8. INTRODUCTION TO THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

8.1. *COMPLEX NUMBERS AND FUNCTIONS*

Definition 8.1.1

A complex number $z,$ is an ordered pair $(x, y),$ where $x \in \mathbb{R}$ is called the real part of z (denoted $\operatorname{Re}(z)$) and $y \in \mathbb{R}$ is called the imaginary part of z (denoted $\operatorname{Im}(z)$). ◊

If we define $1 \equiv (1,0)$ and $i \equiv (0,1),$ then

$$z = (x, y) = x(1,0) + y(0,1) = x + iy.$$

The set of complex numbers is denoted by $\mathbb{C}.$ If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are complex numbers, then the following operations on $\mathbb{C}$ hold:

- $z_1 \pm z_2 = (x_1 \pm x_2, y_1 \pm y_2).$ The additive identity is $(0,0).$
- $z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$ The multiplicative identity is $(1,0).$
- $\alpha z_1 = (\alpha x_1, \alpha y_1),$ for any scalar $\alpha.$

Note that

$$i^2 = (0,1)(0,1) = (0-1, 0+0) = (-1,0),$$

that is, $i^2 = -1$ implying that $i = \sqrt{-1}.$

Definition 8.1.2

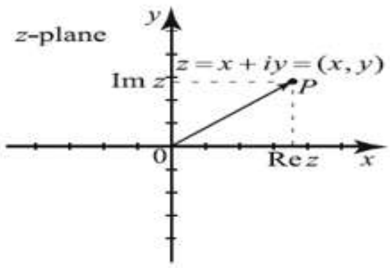
If z is a complex number, then the conjugate of $z,$ denoted by $\bar{z},$ is $\bar{z} = x - iy.$ ◊

Using definition 8.1.2, it can be shown that

- $\dfrac{z_1}{z_2} = \left(\dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \dfrac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \right)$
- $\dfrac{1}{z} = \left(\dfrac{x}{x^2 + y^2}, \dfrac{-y}{x^2 + y^2} \right)$
- $z\bar{z} = x^2 + y^2$
- $\operatorname{Re}(z) = \dfrac{1}{2}(z + \bar{z})$
- $\operatorname{Im}(z) = \dfrac{1}{2i}(z - \bar{z})$

8.1.1. ***The Complex Plane***

We can visualise every element of $\mathbb{C}$ by plotting it as a point on the $xy-$plane. Each complex number $z = x + iy$ corresponds to a point $p(x, y)$.



<u>*Definition 8.1.3*</u>

The absolute value or modulus of a complex number z, denoted $|z|$, is given by

$$|z| = \sqrt{x^2 + y^2}$$

◊

The absolute value of a complex number represents the distance from the point $p(x, y)$ to the origin in the complex plane. It follows that if z_1 and z_2 are complex numbers then $|z_1 - z_2|$ is the distance between z_1 and z_2, i.e.

$$|z_1 - z_2| = \sqrt{(x_1 - x_2) \ + (y_1 - y_2)^2} \ .$$

<u>***Example 8.1.1***</u>

Find and plot all complex numbers z such that

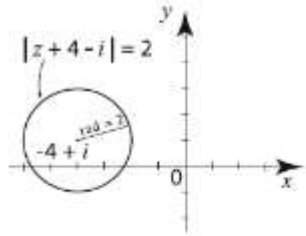
(a) $|z + 4 - i| = 2$ (b) $|z + 4 - i| \le 2$ (c) $|z + 2 + 2i| + |z + 1 + i| = 3\sqrt{2}$

<u>Solutions</u>:

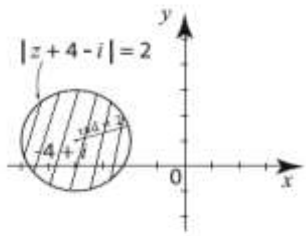
(a) $|z + 4 - i| = 2 \Leftrightarrow |z - (-4 + i)| = 2$

We consider this to be the distance between z and $-4 + i$. Then, z lies on the circle centred at $-4 + i$ with radius $2.$ Note that

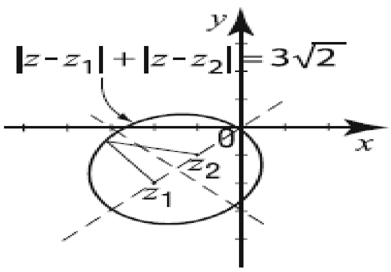
$$\begin{aligned} |z - (-4 + i)| = 2 &\Rightarrow \sqrt{(x+4)^2 + (y-1)^2} = 2 \\ &\Rightarrow (x+4)^2 + (y-1)^2 = 2^2 \end{aligned}$$



(b) z lies on and inside the circle centred at $z_0 = -4 + i$.



(c) Here we have a sum of two distances $|z - z_1| = |z - (-2 - 2i)|$ and $|z - z_2| = |z - (-1 - i)|$, where $z_1 = -2 - 2i$ and $z_2 = -1 - i$. Check that z lies on an ellipse with foci at z_1 and z_2.



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Proposition 8.1.1

Let $z_1, z_2, z_3 \in \mathbb{C}$. Then the following inequalities hold:

(i) $|\operatorname{Re}(z)| \leq |z|$

(ii) $|\operatorname{Im}(z)| \leq |z|$

(iii) $|z| \leq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$

(iv) $|z_1 + z_2| \leq |z_1| + |z_2|$

□

We now go back to figure 1.1. Suppose that OP makes an angle of θ rad with the positive $x-$axis and $r = |z| = \sqrt{x^2 + y^2}$. Then $x = r\cos\theta,\ y = r\sin\theta$ and $\tan\theta = \dfrac{y}{x}$ so that

$$z = x + iy = r(\cos\theta + i\sin\theta) \tag{8.1}$$

Equation (8.1) is called the polar representation of a complex number $z \neq 0$. The angle θ is called an argument of z, denoted $\arg z$. Note that θ plus any multiple of 2π satisfies (8.1). If we restrict the choice of θ to the interval $-\pi < \theta \leq \pi$, then there is a unique value of θ that satisfies (8.1).

