6. NUMERICAL INTEGRATION

We will use the methods called numerical quadrature which use a sum $\sum_{i=0}^{n} a_i f(x_i)$ to approximate

$$\int_a^b f(x)\,dx. \qquad \textbf{(6.1)}$$

The method that involves integrating the Lagrange interpolating polynomial can be used to derive what are known as Newton-Cotes formulas for evaluating (6.1). For example, using the first and second Lagrange interpolating polynomials with equally-spaced nodes gives the Trapezoidal rule and Simpson's rule, respectively. To derive these, we need the following theorem:

Theorem 6.1.1 (Weighted Mean Value Theorem for Integrals)

If $f \in C[a,b]$, g is integrable on $[a,b]$ and $g(x)$ does not change sign on $[a,b]$, then there exists a number $\xi,\ a < \xi < b,$ such that

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx.$$

□

Let $x_0 = a,\ x_1 = b$ and $h = b - a.$ Using the linear Lagrange interpolating polynomial

$$f(x) = \frac{(x-x_1)}{(x_0-x_1)} f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} f(x_1) + \frac{f''(\xi(x))}{2!}(x-x_0)(x-x_1)$$

we get

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_{x_0}^{x_1}\left(\frac{(x-x_1)}{(x_0-x_1)} f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} f(x_1)\right)dx + \int_{x_0}^{x_1}\left(\frac{f''(\xi(x))}{2!}(x-x_0)(x-x_1)\right)dx \\ &= \left.\frac{(x-x_1)^2}{2(x_0-x_1)} f(x_0) + \frac{(x-x_0)^2}{2(x_1-x_0)} f(x_1)\right|_{x_0}^{x_1} + \frac{1}{2}\int_{x_0}^{x_1} f''(\xi(x))(x-x_0)(x-x_1)\,dx \\ &= \frac{h}{2}\left[f(x_1)+f(x_0)\right] + \frac{1}{2}\int_{x_0}^{x_1} f''(\xi(x))(x-x_0)(x-x_1)\,dx.\end{aligned}$$

We can now apply Theorem 6.1.1, since $(x-x_0)(x-x_1)$ does not change sign on $[x_0,x_1]$ to get

$$\int_{x_0}^{x_1} f''(\xi(x))(x-x_0)(x-x_1)\,dx = f''(\xi(x))\int_{x_0}^{x_1}(x-x_0)(x-x_1)\,dx = f''(\xi(x))\left(\frac{x^3}{3}-\frac{(x_1+x_0)x^2}{2}+x_0x_1x\Big|_{x_0}^{x_1}\right)$$

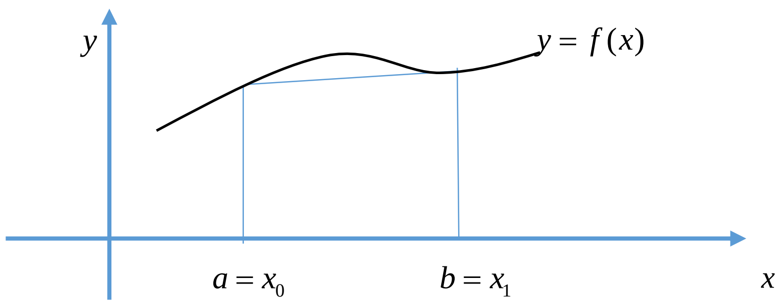
$$= f''(\xi(x))\left[-\frac{x_1^{\,3}}{6}+\frac{x_0x_1^{\,2}}{2}+\frac{x_0^{\,3}}{6}-\frac{x_0^{\,2}x_1}{2}\right]$$

