7. INITIAL-VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

Differential equations are used to model problems in science and engineering that involve the change of some variable with respect to another. This usually leads to Initial-Value Problems (IVP) and Boundary-Value Problems (BVP). In this chapter, we will discuss numerical methods for approximating solutions to such problems with special focus on IVPs. These methods are helpful in situations where the problem is too complicated to solve using analytical methods. For example, the motion of a swinging pendulum under certain simplifying assumptions is described by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\,\theta = 0,$$

where L is the length of the pendulum, g is the gravitational constant of the Earth and θ is the angle the pendulum makes with the vertical. For large values of θ, and initial conditions at $t = t_0$ this leads to the IVP

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\,\theta = 0, \quad \theta(t_0) = \theta_0,\; \theta'(t_0) = \theta_1.$$

This differential equation above can be solved using numerical methods.

When we attempt to solve a differential equation, we must be sure that there really is a solution and that this solution is unique. In addition, we need to determine whether the problem is well-posed, that is, whether small changes in the statement of the problem introduces correspondingly small changes in the solution.

Definition 7.1.1

A function $f(t, y)$ is said to be satisfy a Lipschitz condition in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ (called a Lipschitz constant for f) exists with

$$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|,$$

whenever (t, y_1) and (t, y_2) are in D.

◊

Example 7.1.1

Determine whether $f(t,y)$ satisfies a Lipschitz condition on D in each of the following:

(a) $f(t,y)=\frac{2}{t}y+t^2e^t, \quad D=\{(t,y):1\le t\le 2,\ -\infty<y<\infty\}$

(b) $f(t,y)=ty^2, \quad D=\{(t,y):0\le t\le 2,\ -\infty<y<\infty\}$

(c) $f(t,y)=t\,|\,y\,|, \quad D=\{(t,y):1\le t\le 2,\ -3\le y\le 4\}$

Solutions:

(a) Suppose that (t,y_1) and (t,y_2) are in D. Then

$$|f(t,y_1)-f(t,y_2)|=\left|\frac{2}{t}y_1+t^2e^t-\frac{2}{t}y_2+t^2e^t\right|$$

$$=\frac{2}{|t|}|y_1-y_2|\le 2|y_1-y_2|$$

Thus, f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant $L=2.$

(b) Let (t,y_1) and (t,y_2) are in D. Then

$$|f(t,y_1)-f(t,y_2)|=|ty_1^2-ty_2^2|$$

$$=|t|\,|y_1^2-y_2^2|$$

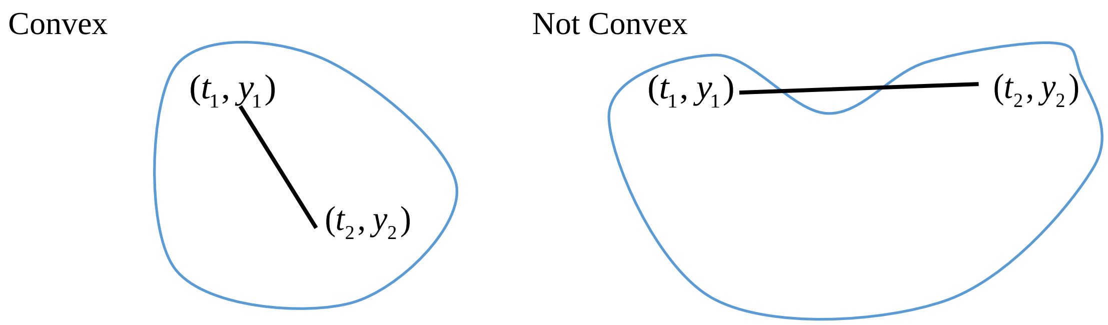
$$\le|t|\big(|y_1|+|y_2|\big)|y_1-y_2|<\infty, \text{ since } |y_1|+|y_2|<\infty \text{ for } -\infty<y_1,y_2<\infty.$$

Therefore, f does not satisfy a Lipschitz condition on D.

(c) Exercise

△

If the set D is convex, i.e. for any two points (t_1,y_1) and (t_2,y_2), the straight line joining these two points lies in D, then we can determine Lipschitz condition by using the following theorem:



Theorem 7.1.1

Suppose that $f(t,y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L>0$ exists with

$$\left|\frac{\partial f}{\partial y}(t,y)\right| \le L, \text{ for all } (t,y)\in D,$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

□

Exercise: Use Theorem 7.1.1 to determine whether the functions in Example 7.1.1 satisfy a Lipschitz condition on the given set D.

Theorem 7.1.2

Suppose that $D=\{(t,y): a\le t\le b,\ -\infty<y<\infty\}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the IVP

$$\frac{dy}{dt}=f(t,y),\ a\le t\le b,\ y(a)=\alpha$$

is well-posed.

□

Example 7.1.2

Show that the IVP

$$\frac{dy}{dt}=y-t^2+1,\ 0\le t\le 2,\ \ y(0)=0.5$$

is well-posed on $D=\{(t,y):0\le t\le 2,\ -\infty<y<\infty\}$.

Solution:

Using Theorem 7.1.2, note that $f(t,y)=y-t^2+1$ is continuous on D and

$$\left|\frac{\partial f}{\partial y}(t,y)\right|=|1|=1,$$

implying that f satisfies a Lipschitz condition in the variable y on D with $L=1$. Hence, the IVP is well-posed.

△

7.1. FIRST-ORDER IVP

We now discuss numerical methods for solving IVPs for first-order ordinary differential equations.

7.1.1 ***EULER'S METHOD***

Suppose we want to approximate the solution to a well-posed IVP

$$\frac{dy}{dt} = f(t, y), \; a \le t \le b, \; y(a) = \alpha. \qquad \textbf{(7.1)}$$

Approximations of y will be obtained at various points called mesh points, in the interval $[a,b]$. We assume that the mesh points are equally distributed, i.e. $t_i = a + ih$ for each $i = 0,1,2,\ldots,N,$ where N is some chosen positive integer and $h = \frac{b-a}{N} = t_{i+1} - t_i$. If $y(t) \in C^2[a,b],$ then by Taylor's theorem

$$y(t_{i+1}) = y(t_i) + y'(t_i)(t_{i+1} - t_i) + \frac{y''(\xi_i)}{2!}(t_{i+1} - t_i)^2,$$

for some $\xi_i \in (t_i, t_{i+1})$. Using (7.1), we have that

$$y(t_{i+1}) = y(t_i) + hf(t_i, y_i) + \frac{h^2}{2!} y''(\xi_i).$$

Euler's method constructs $w_i \approx y(t_i), \; i = 1,2,\ldots,N,$ by deleting the remainder term. Thus, Euler's method is

$$w_0 = \alpha$$
$$w_{i+1} = w_i + hf(t_i, w_i), \text{ for each }, \; i = 1,2,\ldots,N-1 \qquad (7.2)$$

Equation (7.2) is called the difference equation associated with Euler's method.