Definition 8.1.4

The principle value of the argument of a complex number z, denoted by $Arg\, z$, is the unique number with the following properties:

$$-\pi < Arg\, z \leq \pi, \ \tan(Arg\, z) = \frac{y}{x}.$$

◊

Using Definition 8.1.4, we have that

$$\arg z = \{Arg\, z + 2k\pi : k \in \mathbb{Z}\}$$

Example 8.1.2

Find the polar representation of z in each of the following and state the value of $r, Arg\, z$ and $\arg z$:

(a) $z = -1 + i$ (b) $z = \sqrt{3} + i$ (c) $z = 3$

Solutions:

(a) $z = -1 + i \Rightarrow r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1}(\frac{-1}{1}) = -\frac{\pi}{4}$

$$\therefore Arg\, z = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \quad \text{and} \quad \arg z = \frac{3\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\therefore z = \sqrt{2}\left(\cos\tfrac{3\pi}{4} + i \sin\tfrac{3\pi}{4}\right)$$

(b) $z = \sqrt{3} + i \Rightarrow r = 2, \ \theta = \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$

$$\therefore Arg\, z = \frac{\pi}{6} \quad \text{and} \quad \arg z = \frac{\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\therefore z = 2\left(\cos\tfrac{\pi}{6} + i \sin\tfrac{\pi}{6}\right)$$

(c) $z = 3 \Rightarrow r = 3,\ Arg\, z = 0$ and $\arg z = 2k\pi,\ k \in \mathbb{Z}$ implying that $z = 3(\cos 0 + i \sin 0)$

△

When $z_1, z_2 \in \mathbb{C}$ are in the form(8.1), i.e. $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

- $z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$
- $|z_1 z_2| = r_1 r_2$
- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- $\frac{1}{z} = z^{-1} = \frac{1}{r}\left(\cos(-\theta) + i\sin(-\theta)\right) = \frac{1}{r}\left(\cos\theta - i\sin\theta\right)$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$

If $r = 1$, then $z = \cos\theta + i\sin\theta$ and by induction

$$z^n = \cos(n\theta) + i\sin(n\theta) \qquad \textbf{(8.2)}$$

Equation (8.2) is called the De Moivre's identity and in general

$$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right).$$

Definition 8.1.5

Let $w \neq 0$ be a complex number and $n \in \mathbb{Z}^+$. A number z is called an n^{th} root of w if $z^n = w$.

◊

Using Definition 8.1.5, let $w = \rho\left(\cos\alpha + i\sin\alpha\right)$ and $z = r\left(\cos\theta + i\sin\theta\right)$. Then by De Moivre's identity

$$z^n = w \Leftrightarrow r^n\left(\cos(n\theta) + i\sin(n\theta)\right) = \rho\left(\cos\alpha + i\sin\alpha\right).$$

Thus, $r^n = \rho \Rightarrow r = \rho^{\frac{1}{n}}$ and when two complex numbers are equal, their arguments must differ by $2k\pi$, i.e. $n\theta = \alpha + 2k\pi \Rightarrow \theta = \frac{\alpha + 2k\pi}{n}$, $k = 0, 1, 2, \ldots, n-1$.

Therefore, the n^{th} root of a complex number z is given by

$$z_{k+1} = \rho^{\frac{1}{n}}\left(\cos\left(\tfrac{\alpha+2k\pi}{n}\right) + i\sin\left(\tfrac{\alpha+2k\pi}{n}\right)\right),\ k = 0, 1, 2, \ldots, n-1, \qquad \textbf{(8.3)}$$

where $\alpha = Arg\, w$.

Example 8.1.3

1. Calculate $(1+i)^{11}$
2. Find all numbers z such that

   (a) $z^6 = 1$ (b) $(z+1)^3 = 2+2i$

Solutions:

1. Let $z = 1+i$. Then $r = \sqrt{2}$, $Arg\, z = \tan^{-1}(1) = \dfrac{\pi}{4}$

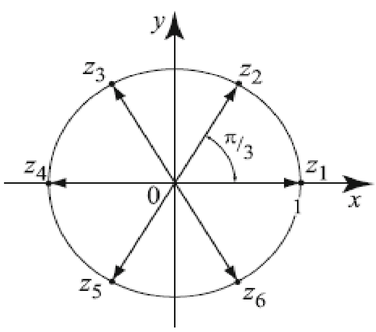
   $\therefore (1+i)^{11} = \left(\sqrt{2}\right)^{11}\left(\cos\left(\frac{11\pi}{4}\right) + i\sin\left(\frac{11\pi}{4}\right)\right)$

2. (a) $z^6 = 1 \Rightarrow z = 1^{\frac{1}{6}}$. Let $w = 1$ and using (8.3), we have that $\rho = 1$ and $\alpha = Arg\, w = 0$ so that

$$z_{k+1} = 1^{\frac{1}{6}}\left(\cos\left(\tfrac{0+2k\pi}{6}\right) + i\sin\left(\tfrac{0+2k\pi}{6}\right)\right),\ k = 0,1,2,3,4,5$$

When $k = 0,\ z_1 = 1$

$$k = 1,\ z_2 = \cos\left(\tfrac{\pi}{3}\right) + i\sin\left(\tfrac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$k = 2,\ z_3 = \cos\left(\tfrac{2\pi}{3}\right) + i\sin\left(\tfrac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$k = 3,\ z_4 = \cos\left(\tfrac{3\pi}{3}\right) + i\sin\left(\tfrac{3\pi}{3}\right) = -1$$
$$k = 4,\ z_5 = \cos\left(\tfrac{4\pi}{3}\right) + i\sin\left(\tfrac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$
$$k = 5,\ z_6 = \cos\left(\tfrac{6\pi}{3}\right) + i\sin\left(\tfrac{6\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$



(b) Let $w = 2+2i$ so that $\rho = \sqrt{2^2+2^2} = \sqrt{8}$ and $\alpha = \dfrac{\pi}{4}$

$$\therefore (z+1)^3 = 2+2i \Leftrightarrow z+1 = (2+2i)^{\frac{1}{3}}$$

$$\Rightarrow z_{k+1}+1 = 8^{\frac{1}{6}}\left(\cos\left(\frac{\frac{\pi}{4}+2k\pi}{3}\right)+i\sin\left(\frac{\frac{\pi}{4}+2k\pi}{3}\right)\right),\ k=0,1,2$$

$$k=0,\ z_1 = -1+8^{\frac{1}{6}}\left(\cos\left(\tfrac{\pi+8(0)\pi}{12}\right)+i\sin\left(\tfrac{\pi+8(0)\pi}{12}\right)\right) = 2^{\frac{1}{2}}\left(\cos\left(\tfrac{\pi}{12}\right)+i\sin\left(\tfrac{\pi}{12}\right)\right)-1$$

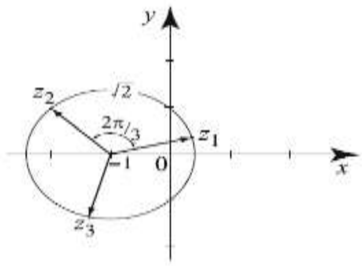
$$k=1,\ z_2 = 2^{\frac{1}{2}}\left(\cos\left(\tfrac{9\pi}{12}\right)+i\sin\left(\tfrac{9\pi}{12}\right)\right)-1 = 2^{\frac{1}{2}}\left(\cos\left(\tfrac{3\pi}{4}\right)+i\sin\left(\tfrac{3\pi}{4}\right)\right)-1$$

$$k=2,\ z_3 = 2^{\frac{1}{2}}\left(\cos\left(\tfrac{17\pi}{12}\right)+i\sin\left(\tfrac{17\pi}{12}\right)\right)-1$$

Expressing $z_i,\ i=1,2,3$ in the form $z_i = a+ib,\ a,b\in\mathbb{R},$ we have that

$z_1 = \left(\sqrt{2}\cos\left(\tfrac{\pi}{12}\right)-1\right)+i\left(\sqrt{2}\sin\left(\tfrac{\pi}{12}\right)\right),\ z_2 = \left(\sqrt{2}\cos\left(\tfrac{3\pi}{4}\right)-1\right)+i\left(\sqrt{2}\sin\left(\tfrac{3\pi}{4}\right)\right)$ and