$$= f''(\xi(x))\left[\frac{-\left(x_1-x_0\right)^3}{6}\right]$$

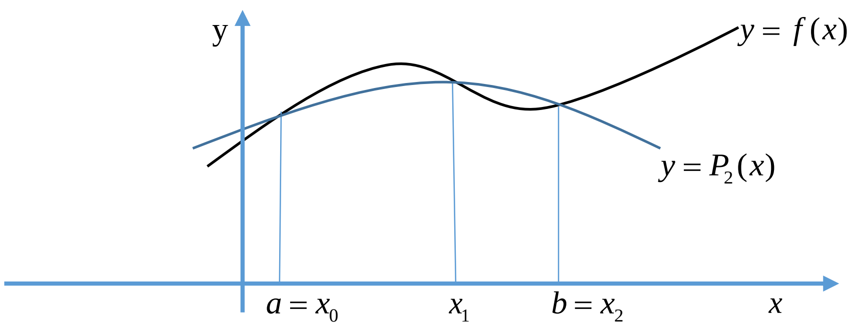
$$= -\frac{h^3}{6}f''(\xi(x)).$$

Thus, the Trapezoidal rule is

$$\int_a^b f(x)\,dx = \frac{h}{2}[f(x_1)+f(x_0)]-\frac{1}{2}\cdot\frac{h^3}{6}f''(\xi(x)) = \frac{h}{2}[f(x_1)+f(x_0)]-\frac{h^3}{12}f''(\xi(x))$$



Letting $x_0=a,\ x_1=a+h$ and $x_2=b$ with $h=\dfrac{b-a}{2}$ and using the second Lagrange polynomial, we can derive the Simpson's rule with an $O(h^4)$ error term involving $f^{(3)}$.



We can use the third Taylor polynomial about x_1 to obtain a higher-order term involving $f^{(4)}$. Then, for each x in $[x_0,x_2]$, there exists a number $\xi_1(x)$ in (x_0,x_2) such that

$$f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2!}(x - x_1)^2 + \frac{f'''(x_1)}{3!}(x - x_1)^3 + \frac{f^{(4)}(\xi_1(x))}{4!}(x - x_1)^4$$

for some number $\xi_1(x)$ in (x_0, x_2), so that

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_{x_0}^{x_2} \left(f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2!}(x - x_1)^2 + \frac{f'''(x_1)}{3!}(x - x_1)^3 \right) dx \\
&\quad + \int_{x_0}^{x_2} \left(\frac{f^{(4)}(\xi_1(x))}{4!}(x - x_1)^4 \right) dx \\
&= \left[f(x_1)(x - x_1) + \frac{f'(x_1)}{2}(x - x_1)^2 + \frac{f''(x_1)}{6}(x - x_1)^3 + \frac{f'''(x_1)}{24}(x - x_1)^4 \right]\Bigg|_{x_0}^{x_2} \\
&\quad + \int_{x_0}^{x_2} \left(\frac{f^{(4)}(\xi_1(x))}{4!}(x - x_1)^4 \right) dx \\
&= \left[f(x_1)(x_2 - x_1) + \frac{f'(x_1)}{2}(x_2 - x_1)^2 + \frac{f''(x_1)}{6}(x_2 - x_1)^3 + \frac{f'''(x_1)}{24}(x_2 - x_1)^4 \right. \\
&\quad \left. + f(x_1)(x_1 - x_0) - \frac{f'(x_1)}{2}(x_1 - x_0)^2 + \frac{f''(x_1)}{6}(x_1 - x_0)^3 - \frac{f'''(x_1)}{24}(x_1 - x_0)^4 \right] \\
&\quad + \int_{x_0}^{x_2} \left(\frac{f^{(4)}(\xi_1(x))}{4!}(x - x_1)^4 \right) dx \\
&= 2hf(x_1) + \frac{f''(x_1)}{3}h^3 + \int_{x_0}^{x_2} \left(\frac{f^{(4)}(\xi_1(x))}{4!}(x - x_1)^4 \right) dx, \quad \text{since } h = x_2 - x_1 = x_1 - x_0.
\end{aligned}$$

Thus, by Theorem 6.1.1, we have that

$$\begin{aligned}
\int_{x_0}^{x_2} \left(\frac{f^{(4)}(\xi_1(x))}{4!}(x - x_1)^4 \right) dx &= \frac{f^{(4)}(\xi_1(x))}{24} \int_{x_0}^{x_2} (x - x_1)^4\,dx = \frac{f^{(4)}(\xi_1(x))}{120}(x - x_1)^5 \Bigg|_{x_0}^{x_2} \\
&= \frac{f^{(4)}(\xi_1(x))}{120}\left[(x_2 - x_1)^5 - (x_0 - x_1)^5 \right] = \frac{f^{(4)}(\xi_1(x))}{120}\left[(x_2 - x_1)^5 + (x_1 - x_0)^5 \right] \\
&= \frac{f^{(4)}(\xi_1(x))}{60}h^5
\end{aligned}$$

$$\Rightarrow \int_a^b f(x)\,dx = 2hf(x_1) + \frac{h^3}{3} f''(x_1) + \frac{h^5}{60} f^{(4)}(\xi_1(x)).$$

