{(k-1)} - 11)$$
$$x_4^{(k)} = \frac{1}{8}(-3x_2^{(k-1)} + x_3^{(k-1)} + 15).$$

Starting with $X^{(0)} = (0,0,0,0)^t$ , we obtain

$$x_1^{(1)} = \frac{1}{10}(x_2^{(0)} - 2x_3^{(0)} + 6) = \frac{3}{5} = 0.6000$$
$$x_2^{(1)} = \frac{1}{11}(x_1^{(0)} + x_3^{(0)} - 3x_4^{(0)} + 25) = \frac{25}{11} = 2.2727$$
$$x_3^{(1)} = \frac{1}{10}(-2x_1^{(0)} + x_2^{(0)} + x_4^{(0)} - 11) = \frac{-11}{10} = -1.1000$$
$$x_4^{(1)} = \frac{1}{8}(-3x_2^{(0)} + x_3^{(0)} + 15) = \frac{15}{8} = 1.8750.$$

Thus, $X^{(1)} = (0.6000, 2.2727, -1.1000, 1.8750)^t$

$$\Rightarrow \frac{\left\| X^{(1)} - X^{(0)} \right\|_\infty}{\left\| X^{(1)} \right\|_\infty} = \frac{Max\{|0.6000|,|2.2727|,|-1.1000|,|1.8750|\}}{Max\{|0.6000|,|2.2727|,|-1.1000|,|1.8750|\}} = 1 > 10^{-3}.$$

Using $X^{(1)}$, we get

$$x_1^{(2)} = \frac{1}{10}(x_2^{(1)} - 2x_3^{(1)} + 6) = \frac{1}{10}(2.2727 - 2(-1.1000) + 6) = 1.0473$$
$$x_2^{(2)} = \frac{1}{11}(x_1^{(1)} + x_3^{(1)} - 3x_4^{(1)} + 25) = \frac{1}{11}(0.6000 - 1.1000 - 3(1.8750) + 25) = 1.7159$$
$$x_3^{(2)} = \frac{1}{10}(-2x_1^{(1)} + x_2^{(1)} + x_4^{(1)} - 11) = \frac{1}{10}(-2(0.6000) + 2.2727 + 1.8750) = -0.8052$$
$$x_4^{(2)} = \frac{1}{8}(-3x_2^{(1)} + x_3^{(1)} + 15) = \frac{1}{8}(-3(2.2727) - 1.1000 + 15) = 0.8852.$$

Thus, $X^{(2)} = (1.0473, 1.7159, -0.8052, 0.8852)^t$

$$\therefore \frac{\left\| X^{(2)} - X^{(1)} \right\|_\infty}{\left\| X^{(2)} \right\|_\infty} = \frac{Max\{|1.0473 - 0.6000|,\ |1.7159 - 2.2727|,\ |-0.8052 + 1.1000|,\ |0.8852 - 1.8750|\}}{Max\{|1.0473|,|1.7159|,|-0.8052|,|0.8852|\}}$$
$$= \frac{0.9898}{1.7159} = 0.5768 > 10^{-3}.$$

Continuing in this manner, we get the following values:

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_1^{(k)}$ | 0 | 0.6000 | 1.0473 | 0.9326 | 1.0152 | 0.9890 | 1.0032 | 0.9981 | 1.0006 | 0.9997 | 1.0001 |
| $x_2^{(k)}$ | 0 | 2.2727 | 1.7159 | 2.0530 | 1.9537 | 2.0114 | 1.9922 | 2.0023 | 1.9987 | 2.0004 | 1.9998 |
| $x_3^{(k)}$ | 0 | −1.1000 | −0.8052 | −1.0493 | −0.9681 | −1.0103 | −0.9945 | −1.0020 | −0.9990 | −1.0004 | −0.9998 |
| $x_4^{(k)}$ | 0 | 1.8750 | 0.8852 | 1.1309 | 0.9739 | 1.0214 | 0.9944 | 1.0036 | 0.9989 | 1.0006 | 0.9998 |