$z_3 = \left(\sqrt{2}\cos\left(\tfrac{17\pi}{12}\right)-1\right)+i\left(\sqrt{2}\sin\left(\tfrac{17\pi}{12}\right)\right)$



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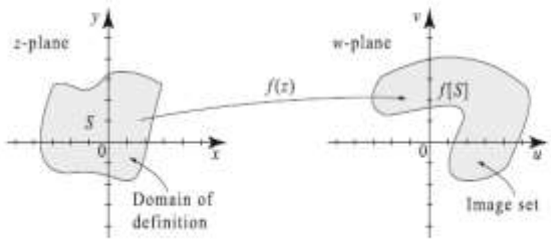
8. 1. 2. *Complex Functions*

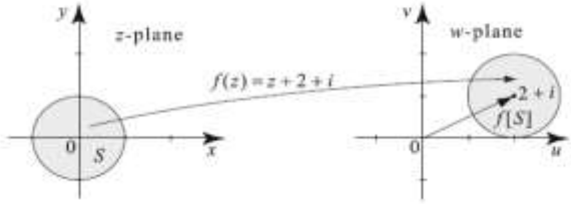
A complex-valued function f is a relation that assigns to each complex number z in a set S a unique complex number $f(z)$. The set S is called the domain of definition of f and the unique number $f(z)$ (sometimes written $w = f(z)$) is called the value of f at z. If we view f as a mapping from $z-$plane to $w-$plane, then $f[S]$ is called the image (or range) of S under f.



Example 8.1.3

1. If $f(z) = 4z^2+2z+1,$ the domain of definition of f is the entire complex plane and the image is also the entire complex plane.

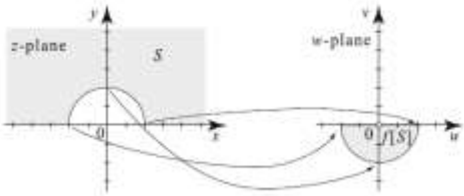
2. If $f(z) = z + 2 + i$ and $S = \{z : |z| \leq 1\}$, then letting $z = x + iy$ we get $f(z) = x + 2 + i(y + 1)$. Thus, the image of S is the set S translated two units to the right and one unit up.



3. If $f(z) = \dfrac{1}{z}$ on $S = \{z : 2 \leq z, 0 \leq \arg z \leq \pi\}$, then as the modulus of z increases from 2 to ∞, the modulus of $\dfrac{1}{z}$ decreases from $\dfrac{1}{2}$ to 0 (but not equal to 0). Letting $z = x + iy$, we get

$$f(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2},$$

where $y \geq 0$. Thus, as argument of z goes from 0 up to π, argument of w goes from 0 up to $-\pi$.



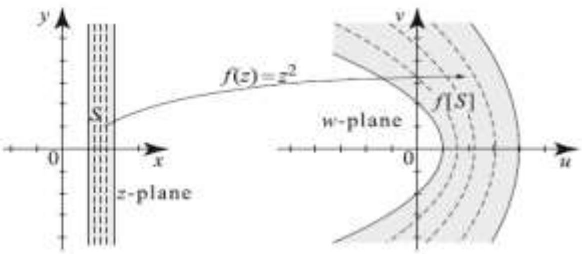
4. If $f(z) = z^2$ on the vertical strip $S = \{z : 1 \leq z \leq 2\}$, then letting $z = x + iy$, we obtain

$$f(z) = (x + iy)^2 = x^2 - y^2 + i2xy.$$

Thus, in the $w-$plane $u = x^2 - y^2$ and $v = 2xy$. Any vertical line $x = a$ gives $u = a^2 - y^2, v = 2ay$, where $1 \leq a \leq b$ and $-\infty \leq y \leq \infty$. Eliminating y we get

$$y = \frac{v}{2a} \Rightarrow u = a^2 - \left(\frac{v}{2a}\right)^2 = a^2 - \frac{v^2}{4a^2}$$

$\Rightarrow v^2 = -4a^2(u - a^2)$, which is a parabola with vertex at $(a^2, 0)$ and $v-$intercepts at $\left(0, \pm 2a^2\right)$.



8.2. ***ELEMENTARY FUNCTIONS***

We consider complex functions involving exponential, logarithmic, trigonometric and hyperbolic functions.

8.2.1. *The Complex Exponential*

Definition 8.2.1

The complex exponential function, denoted $Exp(z)$ or e^z, is defined as

$$e^z = \sum_{n=0}^{\infty} \tfrac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \ldots \text{ for all } z \in \mathbb{C}.$$

◊

If $z = i\theta$, then using Definition 8.2.1, we have that

$$\begin{aligned} e^z = e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} \ldots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \ldots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right) \\ &= \cos\theta + i\sin\theta, \end{aligned}$$

that is,

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \textbf{(8.4)}$$

Equation (8.4) is called Euler's identity. Thus, if $z = x + iy,\ x, y \in \mathbb{R}$, then

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y),$$

$$|e^z| = |e^x(\cos y + i\sin y| = e^x \Rightarrow |e^z| = e^{\mathrm{Re}(z)}$$

and

$$\arg(e^z) = \arg(e^x(\cos y + i\sin y)) = y + 2k\pi,\ k \in \mathbb{Z}.$$

If z is in polar form, i.e. $z = r(\cos\theta + i\sin\theta)$, then using (8.4), we have that $z = re^{i\theta}$

Proposition 8.2.1

Let $z, w \in \mathbb{C}$. Then

(i) $e^{z+w} = e^z.e^w$

(ii) $e^{-z} = \dfrac{1}{e^z}$

(iii) $e^{z-w} = e^z.e^{-w} = \dfrac{e^z}{e^w}$ □

Example 8.2.1

1. For each of the following, compute $e^z, |e^z|$ and $\arg(e^z)$:

(a) $z=3-i\dfrac{\pi}{3}$ (b) $z=i\dfrac{5\pi}{4}$ (c) $z=1+i$ (d) $z=-\pi$

2. Find the exponential form of each of the following:

(a) $z=-7\sqrt{3}+7i$ (b) $z=1+i$

3. Find the image of $f(z)=e^z$ on $S=\{z:-1\le x\le 1,\ 0\le y\le \pi\}$.

Solutions:

1. (a) $e^z=e^{3-\frac{\pi}{3}i}=e^3(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3})=e^3\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$

$$\therefore |e^z|=e^{\mathrm{Re}(z)}=e^3 \text{ and } \arg(e^z)=y+2k\pi=-\frac{\pi}{3}+2k\pi,\ k\in\mathbb{Z}.$$

(b) and (c) Exercise

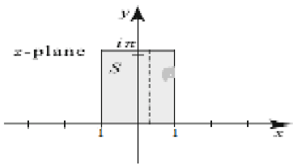
(d) $e^z=e^{-\pi+0i}=e^{-\pi}$, $|e^z|=e^{\mathrm{Re}(z)}=e^{-\pi}$ and $\arg(e^z)=0+2k\pi=2k\pi,\ k\in\mathbb{Z}$.