Using the second derivative midpoint formula, we have that

$$f''(x_1) = \frac{1}{h^2}[f(x_1 - h) - 2f(x_1) + f(x_1 + h)] - \frac{h^2}{12} f^{(4)}(\xi_1(x)) = \frac{1}{h^2}[f(x_0) - 2f(x_1) + f(x_2)] - \frac{h^2}{12} f^{(4)}(\xi_1(x))$$

so that

$$\int_a^b f(x)\,dx = 2hf(x_1) + \frac{h^3}{3}\left[\frac{1}{h^2}[f(x_0) - 2f(x_1) + f(x_2)] - \frac{h^2}{12} f^{(4)}(\xi_1(x)) \right] + \frac{h^5}{60} f^{(4)}(\xi_1(x)).$$

Therefore, the Simpson's rule is

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]-\frac{h^5}{90}f^{(4)}(\xi_1(x)).$$

The error term in Simpson's rule involves the fourth derivative of f, so it gives exact results when applied to any polynomial of degree three or less.

Example 6.1.1

Use the Trapezoidal rule and the Simpson's rule to evaluate $\int_0^2 \sqrt{1+x^2}$.

Solutions:

Using Trapezoidal rule, we have that $x_0=0,\ x_1=2$ and $h=2$ so that

$$\int_0^2 \sqrt{1+x^2} \approx \frac{2}{2}\left[f(x_0)+f(x_2)\right] = f(0)+f(2) = 1+\sqrt{5} = 3.236067977$$

Using Simpson's rule, we have that $x_0=0,\ x_1=1,\ x_2=2$ and $h=\frac{2-0}{2}=1$ so that

$$\int_0^2 \sqrt{1+x^2} \approx \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] = \frac{1}{3}\left[f(0)+4f(1)+f(2)\right]$$

$$\frac{1}{3}\left[1+4\sqrt{2}+\sqrt{5}\right] = 2.964307409.$$

△

The Newton-Cotes formulas, however, are generally unsuitable for use over large integration intervals because of the oscillatory nature of high-degree polynomials.

Theorem 6.1.2

Let $f \in C^2[a,b],\ h=\frac{b-a}{n}$ and $x_j = a+jh$ for each $j=0,1,2,\dots,n.$ There exists a $\mu \in (a,b)$ for which the Composite Trapezoidal rule for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a)+2\sum_{j=1}^{n-1} f(x_j)+f(b)\right]-\frac{b-a}{12}h^2 f''(\mu).$$

□

Example 6.1.2

(a) Find an approximation to the area of the region bounded by the normal curve

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-\left(\frac{x}{\sigma}\right)^2}{2}}$$

and the $x-$ axis on the interval $[-\sigma, \sigma]$ using the Composite Trapezoidal rule with $n = 8$.

(b) Using the Composite Trapezoidal rule, determine the value of n and h required to approximate $\int_0^2 e^{2x} \sin 3x \, dx$ to within 10^{-4}.

Solutions:

(a) Clearly, $f \in C^2\left[-\sigma, \sigma\right]$.

With $a = -\sigma$, $b = \sigma$ and $n = 8$, we have that $h = \frac{\sigma - (-\sigma)}{8} = \frac{\sigma}{4}$ and $x_j = -\sigma + j\left(\frac{\sigma}{4}\right)$

$$\Rightarrow \; x_1 = -\sigma + \frac{\sigma}{4} = -\frac{3\sigma}{4}, \quad x_2 = -\sigma + \frac{\sigma}{2} = -\frac{\sigma}{2}, \; x_3 = -\sigma + \frac{3\sigma}{4} = -\frac{\sigma}{4}, \; x_4 = -\sigma + \sigma = 0,$$

$$x_5 = -\sigma + \frac{5\sigma}{4} = \frac{\sigma}{4}, \; x_6 = -\sigma + \frac{3\sigma}{2} = \frac{\sigma}{2} \text{ and } x_7 = -\sigma + \frac{7\sigma}{4} = \frac{3\sigma}{4}$$

$$\Rightarrow \; f(a) = f(-\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-\left(\frac{-\sigma}{\sigma}\right)^2}{2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}}$$