Example 7.1.3

Use Euler's method to approximate the solution to

$$\frac{dy}{dt} = y - t^2 + 1, \; 0 \le t \le 2, \quad y(0) = 0.5$$

at $t = 2$ when $h = 0.5$.

Solution:

Since $h = 0.5,$ we have that

$t_0 = 0,\ t_1 = 0 + 1(0.5) = 0.5,\ t_2 = 0 + 2(0.5) = 1,\ t_3 = 0 + 3(0.5) = 1.5$ and $t_4 = 0 + 4(0.5) = 2$

$\Rightarrow N = 4, \quad w_0 = y(0) = 0.5$

$$w_1 = w_0 + hf(t_0, w_0) = 0.5 + (0.5)(0.5 - 0^2 + 1) = 1.25,$$

i.e. $y(t_1) = y(0.5) \approx w_1 = 1.25$

$$w_2 = w_1 + (0.5)(w_1 - t^2 + 1) = 1.25 + (0.5)(1.25 - (0.5)^2 + 1) = 2.25,$$

i.e. $y(t_2) = y(1) \approx w_2 = 2.25$

$$w_3 = w_2 + (0.5)(w_2 - t^2 + 1) = 2.25 + (0.5)(2.25 - 1^2 + 1) = 3.375,$$

i.e. $y(t_3) = y(1.5) \approx w_2 = 3.375.$

Thus,

$$y(2) \approx w_4 = w_3 + (0.5)(w_3 - t^2 + 1) = 3.375 + (0.5)(3.375 - (1.5)^2 + 1) = 4.4375$$

Δ

Exercise: Solve the IVP analytically and compare the actual values with the approximated values when $N = 10,\ 20,\ 50$ and so on. For each $N,$ compare the errors as t increases.

7.1.2. ***HIGHER-ORDER TAYLOR METHODS***

Suppose that $y(t) \in C^{(n+1)}[a,b]$ is a solution to the IVP

$$y'(t) = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Using n^{th} Taylor polynomial about t_i and evaluated at $t_{i+1},$ we get

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2!}y''(t_i) + \ldots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i),$$

for some $\xi_i \in (t_i, t_{i+1}).$ Since $y'(t) = f(t,y), \quad y'(t) = f(t,y)$ and generally,

$y^{(k)}(t) = f^{(k-1)}(t,y)$ implying that

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2!} f'(t_i, y(t_i)) + ... + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(t_i, y(\xi_i)).$$

Thus, the difference equation for the Taylor method of order n is

$$w_0 = \alpha$$
$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \;\; i = 0,1,2,..., N-1,$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + ... + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i).$$

NOTE: Euler's method is Taylor's method of order one.

Example 7.1.4

Apply Taylor's method of order (a) two and (b) four with $N = 10$ to the IVP

$$\frac{dy}{dt} = y - t^2 + 1, \; 0 \le t \le 2, \quad y(0) = 0.5$$

at $y(1)$.

Solution:

(a) Since $y' = f(t, y) = y - t^2 + 1, \;\; y'' = \dfrac{d(y - t^2 + 1)}{dt} = y' - 2t = y - t^2 + 1 - 2t = y - t^2 - 2t + 1.$

Thus,

$$T^2(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) = w_i - t_i^{\,2} + 1 + \frac{h}{2}(w_i - t_i^{\,2} - 2t_i + 1)$$
$$\therefore \;\; w_{i+1} = w_i + h\left(w_i - t_i^{\,2} + 1\right) + \frac{h^2}{2}\left(w_i - t_i^{\,2} - 2t_i + 1\right).$$

With $N = 10, \; h = \dfrac{b-a}{N} = \dfrac{2-0}{10} = 0.2$ so that $t_0 = 0, \; t_1 = 0.2, \; t_2 = 0.4, ..., t_{10} = 2.0$

$$w_0 = 0.5$$
$$w_1 = w_0 + h\left(w_0 - t_0^{\,2} + 1\right) + \frac{h^2}{2}\left(w_0 - t_0^{\,2} - 2t_0 + 1\right) = 0.5 + 0.2(0.5 - 0^2 + 1) + \frac{(0.2)^2}{2}(0.5 - 0^2 - 2(0) + 1)$$
$$= 0.83$$

$$w_2 = w_1 + h\left(w_1 - t_1^{\,2} + 1\right) + \frac{h^2}{2}\left(w_1 - t_1^{\,2} - 2t_1 + 1\right)$$
$$= 0.83 + 0.2(0.83 - (0.2)^2 + 1) + \frac{(0.2)^2}{2}(0.83 - (0.2)^2 - 2(0.2) + 1)$$
$$= 1.2158$$

$$w_3 = w_2 + h\left(w_2 - t_2^{\,2} + 1\right) + \frac{h^2}{2}\left(w_2 - t_2^{\,2} - 2t_2 + 1\right)$$
$$= 1.2158 + 0.2(1.2158 - (0.4)^2 + 1) + \frac{(0.2)^2}{2}(1.2158 - (0.4)^2 - 2(0.4) + 1)$$
$$= 1.652076$$

$$w_4 = w_3 + h\left(w_3 - t_3^{\,2} + 1\right) + \frac{h^2}{2}\left(w_3 - t_3^{\,2} - 2t_3 + 1\right)$$
$$= 1.652076 + 0.2(1.652076 - (0.6)^2 + 1) + \frac{(0.2)^2}{2}(1.652076 - (0.6)^2 - 2(0.6) + 1)$$
$$= 2.132333$$

$$w_5 = w_4 + h\left(w_4 - t_4^{\,2} + 1\right) + \frac{h^2}{2}\left(w_4 - t_4^{\,2} - 2t_4 + 1\right)$$
$$= 2.132333 + 0.2(2.132333 - (0.8)^2 + 1) + \frac{(0.2)^2}{2}(2.132333 - (0.8)^2 - 2(0.8) + 1)$$
$$= 2.648646$$