2. (a) $r=\sqrt{\left(-7\sqrt{3}\right)^2+7^2}=\sqrt{196}=14$ and $\tan^{-1}(\frac{y}{x})=\tan^{-1}(\frac{7}{-7\sqrt{3}})=-\dfrac{\pi}{6}$

$$\therefore \theta=Arg\,z=-\frac{\pi}{6}+\pi=\frac{5\pi}{6}\Rightarrow z=re^{i\theta}=14e^{i\frac{5\pi}{6}}$$

(b) Exercise

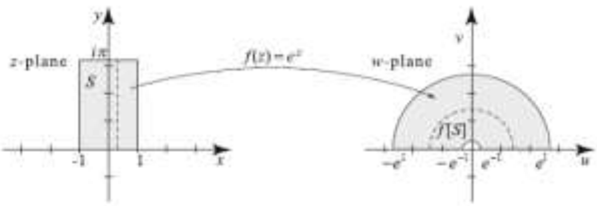
3. Note that S is a rectangular area



Consider any vertical line $x=a,\ -1\le a\le 1$. Then,

$$f(z)=e^a(\cos y+i\sin y)\ \Rightarrow u=e^a\cos y,\ v=e^a\sin y\ \Rightarrow u^2+v^2=\left(e^a\right)^2$$

which is a circle centred at $(0,0)$ with radius e^a. Since $-1\le a\le 1$, it follows that $e^{-1}\le e^a\le e^1$.



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8.2.2. ***The Complex Logarithms and Powers***

To define complex logarithm $w = \log z$, set $w = \log z \Rightarrow e^w = z$. Expressing w and z as $w = u + iv$ and $z = re^{i\theta}$, we get

$$e^u . e^{iv} = re^{i\theta} \Rightarrow e^u = r \text{ and } e^{iv} = e^{i\theta}$$

$$\Rightarrow u = \ln r \text{ and } v = \theta + 2k\pi = \arg z,$$

where $\ln r$ is the usual natural logarithm. Thus,

$$\log z = \ln |z| + i \arg z, \quad z \neq 0 \qquad \textbf{(8.5)}$$

Definition 8.2.2

The principal value or principal branch of the complex logarithm, denoted $Log\, z$, is defined by

$$Log\, z = \ln |z| + i\, Arg\, z, \quad z \neq 0.$$

◊

Example 8.2.2

1. Evaluate the following:

   (a) $\log(1+i)$ (b) $\log(-2)$

2. Evaluate the following:

   (a) $Log(1+i)$ (b) $Log\left(e^{6\pi i}\right)$

Solutions:

1. (a) Let $z = 1 + i$. Then $|z| = \sqrt{2}$ and $\tan^{-1}(1) = \dfrac{\pi}{4}$

   $\therefore \log z = \log(1+i) = \ln\sqrt{2} + i\left(\dfrac{\pi}{4} + 2k\pi\right), \; k \in \mathbb{Z}$

   (b) Letting $z = -2$ gives

   $$\log z = \log(-2) = \ln|-2| + i(\pi + 2k\pi) = \ln 2 + i(\pi + 2k\pi), \; k \in \mathbb{Z}.$$

2. (a) From part 1(a), we get $Arg\, z = \dfrac{\pi}{4} \Rightarrow Log(1+i) = \ln\sqrt{2} + i\dfrac{\pi}{4}.$

   (b) Letting $z = e^{6\pi i} = 1$ gives $Arg\, z = 0$ so that

   $$Log\, z = Log\left(e^{6\pi i}\right) = \ln 1 + 0i = 0.$$

△

Recall that the logarithmic function given in (8.5) is not single-valued because $\arg z$ takes on a different value in a specified range. In fact, for every real number $\alpha,$ we can specify that $\alpha < \arg z \le \alpha + 2\pi.$ We can define the α^{th} branch of $\log z,$ denoted $\log_\alpha z,$ by the identity

$$\log_\alpha z = \ln|z| + i\arg_\alpha z,$$

where $\log_\alpha z \in (\alpha, \alpha + 2\pi).$

NOTE: When $\alpha = -\pi,$ we get the principal value of the logarithm.

Definition 8.2.3

For any non-zero complex number $z,$ we define the complex power as

$$z^a = e^{a\log z},$$

where $\log z$ is as defined in (8.5). If we choose the principal logarithm, then

$$z^a = e^{a\,Log\,z}.$$

◊

Example 8.2.3

1. Evaluate the following using the principal branch of the logarithm:

(i) $(-i)^{1+i}$ (ii) $(-1)^i$

2. Find the solution of the equation $z^{1+i} = 4.$

Solutions:

1. (i) Let and $z = -i$. Then, $Log\, z = \ln 1 + i\left(-\frac{\pi}{2}\right) = -i\dfrac{\pi}{2}$ so that

$$z^a = (-i)^{1+i} = e^{(1+i)\left(-i\frac{\pi}{2}\right)} = e^{\left(-i\frac{\pi}{2}+\frac{\pi}{2}\right)} = e^{\frac{\pi}{2}}\left(\cos\tfrac{\pi}{2} - i\sin\tfrac{\pi}{2}\right) = -ie^{\frac{\pi}{2}}$$

(ii) Let $a = i$ and $z = -1$. Then, $Log\, z = \ln|-1| + i\pi = i\pi$.

$\therefore\ z^a = e^{i(i\pi)} = e^{-\pi}.$

2. $z^{1+i} = 4 \Rightarrow z = 4^{\frac{1}{1+i}} = 4^{\frac{1}{2}-i\frac{1}{2}}$.

Since $\log 4 = \ln 4 + i2k\pi$, we have that

$$\begin{aligned}
z = 4^{\frac{1}{2}-i\frac{1}{2}} &= e^{\left(\frac{1}{2}-i\frac{1}{2}\right)\log 4} = e^{\left(\frac{1}{2}-i\frac{1}{2}\right)(\ln 4 + i2k\pi)} = e^{\left(\frac{1}{2}-i\frac{1}{2}\right)(2\ln 2 + i2k\pi)} = e^{(\ln 2 + k\pi) - i(\ln 2 - k\pi)} \\
&= e^{(\ln 2 + k\pi)}\left(\cos(\ln 2 - k\pi) - i\sin(\ln 2 - k\pi)\right) \\
&= e^{\ln 2}.e^{k\pi}(-1)^k\left(\cos(\ln 2) - i\sin(\ln 2)\right), \qquad \text{by Trigonometric identities} \\
&= (-1)^k.2e^{k\pi}\left(\cos(\ln 2) - i\sin(\ln 2)\right).
\end{aligned}$$

We can also solve this equation by taking logs on both sides.