$$\therefore \frac{\left\|X^{(10)} - X^{(9)}\right\|_\infty}{\left\|X^{(10)}\right\|_\infty} = \frac{Max\{|1.0001-0.9997|, |1.9998-2.0004|, |-0.9998+1.0004|, |0.9998-1.0006|\}}{Max\{|1.0001|,|1.9998|,|-0.9998|,|0.9998|\}}$$

$$= \frac{0.0008}{1.9998} = 0.0004 < 10^{-3}.$$

$\therefore$ we end at $k = 10$

△

_Gauss-Seidel Method_

The Gauss-Seidel method is an improvement of the Jacobi's iterative method where we use the first approximation in the first iteration to approximate other values. For example, in Example 3.1.5, we can use the first approximation $x_1^{(1)} = 0.6000$ to approximate $x_2^{(1)}$ and so on. Thus,

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1}\left(a_{ij}x_j^{(k)}\right) - \sum_{j=i+1}^{n}\left(a_{ij}x_j^{(k-1)}\right)\right], \quad \text{for } i = 1, 2, \ldots, n.$$

***Example 3.1.6***

Solve the system in Example 3.1.5 using Gauss-Seidel method.

Solution:

For $X^{(0)} = (0,0,0,0)^t$, we got $x_1^{(1)} = 0.6000.$ We now use this value to approximate $x_2^{(1)}$.

$$x_2^{(1)} = \frac{1}{11}(x_1^{(1)} + x_3^{(0)} - 3x_4^{(0)} + 25) = \frac{1}{11}\left(0.6000 + 0 - 3(0) + 25\right) = 2.3273$$
$$x_3^{(1)} = \frac{1}{10}(-2x_1^{(1)} + x_2^{(1)} + x_4^{(0)} - 11) = \frac{1}{10}\left(-2(0.6000) + 2.3273 + 0 - 11\right) = -0.9873$$
$$x_4^{(1)} = \frac{1}{8}(-3x_2^{(1)} + x_3^{(1)} + 15) = \frac{1}{8}\left(-3(2.3273) - 0.9873 + 15\right) = 0.8789.$$

△

Exercise: Show that five iterations are required to achieve the required stopping criterion.

## 3.2. NON-LINEAR SYSTEMS

We consider a system of non-linear equations of the form

$$\begin{aligned} &f_1(x_1, x_2, ..., x_n) = 0 \\ &f_2(x_1, x_2, ..., x_n) = 0 \\ &\vdots \\ &f_n(x_1, x_2, ..., x_n) = 0. \end{aligned}$$

The number of equations should be equal to the number of variables.

### 3.2.1. Fixed Point Method for Non-linear systems

*Theorem 3.2.1*

Let $D = \left\{ X = (x_1, x_2, ..., x_n)^t : a_i \le x_i \le b_i,\ i = 1, 2, ..., n \right\}$ be some collection of constants $a_i$ and $b_i$. Suppose $G(X) = \left( g_1(x_1, x_2, ..., x_n),\ g_2(x_1, x_2, ..., x_n), ..., g_n(x_1, x_2, ..., x_n) \right)^t$ is a continuous function from $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ with the property that $G(X) \in D$ whenever $X \in D$. Then, $G$ has a fixed point in $D$.