$\therefore\ y(t_5) = y(1.0) \approx w_5 = 2.648646$

(b) $f(t,y) = y - t^2 + 1,\ \ f'(t,y) = y - t^2 - 2t + 1,\ \ f''(t,y) = y' - 2t - 2 = y - t^2 - 2t - 1$ and

$f'''(t,y) = y - t^2 - 2t - 1$. Thus,

$$T^4(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{3!} f''(t_i, w_i) + \frac{h^3}{4!} f'''(t_i, w_i)$$
$$= w_i - t_i^{\,2} + 1 + \frac{h}{2}\left(w_i - t_i^{\,2} - 2t_i + 1\right) + \frac{h^2}{6}\left(w_i - t_i^{\,2} - 2t_i - 1\right) + \frac{h^3}{24}\left(w_i - t_i^{\,2} - 2t_i - 1\right)$$

$\therefore\ w_{i+1} = w_i + hT^4(t_i, w_i)$

$$= w_i + h\left(w_i - t_i^{\,2} + 1\right) + \frac{h^2}{2}\left(w_i - t_i^{\,2} - 2t_i + 1\right) + \frac{h^3}{6}\left(w_i - t_i^{\,2} - 2t_i - 1\right) + \frac{h^4}{24}\left(w_i - t_i^{\,2} - 2t_i - 1\right)$$

$\therefore\ \ w_0 = 0.5,$

$$w_1 = w_0 + h\left(w_0 - t_0{}^2 + 1\right) + \frac{h^2}{2}\left(w_0 - t_0{}^2 - 2t_0 + 1\right) + \frac{h^3}{6}\left(w_0 - t_0{}^2 - 2t_0 - 1\right) + \frac{h^4}{24}\left(w_0 - t_0{}^2 - 2t_0 - 1\right)$$

$$= 0.5 + (0.2)\left(0.5 - 0^2 + 1\right) + \frac{(0.2)^2}{2}\left(0.5 - 0^2 - 2(0) + 1\right) + \frac{(0.2)^3}{6}\left(0.5 - 0^2 - 2(0) - 1\right) + \frac{(0.2)^4}{24}\left(0.5 - 0^2 - 2(0) - 1\right)$$

$$= 0.8293$$

$$w_2 = w_1 + h\left(w_1 - t_1{}^2 + 1\right) + \frac{h^2}{2}\left(w_1 - t_1{}^2 - 2t_1 + 1\right) + \frac{h^3}{6}\left(w_1 - t_1{}^2 - 2t_1 - 1\right) + \frac{h^4}{24}\left(w_1 - t_1{}^2 - 2t_1 - 1\right)$$

$$= 0.8293 + (0.2)\left(0.8293 - (0.2)^2 + 1\right) + \frac{(0.2)^2}{2}\left(0.8293 - (0.2)^2 - 2(0.2) + 1\right) + \frac{(0.2)^3}{6}\left(0.8293 - (0.2)^2 - 2(0.2) - 1\right) + \frac{(0.2)^4}{24}\left(0.8293 - (0.2)^2 - 2(0.2) - 1\right)$$

$$= 1.214091$$

$$w_3 = w_2 + h\left(w_2 - t_2{}^2 + 1\right) + \frac{h^2}{2}\left(w_2 - t_2{}^2 - 2t_2 + 1\right) + \frac{h^3}{6}\left(w_2 - t_2{}^2 - 2t_2 - 1\right) + \frac{h^4}{24}\left(w_2 - t_2{}^2 - 2t_2 - 1\right)$$

$$= 1.214091 + (0.2)\left(1.214091 - (0.4)^2 + 1\right) + \frac{(0.2)^2}{2}\left(1.214091 - (0.4)^2 - 2(0.4) + 1\right) + \frac{(0.2)^3}{6}\left(1.214091 - (0.4)^2 - 2(0.4) - 1\right) + \frac{(0.2)^4}{24}\left(1.214091 - (0.4)^2 - 2(0.4) - 1\right)$$

$$= 1.648947$$

$$w_4 = w_3 + h\left(w_3 - t_3{}^2 + 1\right) + \frac{h^2}{2}\left(w_3 - t_3{}^2 - 2t_3 + 1\right) + \frac{h^3}{6}\left(w_3 - t_3{}^2 - 2t_3 - 1\right) + \frac{h^4}{24}\left(w_3 - t_3{}^2 - 2t_3 - 1\right)$$

$$= 1.648947 + (0.2)\left(1.648947 - (0.6)^2 + 1\right) + \frac{(0.2)^2}{2}\left(1.648947 - (0.6)^2 - 2(0.6) + 1\right) + \frac{(0.2)^3}{6}\left(1.648947 - (0.6)^2 - 2(0.6) - 1\right) + \frac{(0.2)^4}{24}\left(1.648947 - (0.6)^2 - 2(0.6) - 1\right)$$

$$= 2.127240$$

$$\therefore\ y(t_5) = y(1) \approx w_5 = w_4 + h\left(w_4 - t_4{}^2 + 1\right) + \frac{h^2}{2}\left(w_4 - t_4{}^2 - 2t_4 + 1\right) + \frac{h^3}{6}\left(w_4 - t_4{}^2 - 2t_4 - 1\right) + \frac{h^4}{24}\left(w_4 - t_4{}^2 - 2t_4 - 1\right)$$

$$= 2.127240 + (0.2)\left(2.127240 - (0.8)^2 + 1\right) + \frac{(0.2)^2}{2}\left(2.127240 - (0.8)^2 - 2(0.8) + 1\right) + \frac{(0.2)^3}{6}\left(2.127240 - (0.8)^2 - 2(0.8) - 1\right) + \frac{(0.2)^4}{24}\left(2.127240 - (0.8)^2 - 2(0.8) - 1\right)$$

$$= 2.640874$$

Exercise: Find $y(2)$ in each case.

△

7.1.3. ***RUNGE-KUTTA METHODS***

Euler's method is less efficient in practical problems because it requires h being small for obtaining reasonable accuracy. The Runge-Kutta methods are designed to give greater accuracy and they possess the advantage of requiring only function values at some selected points on the subinterval. We start by deriving Runge-Kutta methods of order two for approximating $y(t_i) \approx w_i,\ i = 1, 2, ..., N.$

Second-order Runge-Kutta methods are obtained by using weighted estimates of the change in y when t advances by h, k_1 and k_2 such that

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + ak_1 + bk_2, \qquad (7.3)\\ k_1 &= hf(t_i, y(t_i))\\ k_2 &= hf\left(t_i + \alpha h, y(t_i) + \beta k_1\right). \end{aligned}$$

We need to devise a scheme of choosing the four parameters a, b, α and β. Before we do that, we need to state Taylor's Theorem in two variables.