$$f(x_1) = f\left(-\frac{3\sigma}{4}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-\left(\frac{-\frac{3\sigma}{4}}{\sigma}\right)^2}{2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{9}{32}}$$

$$f(x_2) = f\left(-\frac{\sigma}{2}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-\left(\frac{-\frac{\sigma}{2}}{\sigma}\right)^2}{2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{8}}$$

$$f(x_3) = f\left(-\frac{\sigma}{4}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-\left(\frac{-\frac{\sigma}{4}}{\sigma}\right)^2}{2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{32}}$$

$$f(x_4) = f(0) = \frac{1}{\sigma\sqrt{2\pi}}$$

$$f(x_5)=f\left(\frac{\sigma}{4}\right)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left(\frac{\frac{\sigma}{4}}{\sigma}\right)^2}{2}}=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{32}}$$

$$f(x_6)=f\left(\frac{\sigma}{2}\right)==\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left(\frac{\frac{\sigma}{2}}{\sigma}\right)^2}{2}}=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{8}}$$

$$f(x_7)=f\left(\frac{3\sigma}{4}\right)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left(\frac{\frac{3\sigma}{4}}{\sigma}\right)^2}{2}}=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{9}{32}} \text{ and } f(b)=f(\sigma)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left(\frac{\sigma}{\sigma}\right)^2}{2}}=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}}.$$

$$\therefore \int_{-\sigma}^{\sigma} f(x)\,dx \approx \frac{h}{2}\left[f(-\sigma)+2\sum_{j=1}^{8-1} f(x_j)+f(\sigma)\right]$$

$$=\frac{\sigma}{8}\left[f(-\sigma)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+2f(x_5)+2f(x_6)+2f(x_7)+f(\sigma)\right]$$

$$=\frac{\sigma}{8}\left[\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}}+2\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{9}{32}}\right)+2\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{8}}\right)+2\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{32}}\right)+2\left(\frac{1}{\sigma\sqrt{2\pi}}\right)+\right.$$
$$\left.2\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{32}}\right)+2\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{8}}\right)+2\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{9}{32}}\right)+\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}}\right]$$

$$=\frac{\sigma}{8}.\frac{2}{\sigma\sqrt{2\pi}}\left[e^{-\frac{1}{2}}+2e^{-\frac{9}{32}}+2e^{-\frac{1}{8}}+2e^{-\frac{1}{32}}+1\right]$$

$$\approx 0.680163689$$

(b) Here, $a=0,\ b=2$ and $f(x)=e^{2x}\sin 3x$

$$\Rightarrow f'(x)=2e^{2x}\sin 3x+3e^{2x}\cos 3x$$

$$f''(x)=2\left[2e^{2x}\sin 3x+3e^{2x}\cos 3x\right]+3\left[2e^{2x}\cos 3x-3e^{2x}\sin 3x\right]=12e^{2x}\cos 3x-5e^{2x}\sin 3x.$$

The error function for the Composite Trapezoidal rule is

$$E(f)=-\frac{b-a}{12}h^2 f''(\mu)$$

$$\therefore \left|-\frac{b-a}{12}h^2 f''(\mu)\right|<10^{-4} \Rightarrow \left(\frac{2}{12}h^2\right)\underset{\mu\in[0,2]}{\text{Max}}\left(12e^{2x}\cos 3x-5e^{2x}\sin 3x\right)<10^{-4}.$$

Since $\underset{\mu\in[0,2]}{\text{Max}}\left(12e^{2x}\cos 3x-5e^{2x}\sin 3x\right)\approx 705.3601029,$ we have that

$$h^2<\frac{6\times 10^{-4}}{705.3601029}$$

$$\Rightarrow h < \sqrt{8.506293417 \times 10^{-7}} \approx 0.0009222956911.$$

Since $h = \dfrac{b-a}{n}$, we have that

$$\frac{2}{n} < 0.0009222956911 \quad \Rightarrow n > \frac{2}{0.0009222956911} = 2168.501945$$
$$\Rightarrow n > 2168.$$

△

Theorem 6.1.3

Let $f \in C^4[a,b]$, n be even, $h = \dfrac{b-a}{n}$ and $x_j = a + jh$ for each $j = 0,1,2,...,n.$ There exists a $\mu \in (a,b)$ for which the Composite Simpson's rule for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(b) \right] - \frac{b-a}{180} h^4 f^{(4)}(\mu).$$

□

Example 6.1.3

(a) A car laps a race track in 84 seconds. The speed of the car at each $6-$second interval is determined by using a radar gun and is given from the beginning of the lap, in feet/second, by the entries in the following table:

Time	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84
Speed (ft/s)	124	134	148	156	147	133	121	109	99	85	78	89	104	116	123

How long is the track?