△

8.2.3. *The Complex Trigonometric and Hyperbolic Functions*

By Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{i}$$

$$\Rightarrow e^{-i\theta} = \cos\theta - i\sin\theta \tag{ii}$$

Solving (i) and (ii) simultaneously for $\cos\theta$ and $\sin\theta$, we obtain

$$\begin{aligned}
\cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
\sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}
\end{aligned} \tag{8.6}$$

We can also write $\cos z$ as

$$\begin{aligned}
\cos z &= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{-y+ix} + e^{y-ix}}{2} \\
&= \frac{e^{-y}(\cos x + i\sin y) + e^{y}(\cos - i\sin y)}{2} \\
&= \cos x\left(\frac{e^{y} + e^{-y}}{2}\right) - i\sin x\left(\frac{e^{y} - e^{-y}}{2}\right) \\
&= \cos x\cosh y - i\sin x\sinh y,
\end{aligned}$$

i.e.

$$\cos z = \cos x\cosh y - i\sin x\sinh y. \tag{8.7}$$

Similarly,

$$\sin z = \sin x\cosh y + i\cos x\sinh y \tag{8.8}$$

Definition 8.2.4

For any a complex number z, the hyperbolic cosine and sine are defined as

$$\cosh z = \frac{e^z + e^{-z}}{2} = \cosh x \cos y + i \sinh x \sin y$$

and

$$\sinh z = \frac{e^z - e^{-z}}{2} = \sinh x \cos y + i \cosh x \sin y$$

◊

Proposition 8.2.2

Let $z, z_1, z_2 \in \mathbb{C}$. Then

(i) $\cos(-z) \equiv \cos z$ and $\sin(-z) \equiv -\sin z$

(ii) $\tan z \equiv \dfrac{\sin z}{\cos z}$, provided $\cos z \neq 0$

(iii) $e^{iz} \equiv \cos z + i \sin z$

(iv) $\cos^2 z + \sin^2 z \equiv 1$

(v) $\cos(z_1 + z_2) \equiv \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

(vi) $\sin(z_1 + z_2) \equiv \sin z_1 \cos z_2 + \sin z_2 \cos z_2$

(vii) $\cos^2 z \equiv \dfrac{1 + \cos 2z}{2}$

(viii) $\sin^2 z \equiv \dfrac{1 - \cos 2z}{2}$

□

Proposition 8.2.3

Let $z, z_1, z_2 \in \mathbb{C}$. Then

(i) $\cosh(iz) \equiv \cos z$ and $\cos(iz) \equiv \cosh z$

(ii) $\sinh(iz) \equiv i \sin z$ and $\sin(iz) \equiv i \sinh z$

(iii) $\cosh^2 z - \sinh^2 z \equiv 1$

(iv) $\cosh z \equiv \cosh x \cos y + i \sinh x \sin y$

(v) $\sinh z \equiv \sinh x \cos y + i \cosh x \sin y$

(vi) $\tanh(iz) \equiv i \tan z$ and $\tan(iz) \equiv i \tanh z$

(vii) $\coth(iz) \equiv -i \cot z$ and $\cot(iz) \equiv -i \coth z$

□

Example 8.2.4

Compute $\cos z$, $\sin z$, $\cosh z$ and $\sinh z$ for each of the following:

(a) $z = 2 + i\pi$ (b) $z = i\,\frac{5\pi}{4}$

Solutions:

(a) $z = 2 + i\pi \quad \Rightarrow \cos z = \cos(2+i\pi) = \cos 2 \cosh \pi - i \sin 2 \sinh \pi$

$$\sin z = \sin(2+i\pi) = \sin 2 \cosh \pi + i \cos 2 \sinh \pi$$
$$\cosh z = \cos h(2+i\pi) = \cosh 2 \cos \pi + i \sinh 2 \sin \pi = -\cosh 2$$
$$\sinh z = \sin h(2+i\pi) = \sinh 2 \cos \pi + i \cosh 2 \sin \pi = -\sinh 2$$

(b) Exercise △

8.2.4. *Inverse Trigonometric and Hyperbolic Functions*

If $w = \sin^{-1} z$, then $z = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$

$$\Rightarrow e^{iw} - e^{-iw} - 2iz = 0$$
$$\Rightarrow \left(e^{iw}\right)^2 - 2ize^{-iw} - 1 = 0$$
$$\Rightarrow e^{iw} = iz \pm \sqrt{1-z^2}$$
$$\Rightarrow iw = \log\left(iz \pm \sqrt{1-z^2}\right),$$
$$\text{i.e.} \quad w = \sin^{-1} z = -i \log\left(iz \pm \sqrt{1-z^2}\right)$$

Similarly,

$$\arccos z = -i \log\left(z \pm \sqrt{z^2-1}\right)$$
$$\arctan z = \frac{i}{2} \log\left(\frac{1-iz}{1+iz}\right), \qquad z \neq \pm i$$
$$\cosh^{-1} z = \log\left(z \pm \sqrt{z^2-1}\right)$$
$$\sinh^{-1} z = \log\left(z \pm \sqrt{z^2+1}\right)$$
$$\tanh^{-1} z = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right),$$

where specific branches of a square root and logarithmic function are used.

<u>Example 8.2.5</u>

Evaluate

(a) $\arcsin(i)$ (b) $\cosh^{-1}(1+i)$

using the principle argument.

<u>Solutions</u>:

(a) $\arcsin z = -i\log\left(iz \pm \sqrt{1-z^2}\right) \Rightarrow \arcsin(i) = -i\log\left(i^2 \pm \sqrt{1-i^2}\right) = -i\log\left(-1\pm\sqrt{2}\right)$

Let $w_1 = -1+\sqrt{2}$ and $w_2 = -1-\sqrt{2}$. Then, $Arg\ w_1 = 0$ and $Arg\ w_2 = \pi$ so that

$$Log\ w_1 = \ln\left|-1+\sqrt{2}\right| \quad \text{and} \quad Log\ w_2 = \ln\left|-1-\sqrt{2}\right| + i\pi,$$

i.e. $\arcsin(i) = -i\ln\left|-1+\sqrt{2}\right|$ or $\arcsin(i) = -i\left(\ln\left|-1-\sqrt{2}\right| + i\pi\right) = \pi - i\ln\left|1+\sqrt{2}\right|$

(b)

$\cosh^{-1} z = \log\left(z \pm \sqrt{z^2-1}\right) \Rightarrow \cosh^{-1}(1+i) = \log\left((1+i) \pm \sqrt{(1+i)^2 - 1}\right) = \log\left(1+i\pm\sqrt{-1+2i}\right)$

Let $w = -1+2i$. Then, $Arg\ w = \pi - 1.107 = 2.03$. Thus,

$Log\, w = \ln\sqrt{5} + 2.03i = \frac{1}{2}\ln 5 + 2.03i$ and

$$\sqrt{-1+2i} = e^{\frac{1}{2}Log\, w} = e^{\frac{1}{2}\left(\frac{1}{2}\ln 5 + 2.03i\right)} = e^{\ln\sqrt[4]{5} + 1.015i} = \sqrt[4]{5}\cos 1.015 + i\ \sqrt[4]{5}\sin 1.015$$

$$\begin{aligned}
\therefore \quad \cosh^{-1}(1+i) = Log\left(1+i+\sqrt[4]{5}\cos 1.015 + i\ \sqrt[4]{5}\sin 1.015\right) &= Log\left(1+i+0.789+1.270\ i\right) \\
&= Log\left(1.789 + 2.270\ i\right) \\
&= \ln 2.890 + 0.903\ i
\end{aligned}$$

$$\begin{aligned}
\text{Or} \quad \cosh^{-1}(1+i) &= Log\left(1+i-\left(\sqrt[4]{5}\cos 1.015 + i\ \sqrt[4]{5}\sin 1.015\right)\right) \\
&= Log\left(1+i-\left(0.789+1.270\ i\right)\right) \\
&= Log\left(0.211 - 0.247\ i\right) \\
&= \ln 0.325 - 0.864\ i.
\end{aligned}$$