<u>*Theorem 7.1.3*</u>

Suppose that $f(t, y)$ and all its partial derivatives of order less than or equal to $n+1$ are continuous on $D = \{(t, y) : a \le t \le b,\ c \le y \le d\}$, and let $(t_0, y_0) \in D.$ For every $(t, y) \in D,$ there exists ξ between t and t_0, and μ between y and y_0 with

$$f(t, y) = P_n(t, y) + R_n(t, y),$$

where

$$\begin{aligned} P_n((t, y) = f(t_0, y_0) &+ \left[(t - t_0)\frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(t_0, y_0)\right]\\ &+ \left[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) + \frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0, y_0)\right] + \ldots\\ &+ \left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^j\,\frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0, y_0)\right] \end{aligned}$$

and

$$R_n(t, y) = \frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t-t_0)^{n+1-j}(y-y_0)^j\,\frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(\xi, \mu)$$

The function $P_n(t, y)$ is called the n^{th} Taylor polynomial in two variables for the

function f about (t_0, y_0), and $R_n(t, y)$ is the remainder term associated with $P_n(t, y)$.

□

Example 7.1.5

Find the second Taylor polynomial for

$$f(t,y)=\sqrt{4t+12y-t^2-2y^2-6}$$

about $(2,3)$.

Solution:

$$P_2((t,y)= f(t_0,y_0)+\left[(t-t_0)\frac{\partial f}{\partial t}(t_0,y_0)+(y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)\right]$$
$$+\left[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0)+(t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t\partial y}(t_0,y_0)+\frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right]$$

$$f(2,3)=\sqrt{4(2)+12(3)-(2^2)-2(3)^2-6}=4$$

$$\frac{\partial f}{\partial t}(t,y)=\frac{1}{2}(4-2t)\left(4t+12y-t^2-2y^2-6\right)^{-\frac{1}{2}} \qquad \Rightarrow \frac{\partial f}{\partial t}(2,3)=0$$
$$\frac{\partial f}{\partial y}(t,y)=\frac{1}{2}(12-4y)\left(4t+12y-t^2-2y^2-6\right)^{-\frac{1}{2}} \qquad \Rightarrow \frac{\partial f}{\partial y}(2,3)=0$$

$$\frac{\partial^2 f}{\partial t\partial y}(t,y)=\frac{1}{2}(4-2t)\left(-\frac{1}{2}\right)(12-4y)\left(4t+12y-t^2-2y^2-6\right)^{-\frac{3}{2}}$$
$$\Rightarrow \frac{\partial^2 f}{\partial t\partial y}(2,3)=-\frac{1}{4}(0)(0)(4^{-3})=0$$

$$\frac{\partial^2 f}{\partial t^2}(t,y)=\frac{1}{2}\left(-\frac{1}{2}\right)(4-2t)^2\left(4t+12y-t^2-2y^2-6\right)^{-\frac{3}{2}}+\frac{1}{2}(-2)\left(4t+12y-t^2-2y^2-6\right)^{-\frac{1}{2}}$$
$$\Rightarrow \frac{\partial^2 f}{\partial t^2}(2,3)=-\frac{1}{4}(0)^2(4^{-3})-4^{-1}=-\frac{1}{4}$$
$$\frac{\partial^2 f}{\partial y^2}(t,y)=\frac{1}{2}\left(-\frac{1}{2}\right)(12-4y)^2\left(4t+12y-t^2-2y^2-6\right)^{-\frac{3}{2}}+\frac{1}{2}(-4)\left(4t+12y-t^2-2y^2-6\right)^{-\frac{1}{2}}$$
$$\Rightarrow \frac{\partial^2 f}{\partial y^2}(2,3)=-\frac{1}{4}(0)^2(4^{-3})-2(4^{-1})=-\frac{1}{2}$$

$$\therefore \quad P_2((t,y)=4-\frac{1}{4}.\frac{(t-t_0)^2}{2}-\frac{1}{2}.\frac{(y-y_0)^2}{2}=4-\frac{(t-t_0)^2}{8}-\frac{(y-y_0)^2}{4}$$

△

We now turn our attention to determining the four parameters a, b, α and β of (7.3). Using $T^2(t_i, y(t_i)),$ we have that

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i))$$

and since $\dfrac{df(t,y)}{dt} = \dfrac{\partial f(t,y)}{\partial t} + \dfrac{\partial f(t,y)}{\partial y} y'(t),$ it follows that

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}\left(f_t(t_i, y(t_i)) + f_y(t_i, y(t_i)).f(t_i, y(t_i))\right) \\ &= y(t_i) + hf(t_i, y(t_i)) + h^2\left(\tfrac{1}{2} f_t(t_i, y(t_i)) + \tfrac{1}{2} f_y(t_i, y(t_i)) f(t_i, y(t_i))\right) \end{aligned} \tag{7.4}$$

Using Taylor polynomial of degree one about $(t, y),$ it can be shown that

$$f\left(t+\alpha h, y+\beta k_1\right) = P_1\left(t+\alpha h, y+\beta k\right) + R_1\left(t+\alpha h, y+\beta k\right),$$

where

$$\begin{aligned} P_1\left(t+\alpha h, y+\beta k_1\right) &= f(t,y) + \left[(t+\alpha h - t)\frac{\partial f}{\partial t}(t,y) + (y+\beta k_1 - y)\frac{\partial f}{\partial y}(t,y)\right] \\ &= f(t,y) + \alpha h f_t(t,y) + \beta k_1 f_y(t,y). \end{aligned}$$

Thus, $f\left(t+\alpha h, y+\beta k_1\right) \approx f(t,y) + \alpha h f_t(t,y) + \beta k_1 f_y(t,y)$ so that substituting k_1 and k_2 into (7.3), we get

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + ahf(t_i, y(t_i)) + bh\left[f(t_i, y(t_i)) + \alpha h f_t(t_i, y(t_i)) + \beta h f_y(t_i, y(t_i)).f(t_i, y(t_i))\right] \\ &= y(t_i) + (a+b)hf(t_i, y(t_i)) + \alpha b h^2 f_t(t_i, y(t_i)) + b\beta h^2 f_y(t_i, y(t_i)).f(t_i, y(t_i)) \end{aligned} \tag{7.5}$$