Moreover, suppose that each $g_i$ has continuous partial derivatives and a constant $M < 1$ exists with

$$\left| \frac{\partial \left( g_i(X) \right)}{\partial x_j} \right| \le \frac{M}{n}, \text{ whenever } X \in D$$

for each $j = 1, 2, ..., n.$ Then the sequence $\left\{ X^{(k)} \right\}_{k=0}^{\infty}$ defined by an arbitrarily selected $X^{(0)}$ in $D$ and generated by

$$X^{(k)} = G(X^{(k-1)}), \quad k \ge 1$$

converges to the unique fixed point $p \in D$ and

$$\left\| X^{(k)} - p \right\|_\infty \le \frac{M^k}{1 - M} \left\| X^{(1)} - X^{(0)} \right\|_\infty .$$

□

***Example 3.2.1***

Show that the system

$$3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0$$
$$x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0,$$
$$e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0$$

has a unique solution on $D = \left\{ X = (x_1, x_2, x_3)^t : -1 \le x_i \le 1,\ i = 1,2,3 \right\}$ and iterate starting with $X^{(0)} = (0.1, 0.1, -0.1)^t$ until

$$\frac{\left\| X^{(k)} - X^{(k-1)} \right\|_\infty}{\left\| X^{(k)} \right\|_\infty} < 10^{-5}.$$

Solution:

For each equation, we solve for $x_i$

$$x_1 = \frac{1}{3}\cos x_2 x_3 + \frac{1}{6}$$
$$x_2 = \frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1$$
$$x_3 = -\frac{1}{20}e^{-x_1 x_2} - \frac{10\pi - 3}{60}$$

Then, $G(X) = \left( g_1(X), g_2(X), g_3(X) \right)^t$, where

$$g_1(X) = \frac{1}{3}\cos x_2 x_3 + \frac{1}{6}$$
$$g_2(X) = \frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1$$
$$g_3(X) = -\frac{1}{20}e^{-x_1 x_2} - \frac{10\pi - 3}{60}.$$

If each $g_i \in D,$ then $G \in D.$

$$\left| g_1(X) \right| = \left| \frac{1}{3}\cos x_2 x_3 + \frac{1}{6} \right| \le \frac{1}{2}$$
$$\left| g_2(X) \right| = \left| \frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1 \right| \le \frac{1}{9}\sqrt{1 + \sin 1 + 1.06} - 0.1 < 0.09$$
$$\left| g_3(X) \right| = \left| -\frac{1}{20}e^{-x_1 x_2} - \frac{10\pi - 3}{60} \right| \le \frac{1}{20}e^{-1(-1)} + \frac{10\pi - 3}{60} < 0.61.$$

Thus, $-1 \le g_i \le 1,$ for $i = 1,2,3$ and $G \in D$ implying that $G$ has a fixed point.

Also,

$$\left|\frac{\partial g_1}{\partial x_1}\right| = 0,$$

$$\left|\frac{\partial g_1}{\partial x_2}\right| = \left|-\frac{1}{3}x_3 \sin x_2 x_3\right| \le \frac{1}{3}.1.\sin 1 < 0.281$$

$$\left|\frac{\partial g_1}{\partial x_3}\right| = \left|-\frac{1}{3}x_2 \sin x_2 x_3\right| \le \frac{1}{3}.1.\sin 1 < 0.281$$

$$\left|\frac{\partial g_2}{\partial x_1}\right| = \left|\frac{1}{9}.\frac{x_1}{\sqrt{x_1^{\,2} + \sin x_3 + 1.06}}\right| < \frac{1}{9.\sqrt{1 - \sin 1 + 1.06}} < 0.1006$$

$$\left|\frac{\partial g_2}{\partial x_2}\right| = 0$$

$$\left|\frac{\partial g_2}{\partial x_3}\right| = \left|\frac{\cos x_3}{18.\sqrt{x_1^{\,2} + \sin x_3 + 1.06}}\right| < \frac{1}{18.\sqrt{1.219}} = 0.050$$

$$\left|\frac{\partial g_3}{\partial x_1}\right| = \left|\frac{x_2}{20}e^{-x_1 x_2}\right| \le \frac{1}{20}e^{1} < 0.1359$$

$$\left|\frac{\partial g_3}{\partial x_2}\right| = \left|\frac{x_1}{20}e^{-x_1 x_2}\right| \le \frac{1}{20}e^{1} < 0.1359$$

$$\left|\frac{\partial g_3}{\partial x_3}\right| = 0$$

$$\therefore \left|\frac{\partial g_i}{\partial x_j}\right| \le 0.281, \quad \text{for each } \; i, j = 1, 2, 3.$$

$$\therefore \frac{M}{n} = 0.281 \;\Rightarrow M = 3(0.281) = 0.843 < 1.$$

Therefore, $G$ has a unique fixed point in $D$.