(b) Determine the value of n and h required to approximate

$$\int_0^2 \left(\frac{1}{x+4} \right) dx$$

to within 10^{-5} and compute the approximation using Composite Simpson's rule.

Solutions:

(a) Here $h=6,\ x_0=0,\ x_1=6,\ x_2=12,\ldots,x_{13}=78,\ x_{14}=84.$

Since $n=14,$ we can use Composite Simpson's rule:

$\therefore$ Length of the track, $l=\int_0^{84} f(x)\,dx \approx \frac{h}{3}\left[f(x_0)+2\sum_{j=1}^{\frac{14}{2}-1} f(x_{2j})+4\sum_{j=1}^{\frac{14}{2}} f(x_{2j-1})+f(x_{14})\right]$

$$=\frac{h}{3}[f(x_0)+2f(x_2)+2f(x_4)+2f(x_6)+2f(x_8)+2f(x_{10})+2f(x_{12})+4f(x_1)+4f(x_3)+4f(x_5)+$$
$$4f(x_7)+4f(x_9)+4f(x_{11})+4f(x_{13})+f(x_{14})]$$
$$=\frac{h}{3}[124+2(148)+2(147)+2(121)+2(99)+2(78)+2(104)+4(134)+4(156)+4(133)+$$
$$4(109)+4(85)+4(89)+4(116)+(123)]$$
$$=4232 \text{ ft.}$$

(b) $E(f)=-\frac{b-a}{180}h^4 f^{(4)}(\mu),$ where $f(x)=\frac{1}{x+4} \Rightarrow f'(x)=-\frac{1}{(x+4)^2},\ f''(x)=\frac{2}{(x+4)^3},$

$f'''(x)=-\frac{6}{(x+4)^4}$ and $f^{(4)}(x)=\frac{24}{(x+4)^5}.$

$$\Rightarrow f^{(4)}(\mu)=\frac{24}{(\mu+4)^5}$$

$$\therefore |E(f)|=\left|\left(\frac{2}{180}h^4\right)f^{(4)}(\mu)\right|<10^{-5}$$
$$\Rightarrow \frac{h^4}{90}\operatorname*{Max}_{\mu\in[0,2]}\left(\frac{24}{(\mu+4)^5}\right)<10^{-5}$$
$$\Rightarrow \frac{h^4}{15}\left(\frac{4}{4^5}\right)<10^{-5}$$
$$\Rightarrow h^4<10^{-5}(3840)=0.0384$$
$$\Rightarrow h<0.442672767.$$

$$\therefore h=\frac{b-a}{n}<0.442672767$$
$$\Rightarrow n>\frac{2}{0.442672767}=4.518$$
$$\Rightarrow n>4.$$

Using Composite Simpson's rule, we take $n=6$ so that

$$\int_0^2 \left(\frac{1}{x+4}\right) dx \approx \frac{h}{3}[f(x_0)+2f(x_2)+2f(x_4)+4f(x_1)+4f(x_3)+4f(x_5)+f(x_6)]$$

$$= \frac{\frac{2}{6}}{3}\left[\frac{1}{4}+2\left(\frac{1}{\frac{2}{3}+4}\right)+2\left(\frac{1}{\frac{4}{3}+4}\right)+4\left(\frac{1}{\frac{1}{3}+4}\right)+4\left(\frac{1}{1+4}\right)+4\left(\frac{1}{\frac{5}{3}+4}\right)+\frac{1}{2+4}\right]$$

$$= 0.405466374.$$

The exact value is $\int_0^2 \left(\frac{1}{x+4}\right) dx = \ln|x+4|\Big|_0^2 = \ln 6 - \ln 4 \simeq 0.405465108$

giving the error $|\ln\frac{3}{2} - 0.405466374| = 0.00000126589184 < 10^{-5}$.

△

THE END!