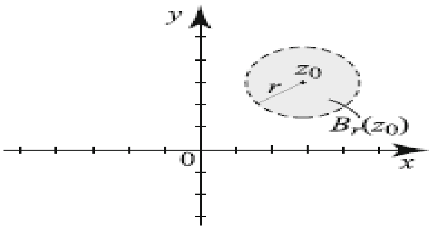
△

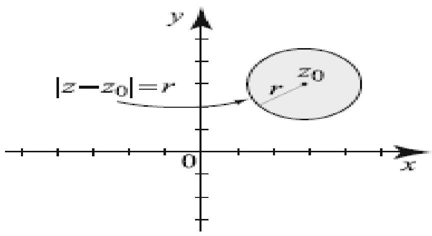
8.3. ***ANALYTIC FUNCTIONS***

Before we develop the theory of functions of a complex variable, we give basic properties of subsets of the complex plane.

Definition 8.3.1

Let $r > 0$ be a positive real number and z_0 be a point in the plane. The $r-$neighbourhood of z_0, denoted $B_r(z_0)$, is the set of all complex numbers z satisfying $|z - z_0| < r$. It is sometimes called open disc.



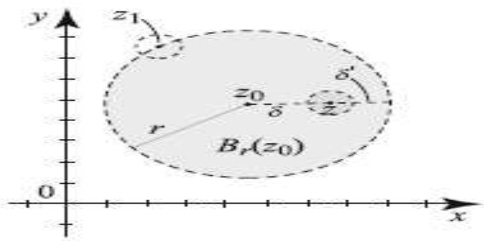


Open disk Closed disk

◊

Definition 8.3.2

Let S be a subset of $\mathbb{C}$. A point z_0 in S is called an interior point of S if we can find a neighbourhood of z_0 that is wholly contained in S. A point z in the complex plane is called a boundary point of S if every neighbourhood of z contains at least one point in S and at least one point not in S. The set of all boundary point of S is called the boundary of S.



z is an interior point of $B_r(z_0)$ while z_1 is a boundary point

◊

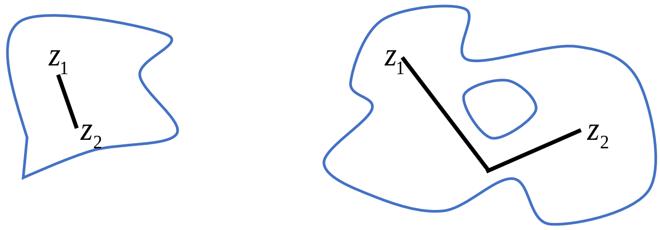
Definition 8.3.3

A subset S of the complex number is called open if every point in S is an interior point of S.

◊

Definition 8.3.4

A set S is connected if every pair of points z_1 and z_2 in S can be joined by an unbroken line consisting of a series of straight lines joining end-to-end each lying entirely within S.



◊

Definition 8.3.5

1. Let f be a complex-valued function defined on a subset S. We say that a complex number L is the limit of f as z approaches z_0 and write

$$\lim_{z\to z_0} f(z) = L \quad \text{or} \quad f(z)\to L \text{ as } z\to z_0,$$

if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$z\in S \text{ and } 0<|z-z_0|<\delta \Rightarrow |f(z)-L|<\varepsilon.$$

2. The function f is continuous at z_0 if and only if $f(z_0)$ exists and

$$\lim_{z\to z_0} f(z) = f(z_0).$$

◊

Proposition 8.3.1

(i) If the limit of a function f exists at a point z_0, then it is unique.

(ii) If $f(z) = u(x,y) + i\,v(x,y)$ and $z_0 = x_0 + i\,y_0$, then

$$\lim_{z\to z_0} f(z) = w_0 = u_0 + i\,v_0, \text{ if and only if}$$

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0.$$

(iii) $\lim_{z\to z_0}\left(f(z)\pm g(z)\right) = \lim_{z\to z_0} f(z) \pm \lim_{z\to z_0} g(z)$

(iv) $\lim_{z\to z_0}\left(cf(z)\right) = c\lim_{z\to z_0} f(z)$

(v) $\lim_{z\to z_0}\left(f(z).g(z)\right) = \lim_{z\to z_0} f(z).\lim_{z\to z_0} g(z)$

□

Example 8.3.1

1. Evaluate the following limits:

(a) $\lim_{z\to(1+2i)}\left(z^2 - z\right)$ (b) $\lim_{z\to 0}\left(\frac{\overline{z}}{z}\right)$ (c) $\lim_{z\to\infty}\frac{2z+3i}{z^2+z+1}$

2. Determine the point where $f(z)=\frac{z^2+4}{z^2+9}$ is not continuous.

Solutions:

1. (a) Using Proposition 8.3.1, let $z = x + i\,y$. Then,

$$f(z)=(x+i\,y)^2-(x+i\,y)=\left(x^2-x-y\right)+i\left(2xy-y\right)$$

$\therefore\ u(x,y)=x^2-x-y^2$ and $v(x,y)=2xy-y$

$\Rightarrow \lim_{(x,y)\to(1,2)}u(x,y)=\lim_{(x,y)\to(1,2)}\left(x^2-x-y^2\right)=-4$ and $\lim_{(x,y)\to(1,2)}v(x,y)=\lim_{(x,y)\to(1,2)}\left(2xy-y\right)=2$

$\therefore\ \lim_{z\to(1+2i)}\left(z^2-z\right)=-4+2i$

Note also that, we can evaluate the limit 'directly'.

$$\lim_{z\to(1+2i)}\left(z^2-z\right)=(1+2i)^2-(1+2i)=-4+2i\,.$$

(b) If $z = x + i\,y$, then

$$\frac{\overline{z}}{z}=\frac{x-i\,y}{x+i\,y}\times\frac{x-i\,y}{x-i\,y}=\frac{x^2-2xy\,i-y^2}{x^2+y^2}.$$

Note that

$$\lim_{(x,y)\to(0,0)}\left(\frac{x^2-y^2}{x^2+y^2}\right) \text{ and } \lim_{(x,y)\to(0,0)}\left(\frac{-2xy}{x^2+y^2}\right)$$

do not exist along different paths. Therefore, $\lim_{z\to 0}\left(\frac{\overline{z}}{z}\right)$ does not exist.

(c) $\lim_{z\to\infty}\frac{2z+3i}{z^2+z+1}=\lim_{z\to\infty}\frac{\frac{2}{z}+\frac{3i}{z^2}}{1+\frac{1}{z}+\frac{1}{z^2}}=\frac{0+0}{1+0+0}=\frac{0}{1}=0$

2. The function $f(z)=\frac{z^2+4}{z^2+9}$ is not continuous at point(s) where $z^2+9=0 \Rightarrow z=\pm 3i$.

△

Definition 8.3.6

A complex-valued function f is said to be differentiable at $z = z_0$ in a domain D if

$$\lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)=\lim_{\Delta z\to 0}\left(\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\right)$$

exists and is denoted by $f'(z_0)$. If f is differentiable at every point of the domain D, then f is said to be analytic in D. A function analytic on the whole complex plane is called an entire function.