Starting with $X^{(0)} = (0.1, 0.1, -0.1)^t$, we have that

$$x_1^{(1)} = \frac{1}{3}\cos x_2^{(0)} x_3^{(0)} + \frac{1}{6} = \frac{1}{3}\cos\big(0.1 \times (-0.1)\big) + \frac{1}{6} = 0.49998333$$

$$x_2^{(1)} = \frac{1}{9}\sqrt{\left(x_1^{\,2}\right)^{(0)} + \sin x_3^{(0)} + 1.06} - 0.1 = \frac{1}{9}\sqrt{(0.1)^2 - \sin 0.1 + 1.06} - 0.1 = 0.00944115$$

$$x_3^{(1)} = -\frac{1}{20}e^{-x_1^{(0)} x_2^{(0)}} - \frac{10\pi - 3}{60} = -\frac{1}{20}e^{-(0.1 \times 0.1)} - \frac{10\pi - 3}{60} = -0.52310127$$

$$\therefore \frac{\left\|X^{(1)} - X^{(0)}\right\|_\infty}{\left\|X^{(1)}\right\|_\infty} = \frac{0.42310127}{0.52310127} = 0.8088 > 10^{-5}.$$

Other values are given the table below:

| $k$ | $x_1^{(k)}$ | $x_2^{(k)}$ | $x_3^{(k)}$ | $\left\|X^{(k)}-X^{(k-1)}\right\|_\infty \big/ \left\|X^{(k)}\right\|_\infty$ |
|---|---|---|---|---|
| 0 | 0.10000000 | 0.10000000 | –0.10000000 | |
| 1 | 0.49998333 | 0.00944115 | –0.52310127 | 0.8088 |
| 2 | 0.49999593 | 0.00002557 | –0.52336331 | 0.018 |
| 3 | 0.50000000 | 0.00001234 | –0.52359814 | 0.00044 |
| 4 | 0.50000000 | 0.00000003 | –0.52359847 | 0.000023 |
| 5 | 0.50000000 | 0.00000002 | –0.52359877 | 0.00000059 |

△

### 3.2.1. Newton's Method for Non-linear systems

Newton's method for non-linear systems involves selecting the initial point $X^{(0)}$ in $\mathbb{R}^n$ and generating

$$X^{(k)} = G(X^{(k-1)}) = X^{(k-1)} - \left[J\left(X^{(k-1)}\right)\right]^{-1} F\left(X^{(k-1)}\right), \qquad \textbf{(3.3)}$$

where $J(X)$ is the Jacobian matrix defined by

$$J(X) = \begin{pmatrix} \dfrac{\partial f_1(X)}{\partial x_1} & \dfrac{\partial f_1(X)}{\partial x_2} & \cdots & \dfrac{\partial f_1(X)}{\partial x_n} \\ \dfrac{\partial f_2(X)}{\partial x_1} & \dfrac{\partial f_2(X)}{\partial x_2} & \cdots & \dfrac{\partial f_2(X)}{\partial x_n} \\ \vdots & & & \\ \dfrac{\partial f_n(X)}{\partial x_1} & \dfrac{\partial f_n(X)}{\partial x_2} & \cdots & \dfrac{\partial f_n(X)}{\partial x_n} \end{pmatrix}.$$