Comparing (7.4) and (7.5), we have that $a+b=1,\ \alpha b = \dfrac{1}{2}$ and $b\beta = \dfrac{1}{2}.$

We can choose one value arbitrarily and get other values. Some choices can be

- $a=0,\ b=1;\ \alpha = \dfrac{1}{2},\ \beta = \dfrac{1}{2}$ Midpoint method
- $a=\dfrac{1}{2},\ b=\dfrac{1}{2};\ \alpha=1,\ \beta=1$ Modified Euler's method
- $a=\dfrac{1}{3},\ b=\dfrac{2}{3};\ \alpha=\dfrac{3}{4},\ \beta=\dfrac{3}{4}$

All these are special cases of second-order Runge-Kutta methods. Higher-order methods can be derived in a similar way. For example, the fourth-order Runge-Kutta methods are defined by the formulae

$$\begin{aligned}
&y(t_{i+1}) = y(t_i) + ak_1 + bk_2 + ck_3 + dk_4,\\
&k_1 = hf(t_i, y(t_i))\\
&k_2 = hf(t_i + \alpha_0 h, y(t_i) + \beta_0 k_1)\\
&k_3 = hf(t_i + \alpha_1 h, y(t_i) + \beta_1 k_1 + \gamma_1 k_2)\\
&k_4 = hf(t_i + \alpha_2 h, y(t_i) + \beta_2 k_1 + \gamma_2 k_2 + \delta_1 k_3),
\end{aligned}$$

where the choice of the parameters is arbitrary. The most common and widely used fourth-order Runge-Kutta method for approximating the solution $y(t_i) \approx w_i,\ i = 1, 2, ..., N$, to the IVP

$$y'(t) = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

is given as follows:

$$\begin{aligned}
&w_0 = \alpha,\\
&k_1 = hf(t_i, w_i)\\
&k_2 = hf(t_i + \tfrac{h}{2}, w_i + \tfrac{1}{2}k_1)\\
&k_3 = hf(t_i + \tfrac{h}{2}, w_i + \tfrac{1}{2}k_2)\\
&k_4 = hf(t_{i+1}, w_i + k_3),\\
&w_{i+1} = w_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \qquad i = 0, 1, 2, ..., N-1
\end{aligned}$$

Example 7.1.5

Use the Runge-Kutta method of order four with $h = 0.2$ and $N = 10$ to obtain approximations to the solution of the IVP

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Solutions:

With $h = 0.2$, we have $t_0 = 0,\ t_1 = 0.2,\ t_2 = 0.4, ..., t_{10} = 2$ and $f(t, y) = y - t^2 + 1.$

$$\begin{aligned}
&w_0 = 0.5,\\
&k_1 = hf(t_0, w_0) = 0.2(0.5 - 0^2 + 1)\\
&\quad = 0.3\\
&k_2 = hf(t_0 + \tfrac{h}{2}, w_0 + \tfrac{1}{2}k_1) = 0.2f(0 + \tfrac{0.2}{2}, 0.5 + \tfrac{0.3}{2}) = 0.2f(0.1, 0.65)\\
&\qquad = 0.2(0.65 - (0.1)^2 + 1)\\
&\qquad = 0.328
\end{aligned}$$

$$k_3 = hf(t_0 + \tfrac{h}{2}, w_0 + \tfrac{1}{2}k_2) = 0.2f(0 + \tfrac{0.2}{2}, 0.5 + \tfrac{0.328}{2})$$
$$= 0.2f(0.1, 0.664)$$
$$= 0.2(0.664 - (0.1)^2 + 1)$$
$$= 0.3308$$

$$k_4 = hf(t_1, w_0 + k_3) = 0.2f(0.2, 0.5 + 0.3308)$$
$$= 0.2f(0.2, 0.8308)$$
$$= 0.2(0.8308 - (0.2)^2 + 1)$$
$$= 0.35816$$

$$\therefore w_1 = w_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$= 0.5 + \frac{1}{6}\left(0.3 + 2(0.328) + 2(0.3308) + 0.35816\right)$$
$$= 0.8292933.$$

$$k_1 = hf(t_1, w_1) = 0.2f(0.2, 0.8292933)$$
$$= 0.2(0.8292933 - (0.2)^2 + 1)$$
$$= 0.35785866$$

$$k_2 = hf(t_1 + \tfrac{h}{2}, w_1 + \tfrac{1}{2}k_1) = 0.2f(0.2 + \tfrac{0.2}{2}, 0.8292933 + \tfrac{0.35785866}{2})$$
$$= 0.2f(0.3, 1.00822263)$$
$$= 0.2(1.00822263 - (0.3)^2 + 1)$$
$$= 0.383644526$$

$$k_3 = hf(t_1 + \tfrac{h}{2}, w_1 + \tfrac{1}{2}k_2) = 0.2f(0.2 + \tfrac{0.2}{2}, 0.8292933 + \tfrac{0.383644526}{2})$$
$$= 0.2f(0.3, 1.021115563)$$
$$= 0.2(1.021115563 - (0.3)^2 + 1)$$
$$= 0.386223112$$

$$k_4 = hf(t_2, w_1 + k_3) = 0.2f(0.4, 0.8292933 + 0.386223112)$$
$$= 0.2f(0.4, 1.215516412)$$
$$= 0.2(1.215516412 - (0.4)^2 + 1)$$
$$= 0.411103282$$

$$\therefore w_2 = w_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$= 0.8292933 + \frac{1}{6}\left(0.3578587 + 2(0.3836445) + 2(0.3862231) + 0.4111033\right)$$
$$= 1.2140762$$

The remaining results are shown in the table below:

t_i	w_i
0.0	0.5
0.2	0.8292933
0.4	1.2140762
0.6	1.6489220
0.8	2.1272027
1.0	2.6408227
1.2	3.1798942
1.4	3.7323401
1.6	4.2834095
1.8	4.8150857
2.0	5.3053630

△

7.2. SYSTEMS OF ODEs AND HIGHER-ORDER EQUATIONS

We consider an m^{th} – order system of first-order IVP

$$\left.\begin{aligned} \frac{du_1}{dt} &= f_1(t, u_1, u_2, ..., u_n) \\ \frac{du_2}{dt} &= f_2(t, u_1, u_2, ..., u_n) \\ &\vdots \\ \frac{du_m}{dt} &= f_m(t, u_1, u_2, ..., u_n) \end{aligned}\right\} \quad \textbf{(7.6)}$$

for $a \le t \le b,$ with the initial conditions

$$u_1(a) = \alpha_1,\ u_2(a) = \alpha_2, ..., u_m(a) = \alpha_m. \quad \textbf{(7.7)}$$