◊

Theorem 8.3.1 (Cauchy-Riemann Equations)

Suppose that $f = u + iv$ is analytic on a domain D. Then throughout D, we have

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}. \qquad \textbf{(8.9)}$$

□

Corollary 8.3.1

If $f(z) = u + iv$ and the partial derivatives are continuous on D and satisfy Cauchy-Riemann equations (8.9), then f is analytic on D.

□

Remark 8.3.1

Cauchy-Riemann equations (8.9) imply that

$$f'(z)=u_x+iv_x \quad \textit{or} \quad f'(z)=v_y-iu_y.$$

□

Example 8.3.2

1. Show that

   (a) $f(z)=e^z$ (b) $f(z)=\sin z$

   are entire functions.

2. Show that

   (a) $f(z)=\bar{z}$ (b) $f(x+iy)=x^2+i(2y+x)$

   are not analytic on $\mathbb{C}$.

3. Determine the set on which the following functions are analytic and compute their complex derivatives:

   (a) $f(z)=\dfrac{1}{z+1}$ (b) $f(z)=|z|^2$

Solutions:

1. (a) $f(z)=e^z=e^x(\cos y+i\sin y)\Rightarrow u(x,y)=e^x\cos y$ and $v(x,y)=e^x\sin y$.

Using Cauchy-Riemann equations, we have that

$$\frac{\partial u}{\partial x}=e^x\cos y,\ \frac{\partial u}{\partial y}=-e^x\sin y,\ \frac{\partial v}{\partial x}=e^x\sin y \quad \text{and} \quad \frac{\partial v}{\partial y}=e^x\cos y$$

$$\therefore\ \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Therefore, $f(z)=e^z$ is entire.

(b) $f(z)=\sin z=\sin x\cosh y+i\cos x\sinh y$

$\Rightarrow u_x(x,y)=\cos x\cosh y;\ u_y(x,y)=\sin x\sinh y;\ v_x(x,y)=-\sin x\sinh y$ and $v_y(x,y)=\cos x\cosh y$

$$\therefore\ u_x(x,y)=v_y(x,y) \quad \text{and} \quad u_y(x,y)=-v_x(x,y)$$

Therefore, $f(z)=\sin z$ is entire.

2. (a) $f(z)=\overline{z}=x-iy \quad \Rightarrow \frac{\partial u}{\partial x}=1,\ \frac{\partial u}{\partial y}=0,\ \frac{\partial v}{\partial x}=0$ and $\frac{\partial v}{\partial y}=-1$. Since $\therefore\ \frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$, the function $f(z)=\overline{z}$ is not analytic on the whole $\mathbb{C}$.

(b) $f(x+iy)=x^2+i(2y+x) \quad \Rightarrow u(x,y)=x^2$ and $v(x,y)=2y+x$.

$$\Rightarrow u_x(x,y)=2x;\ u_y(x,y)=0;\ v_x(x,y)=1 \quad \text{and} \quad v_x(x,y)=2.$$

Clearly, $u_x\neq v_y$ and $u_y\neq -v_x$ implying that $f(x+iy)=x^2+i(2y+x)$ is not analytic on the whole $\mathbb{C}$.

3. (a) $f(z)=\dfrac{1}{z+1}$ is analytic everywhere except at $z=-1$ and $f'(z)=\dfrac{-1}{(z+1)^2}$.

(b) $f(z)=|z|^2=x^2+y^2+0i \quad \Rightarrow u(x,y)=x^2+y^2$ and $v(x,y)=0$

$$\Rightarrow\ u_x(x,y)=2x;\ u_y(x,y)=2y;\ v_x(x,y)=0=v_y(x,y).$$

Therefore, Cauchy-Riemann equations are only satisfied at $z=0$ and at that point $f'(z)=0$.

$\triangle$

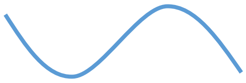
8.4. ***COMPLEX INTEGRATION***

The integral of a complex function $f(z)$ with respect to the complex variable z involves integrating a function $f(z)$ along a curve C in the complex plane. The curves we generally consider are unbroken paths in the complex plane. If these paths are of finite length, we call them arcs.

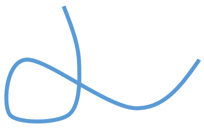
Definition 8.4.1

1. A smooth arc is an arc that does not intersect itself.

Smooth arc

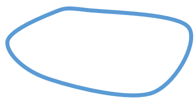


Smooth arc

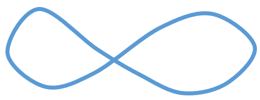


Not smooth arc

2. A smooth closed curve is a closed curve that consists of only one loop.



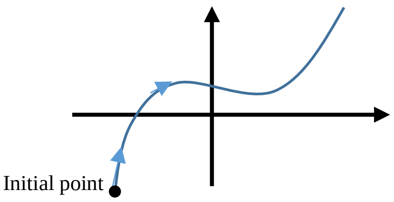
Smooth closed curve



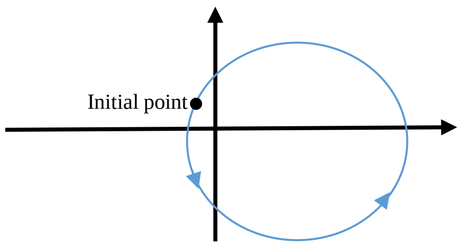
Not smooth closed curve ◊

In short, smooth curves fall into two categories: smooth arcs, which have distinct endpoints, and smooth closed curves, whose endpoints coincide.

For any smooth arc or smooth closed curve, we can specify which endpoint is the initial point thereby specifying the ordering of points. In this case, we have a directed smooth arc or a directed smooth closed curve.



Directed smooth arc



Directed smooth closed curve

Definition 8.4.2

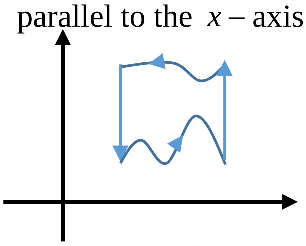
A contour Γ is either a single point z_0 or finite sequence of directed smooth curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for each $k = 1, 2, \ldots, n-1$. That is, $\Gamma = \gamma_1, \gamma_2, \ldots, \gamma_n$. ◊

NOTE: 1. A single directed smooth curve is a contour with $n = 1$.

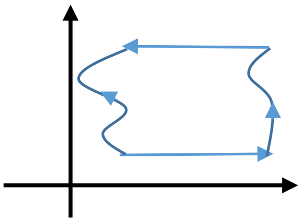
2. Some nonsmooth closed curves can be broken into smooth pieces.

Definition 8.4.3

1. A region Ω_x is said to be $x-$simple region if any line in Ω_x drawn parallel to the $y-$axis cuts Ω_x only twice. Similarly, a region Ω_y is $y-$simple if any line in Ω_y drawn parallel to the $x-$axis cuts Ω_y only twice.