Computing $\left[J\left(X^{(k-1)}\right)\right]^{-1}$ at each stage can be avoided by finding a vector $Y^{(k-1)}$ that satisfies

$$\left[J\left(X^{(k-1)}\right)\right] Y^{(k-1)} = -F(X^{(k-1)})$$

so that the new approximation is given by

$$X^{(k)} = X^{(k-1)} + Y^{(k-1)} \qquad \textbf{(3.4)}$$

***<u>Example 3.2.2</u>***

Use Newton's method to solve the system in Example 3.2.1

Solution:

Here $X=(x_1,x_2,x_3)^t$ and $F(X)=\left(f_1(X),\, f_2(X),\, f_3(X)\right)^t$, where

$$f_1(X)=3x_1-\cos x_2x_3-\frac{1}{2}$$
$$f_2(X)=x_1^{\,2}-81(x_2+0.1)^2+\sin x_3+1.06$$
$$f_3(X)=e^{-x_1x_2}+20x_3+\frac{10\pi-3}{3}.$$

The Jacobian matrix is

$$J(X)=\begin{pmatrix} 3 & x_3\sin x_2x_3 & x_2\sin x_2x_3 \\ 2x_1 & -162(x_2+0.1) & \cos x_3 \\ -x_2e^{-x_1x_2} & -x_1e^{-x_1x_2} & 20 \end{pmatrix}$$

and using the initial point $X^{(0)}=\left(0.1,0.1,-0.1\right)^t$, we get

$$F(X^{(0)})=\left(-0.199995,-2.269833417,8.462025346\right)^t$$

$$J(X^{(0)})=\begin{pmatrix} 3 & 0.0009999833334 & 0.0009999833334 \\ 0.2 & -32.4 & 0.9950041653 \\ -0.9900498337 & -0.9900498337 & 20 \end{pmatrix}.$$

We can find $\left[J(X^{(0)})\right]^{-1}$ and use (3.3) to approximate $X^{(1)}$. If we want to use (3.4), then we let $Y^{(0)}=\left(y_1^{(0)}(0),\, y_2^{(0)}(0),\, y_3^{(0)}(0)\right)^t$ so that $\left[J\left(X^{(0)}\right)\right]Y^{(0)}=-F(X^{(0)})$ gives the linear system

$$3y_1^{(0)}+0.0009999833334y_2^{(0)}+0.0009999833334y_3^{(0)}=0.199995$$
$$0.2y_1^{(0)}-32.4y_2^{(0)}+0.9950041653y_3^{(0)}=2.269833417$$
$$-0.9900498337y_1^{(0)}-0.9900498337y_2^{(0)}+20y_3^{(0)}=-8.462025346$$

Solving this system, we get $Y^{(0)}=\left(0.3998696728,-0.08053315147,\,-0.4215204718\right)^t$.

$$\therefore\quad X^{(1)}=X^{(0)}+Y^{(0)}=(0.4998696728,\ 0.01946684853,\ -0.5215204718)^t$$

Other values for $k=2,3,\ldots,$ are given in the table below:

| $k$ | $x_1^{(k)}$ | $x_2^{(k)}$ | $x_3^{(k)}$ | $\left\|X^{(k)} - X^{(k-1)}\right\|_\infty \Big/ \left\|X^{(k)}\right\|_\infty$ |
|---|---|---|---|---|
| 0 | 0.1 | 0.1 | –0.1 | |
| 1 | 0.4998696728 | 0.0194668485 | –0.5215204718 | 0.8083 |
| 2 | 0.5000142403 | 0.0015885914 | –0.5235569638 | 0.0342 |
| 3 | 0.5000000113 | 0.0000124448 | –0.5235984500 | $3.0099\times10^{-3}$ |
| 4 | 0.5 | $8.516\times10^{-10}$ | –0.5235987755 | $2.376\times10^{-5}$ |
| 5 | 0.5 | $-1.375\times10^{-11}$ | –0.5235987756 | $1.653\times10^{-9}$ |

△

**THE END!**