Definition 7.2.1

The function $f(t, y_1, y_2, ..., y_m),$ defined on the set

$$D = \{(t, u_1, u_2, ..., u_m) : a \le t \le b \text{ and } -\infty < u_i < \infty,\ i = 1, 2, .., m\}$$

is said to satisfy Lipschitz condition on D in the variables $(u_1, u_2, ..., u_m)$ if a constant $L > 0$ exists with

$$\left| f(t, u_1, u_2, ..., u_m) - f(t, z_1, z_2, ..., z_m) \right| \le L \sum_{j=1}^{m} \left| u_i - z_i \right|$$

for all $(t, u_1, u_2, ..., u_m)$ and $(t, u_1, u_2, ..., u_m)$ in D. ◇

Theorem 7.2.1

Suppose that

$$D = \left\{ (t, u_1, u_2, ..., u_m) : a \le t \le b \text{ and } -\infty < u_i < \infty,\ i = 1, 2, .., m \right\}$$

and let $f_i(t, u_1, u_2, ..., u_m)$, for each $i = 1, 2, ..., m$, be continuous and satisfy a Lipschitz condition on D. The system (7.6), subject to the initial condition (7.7), has a unique solution $u_1(t), u_2(t), ..., u_m(t),\ a \le t \le b.$

□

Methods to solve systems of first-order IVPs are generalisations of the methods for a single first-order equation discussed earlier. For example, if $w_{i,j}$ approximates the i^{th} solution $u_i(t)$ at the j^{th} mesh point j, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, N-1$, then the difference equations associated with Euler's method with initial conditions

$$u_1(t_0) = \alpha_1,\ u_2(t_0) = \alpha_2, ..., u_m(t_0) = \alpha_m$$

can be written as follows:

$$\begin{aligned}
&w_{1,0} = \alpha_1,\ w_{2,0} = \alpha_2, ..., w_{m,0} = \alpha_m \\
&w_{1,j+1} = w_{1,j} + hf_1(t_j, w_{1,j}, w_{2,j}, ..., w_{m,j}) \\
&w_{2,j+1} = w_{2,j} + hf_2(t_j, w_{1,j}, w_{2,j}, ..., w_{m,j}) \\
&\vdots \\
&w_{m,j+1} = w_{m,j} + hf_m(t_j, w_{1,j}, w_{2,j}, ..., w_{m,j}).
\end{aligned}$$

Fourth-order Runge-Kutta method can be written as

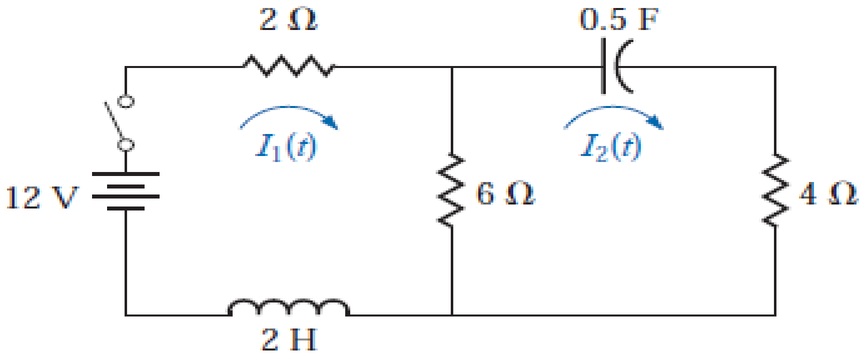
$$k_{1,i} = hf_i(t_j, w_{1,j}, w_{2,j}, \ldots, w_{m,j})$$
$$k_{2,i} = hf_i(t_j + \tfrac{h}{2}, w_{1,j} + \tfrac{1}{2}k_{1,1}, w_{2,j} + \tfrac{1}{2}k_{1,2}, \ldots, w_{m,j} + \tfrac{1}{2}k_{1,m})$$
$$k_{3,i} = hf_i(t_j + \tfrac{h}{2}, w_{1,j} + \tfrac{1}{2}k_{2,1}, w_{2,j} + \tfrac{1}{2}k_{2,2}, \ldots, w_{m,j} + \tfrac{1}{2}k_{2,m})$$
$$k_{4,i} = hf_i(t_j + h, w_{1,j} + k_{3,1}, w_{2,j} + k_{3,2}, \ldots, w_{m,j} + k_{3,m})$$
$$w_{i,j+1} = w_{i,j} + \frac{1}{6}\left(k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4,i}\right),$$

for each $i = 1, 2, \ldots, m.$

Example 7.2.1

The current $I_1(t)$ and $I_2(t)$ in the left and right loop of the circuit shown below satisfy the system of equations

$$I_1' = -4I_1 + 3I_2 + 6, \qquad I_1(0) = 0$$
$$I_2' = -2.4I_1 + 1.6I_2 + 3.6, \quad I_2(0) = 0$$



Use (a) Euler's method (b) Runge-Kutta of order four to this system with $h = 0.1$.

Solution:

(a) $w_{1,0} = 0 = w_{2,0}$,

$f_1(t_j, w_{1,j}, w_{2,j}) = -4w_{1,j} + 3w_{2,j} + 6$ and $f_2(t_j, w_{1,j}, w_{2,j}) = -2.4w_{1,j} + 1.6w_{2,j} + 3.6$

$$I_1(0.1) \approx w_{1,1} = w_{1,0} + h(-4w_{1,0} + 3w_{2,0} + 6) = 0 + 0.1(-4(0) + 3(0) + 6) = 0.6$$
$$I_2(0.1) \approx w_{2,1} = w_{2,0} + h(-2.4w_{1,0} + 1.6w_{2,0} + 3.6) = 0 + 0.1(-2.4(0) + 1.6(0) + 3.6) = 0.36$$
$$I_1(0.2) \approx w_{1,2} = w_{1,1} + h(-4w_{1,1} + 3w_{2,1} + 6) = 0.6 + 0.1(-4(0.6) + 3(0.36) + 6) = 1.068$$
$$I_2(0.2) \approx w_{2,2} = w_{2,1} + h(-2.4w_{1,1} + 1.6w_{2,1} + 3.6) = 0.36 + 0.1(-2.4(0.6) + 1.6(0.36) + 3.6) = 0.6336$$

Other results are shown in the table below:

t_j	$w_{1,j}$	$w_{2,j}$
0.0	0	0
0.1	0.6	0.36
0.2	1.068	0.6336
0.3	1.43088	0.838656
0.4	1.7101248	0.98942976
0.5	1.922903808	1.09730857

(b) $k_{1,1} = hf_1(t_0, w_{1,0}, w_{2,0}) = 0.1(-4w_{1,0} + 3w_{2,0} + 6) = 0.1(-4(0) + 3(0) + 6) = 0.6$

$$k_{1,2} = hf_2(t_0, w_{1,0}, w_{2,0}) = 0.1(-2.4w_{1,0} + 1.6w_{2,0} + 3.6) = 0.1(-2.4(0) + 1.6(0) + 3.6) = 0.36$$

$$\begin{aligned}
k_{2,1} = hf_1(t_0 + \tfrac{h}{2}, w_{1,0} + \tfrac{1}{2}k_{1,1}, w_{2,0} + \tfrac{1}{2}k_{1,2}) &= 0.1 f_1(0.05, 0.3, 0.18) \\
&= 0.1(-4(0.3) + 3(0.18) + 6) \\
&= 0.534
\end{aligned}$$

$$\begin{aligned}
k_{2,2} = hf_2(t_0 + \tfrac{h}{2}, w_{1,0} + \tfrac{1}{2}k_{1,1}, w_{2,0} + \tfrac{1}{2}k_{1,2}) &= 0.1 f_2(0.05, 0.3, 0.18) \\
&= 0.1(-2.4(0.3) + 1.6(0.18) + 3.6) \\
&= 0.3168
\end{aligned}$$

$$\begin{aligned}
k_{3,1} = hf_1(t_0 + \tfrac{h}{2}, w_{1,0} + \tfrac{1}{2}k_{2,1}, w_{2,0} + \tfrac{1}{2}k_{2,2}) &= 0.1 f_1(0.05, 0.267, 0.1584) \\
&= 0.1(-4(0.267) + 3(0.1584) + 6) \\
&= 0.54072
\end{aligned}$$

$$\begin{aligned}
k_{3,2} = hf_2(t_0 + \tfrac{h}{2}, w_{1,0} + \tfrac{1}{2}k_{2,1}, w_{2,0} + \tfrac{1}{2}k_{2,2}) &= 0.1 f_2(0.05, 0.267, 0.1584) \\
&= 0.1(-2.4(0.267) + 1.6(0.1584) + 3.6) \\
&= 0.321264
\end{aligned}$$

$$\begin{aligned}
k_{4,1} = hf_1(t_0 + h, w_{1,0} + k_{3,1}, w_{2,0} + k_{3,2}) &= 0.1 f_1(0.1, 0.54072, 0.321264) \\
&= 0.1(-4(0.54072) + 3(0.321264) + 6) \\
&= 0.4800912
\end{aligned}$$

$$\begin{aligned}
k_{4,2} = hf_2(t_0 + h, w_{1,0} + k_{3,1}, w_{2,0} + k_{3,2}) &= 0.1 f_2(0.1, 0.54072, 0.321264) \\
&= 0.1(-2.4(0.54072) + 1.6(0.321264) + 3.6) \\
&= 0.28162944
\end{aligned}$$

$$\begin{aligned}
\therefore I_1(0.1) \approx w_{1,1} &= w_{1,0} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}) \\
&= 0 + \frac{1}{6}(0.6 + 2(0.534) + 2(0.54072) + 0.4800912) \\
&= 0.5382552
\end{aligned}$$

$$I_2(0.1) \approx w_{2,1} = w_{2,0} + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2})$$
$$= 0 + \frac{1}{6}(0.36 + 2(0.3168) + 2(0.321264) + 0.28162944)$$
$$= 0.31962624$$

$$k_{1,1} = hf_1(t_1, w_{1,1}, w_{2,1}) = 0.1 f_1(0.1, 0.5382552, 0.31962624)$$
$$= 0.1(-4(0.5382552) + 3(0.31962624) + 6)$$
$$= 0.480585792$$
$$k_{1,2} = hf_2(t_1, w_{1,1}, w_{2,1}) = 0.1 f_2(0.1, 0.5382552, 0.31962624)$$
$$= 0.1(-2.4(0.5382552) + 1.6(0.31962624) + 3.6)$$
$$= 0.28195895$$
$$k_{2,1} = hf_1(t_1 + \tfrac{h}{2}, w_{1,1} + \tfrac{1}{2}k_{1,1}, w_{2,1} + \tfrac{1}{2}k_{1,2}) = 0.1 f_1(0.15, 0.778548096, 0.460605715)$$
$$= 0.1(-4(0.778548096) + 3(0.460605715) + 6)$$
$$= 0.426762476$$
$$k_{2,2} = hf_2(t_1 + \tfrac{h}{2}, w_{1,1} + \tfrac{1}{2}k_{1,1}, w_{2,1} + \tfrac{1}{2}k_{1,2}) = 0.1 f_2(0.15, 0.778548096, 0.460605715)$$
$$= 0.1(-2.4(0.778548096) + 1.6(0.460605715) + 3.6)$$
$$= 0.246845371$$
$$k_{3,1} = hf_1(t_1 + \tfrac{h}{2}, w_{1,1} + \tfrac{1}{2}k_{2,1}, w_{2,1} + \tfrac{1}{2}k_{2,2}) = 0.1 f_1(0.15, 0.751636438, 0.443048925)$$
$$= 0.1(-4(0.751636438) + 3(0.443048925) + 6)$$
$$= 0.432260102$$
$$k_{3,2} = hf_2(t_1 + \tfrac{h}{2}, w_{1,1} + \tfrac{1}{2}k_{2,1}, w_{2,1} + \tfrac{1}{2}k_{2,2}) = 0.1 f_2(0.15, 0.751636438, 0.443048925)$$
$$= 0.1(-2.4(0.751636438) + 1.6(0.443048925) + 3.6)$$
$$= 0.250495082$$
$$k_{4,1} = hf_1(t_1 + h, w_{1,1} + k_{3,1}, w_{2,1} + k_{3,2}) = 0.1 f_1(0.2, 0.970515302, 0.570121322)$$
$$= 0.1(-4(0.970515302) + 3(0.570121322) + 6)$$
$$= 0.382830275$$
$$k_{4,2} = hf_2(t_1 + h, w_{1,1} + k_{3,1}, w_{2,1} + k_{3,2}) = 0.1 f_2(0.2, 0.970515302, 0.570121322)$$
$$= 0.1(-2.4(0.970515302) + 1.6(0.570121322) + 3.6)$$
$$= 0.218295739$$

$$\therefore\ I_1(0.2) \approx w_{1,2} = w_{1,1} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$$
$$= 0.5382552 + \frac{1}{6}(0.480585792 + 2(0.426762476) + 2(0.432260102) + 0.382830275)$$
$$= 0.968498737$$

$$I_2(0.2) \approx w_{2,2} = w_{2,1} + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2})$$
$$= 0.31962624 + \frac{1}{6}(0.28195895 + 2(0.246845371) + 2(0.250495082) + 0.218295739)$$
$$= 0.568782172$$

Other values are given in the table below:

t_j	$w_{1,j}$	$w_{2,j}$
0.0	0	0
0.1	0.538255	0.319626
0.2	0.968499	0.568782
0.3	1.310717	0.760733
0.4	1.581263	0.906321
0.5	1.793505	1.014402

△

We now consider a general m^{th} – order initial-value problem

$$y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)}), \quad a \le t \le b,$$

with initial conditions

$$y(a) = \alpha_1, y'(a) = \alpha_2, \ldots, y^{(m-1)}(a) = \alpha_m$$

Letting

$$u_1(t) = y(t), u_2(t) = y'(t), \ldots, \text{and } u_m(t) = y^{(m-1)}(t)$$

we can convert the IVP to a first-order system

$$\frac{du_1}{dt} = \frac{dy}{dt} = u_2, \quad \frac{du_2}{dt} = \frac{dy'}{dt} = u_3, \quad \cdots, \quad \frac{du_{m-1}}{dt} = \frac{dy^{(m-2)}}{dt} = u_m,$$

and

$$\frac{du_m}{dt} = \frac{dy^{(m-1)}}{dt} = y^{(m)} = f(t, y, y', \ldots, y^{(m-1)}) = f(t, u_1, u_2, \ldots, u_m),$$

with initial conditions

$$u_1(a) = y(a) = \alpha_1, \quad u_2(a) = y'(a) = \alpha_2, \quad \ldots, \quad u_m(a) = y^{(m-1)}(a) = \alpha_m.$$

Example 7.2.2

Transform the second-order initial-value problem

$$y'' - 2y' + 2y = e^{2t}\sin t, \qquad \text{for } 0 \le t \le 1, \quad \text{with } y(0) = -0.4,\ y'(0) = -0.6$$

into a system of first order initial-value problems, and use the Runge-Kutta method with $h = 0.1$ to approximate the solution.

Solutions:

Let $u_1(t) = y(t)$ and $u_2(t) = y'(t)$ so that

$$u_1'(t) = u_2(t),$$
$$u_2'(t) = e^{2t}\sin t - 2u_1(t) + 2u_2(t),$$

with initial conditions $u_1(0) = -0.4$ and $u_2(0) = -0.6$. Using the Runge-Kutta method, we get

$$w_{1,0} = -0.4 \text{ and } w_{2,0} = -0.6$$

$$k_{1,1} = hf_1(t_0, w_{1,0}, w_{2,0}) = hw_{2,0} = -0.06,$$
$$k_{1,2} = hf_2(t_0, w_{1,0}, w_{2,0}) = h\left[e^{2t_0}\sin t_0 - 2w_{1,0} + 2w_{2,0}\right] = -0.04,$$
$$k_{2,1} = hf_1\left(t_0 + \frac{h}{2}, w_{1,0} + \frac{1}{2}k_{1,1}, w_{2,0} + \frac{1}{2}k_{1,2}\right) = h\left[w_{2,0} + \frac{1}{2}k_{1,2}\right] = -0.062,$$
$$k_{2,2} = hf_2\left(t_0 + \frac{h}{2}, w_{1,0} + \frac{1}{2}k_{1,1}, w_{2,0} + \frac{1}{2}k_{1,2}\right)$$
$$= h\left[e^{2(t_0+0.05)}\sin(t_0 + 0.05) - 2\left(w_{1,0} + \frac{1}{2}k_{1,1}\right) + 2\left(w_{2,0} + \frac{1}{2}k_{1,2}\right)\right]$$
$$= -0.03247644757,$$
$$k_{3,1} = h\left[w_{2,0} + \frac{1}{2}k_{2,2}\right] = -0.06162832238,$$
$$k_{3,2} = h\left[e^{2(t_0+0.05)}\sin(t_0 + 0.05) - 2\left(w_{1,0} + \frac{1}{2}k_{2,1}\right) + 2\left(w_{2,0} + \frac{1}{2}k_{2,2}\right)\right]$$
$$= -0.03152409237,$$
$$k_{4,1} = h\left[w_{2,0} + k_{3,2}\right] = -0.06315240924,$$

and

$$k_{4,2} = h\left[e^{2(t_0+0.1)}\sin(t_0 + 0.1) - 2(w_{1,0} + k_{3,1}) + 2(w_{2,0} + k_{3,2})\right] = -0.02178637298.$$

Therefore,

$$w_{1,1} = w_{1,0} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}) = -0.4617333423$$

and

$$w_{2,1} = w_{2,0} + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}) = -0.6316312421.$$

Other values of $w_{1,j}$ and $w_{2,j}$ for $j = 0, 1, 2, ..., 10$ are given in the following table:

t_j	$y(t_j) = u_1(t_j)$	$w_{1,j}$	$y'(t_j) = u_2(t_j)$	$w_{2,j}$
0.0	−0.40000000	−0.40000000	−0.6000000	−0.60000000
0.1	−0.46173297	−0.46173334	−0.6316304	−0.63163124
0.2	−0.52555905	−0.52555988	−0.6401478	−0.64014895
0.3	−0.58860005	−0.58860144	−0.6136630	−0.61366381
0.4	−0.64661028	−0.64661231	−0.5365821	−0.53658203
0.5	−0.69356395	−0.69356666	−0.3887395	−0.38873810
0.6	−0.72114849	−0.72115190	−0.1443834	−0.14438087
0.7	−0.71814890	−0.71815295	0.2289917	0.22899702
0.8	−0.66970677	−0.66971133	0.7719815	0.77199180
0.9	−0.55643814	−0.55644290	1.534764	1.5347815
1.0	−0.35339436	−0.35339886	2.578741	2.5787663

△

THE END!