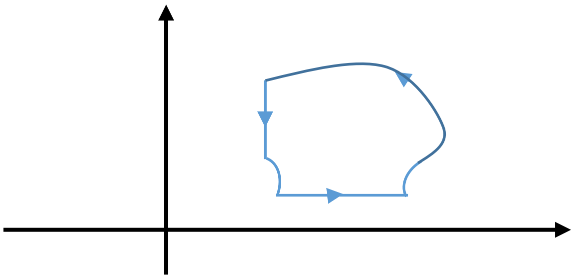


$x-$simple



$y-$simple

2. A region Ω which is both $x-$simple and $y-$simple is called a simple region.

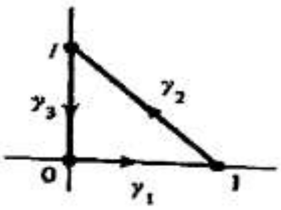


Simple region ◊

Example 8.4.1

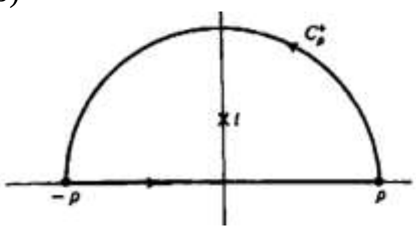
1. Find an admissible parametrisation of each of the following smooth curves:
   (a) C from $z=2-2i$ to $z=2+2i$
   (b) the circle of radius 2 centred at $1-i$.
2. Parametrise the contours given below:

(a)



$0 \le t \le 1$

(b)



Solutions:

1. (a) C is a vertical straight line from $z=2-2i$ to $z=2+2i$ implying that

$$z(y)=2+i\,y, \qquad -2 \le y \le 2.$$

   (b) Using the angle θ as the parameter, we have that

$$z(\theta)=1-i+2e^{i\theta}, \quad 0 \le \theta \le 2\pi.$$

2. (a) Since the initial point of γ_1 is $(0,0)$ and the terminal point is $(1,0)$, we have that

$$\gamma_1 : z_1(t) = t, \quad 0 \le t \le 1.$$

Similarly, taking into consideration the direction, we note that γ_2 is a straight line $y = 1 - x$ implying that $x = 1 - y$. Letting $y = t$ gives $x = 1 - t$.

$$\therefore \gamma_2 : z_2(t) = 1 - t + it, \quad 0 \le t \le 1$$

and

$$\gamma_3 : z_3(t) = (1-t)i, \quad 0 \le t \le 1.$$

Therefore,

$$z(t) = \begin{cases} z_1(t) = t \\ z_2(t) = 1 - t + it \\ z_3(t) = (1-t)i \end{cases}$$

for $0 \le t \le 1$.

(b) Note that the contour is a semicircle centred at the origin with radius ρ. Thus,

$$z(\theta) = \begin{cases} \rho e^{i\theta} \\ -\rho \cos\theta \end{cases}$$

for $0 \le t \le \pi$.

△

8.4.1. *Line Integrals*

A popular method for evaluating complex line integrals consists of breaking everything up into real and imaginary parts. Writing $f(z)$ as $f(z) = u(x,y) + i\,v(x,y)$, where $z = x + iy$ gives

$$\begin{aligned}\int_C f(z)dz &= \int_C \left[u(x,y) + i\,v(x,y)\right] d\left(x + iy\right) \\ &= \int_C u(x,y)dx - v(x,y)dy + i\int_C v(x,y)dx + u(x,y)dy \qquad \textbf{(8.10)}\end{aligned}$$

<u>NOTE</u>: Evaluation of (8.10) depends on the specified path.

From the definition of the line integral, we have the following properties:

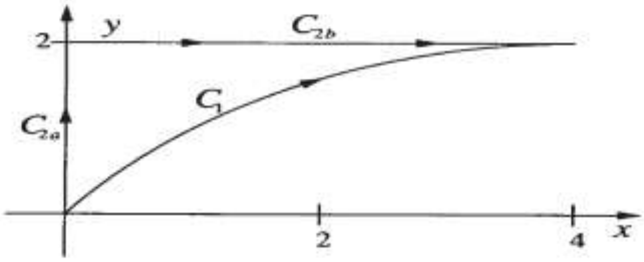
1. $\int_C f(z)\,dz = -\int_{C'} f(z)\,dz$

   where C' is the contour taken in the opposite direction of C.

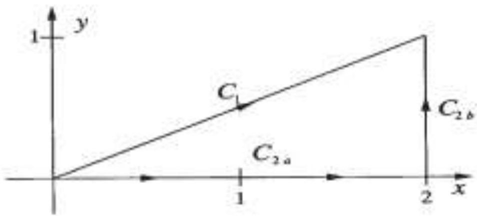
2. $\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$

<u>Example 8.4.2</u>

1. Evaluate $\int_C \bar{z}\,dz$ from $z=0$ to $z=4+2i$ along two contours C_1 (consisting of the curve $x=y^2$) and C_2 (consisting of C_{2a} and C_{2b})



2. Evaluate $\int_C z^2\,dz$ from $z=0$ to $z=2+i$ along two contours C_1 and C_2 (consisting of C_{2a} and C_{2b})



<u>Solutions</u>:

1. Parametrising $x=y^2$, we have that letting $y=t$ implies $x=t^2$ and $z=x+iy=t^2+it$.

   The point $z=0$ corresponds to $y=t \Rightarrow t=0$ or $x=t^2 \Rightarrow t=0$ and point $z=4+2i$ corresponds to $y=t \Rightarrow t=2$ or $x=t^2 \Rightarrow t=2$.

   Thus, along C_1,

$$\begin{aligned}\int_{C_1} \bar{z}\,dz = \int_{C_1} (t^2-it)\,d(t^2+it) &= \int_{C_1} t^2dx+tdy+i\int_{C_1} -tdx+t^2dy \\ &= \int_{C_1} t^2(2t)dt+tdt+i\int_{C_1} -t(2t)dt+t^2dt \\ &= \int_0^2 (2t^3+t)dt+i\int_0^2 -t^2dt \\ &= \frac{1}{2}(t^4+t^2)\Big|_0^2+i\left(\frac{-t^3}{3}\right)\Big|_0^2 \\ &= 10-\frac{8}{3}i.\end{aligned}$$

Along C_{2a}, $x=0,\ 0\le y\le 2 \Rightarrow z=iy,\ 0\le y\le 2$.

$$\therefore \int_{C_{2a}} \overline{z}\, dz = \int_{C_{2a}} -iy\, d\left(iy\right) = \int_{C_{2a}} -i(-y)\, dy + i\int_{C_{2a}} -y\, dx$$
$$= \int_0^2 ydy$$
$$= 2$$

Along C_{2b}, $0\le x\le 4,\ y=2 \Rightarrow z=x+2i.$

$$\therefore \int_{C_{2b}} \overline{z}\, dz = \int_{C_{22}} (x-2i)\, d\left(x+2i\right) = \int_{C_{2a}} x\, dx + 2dy\ +\ i\int_{C_{2a}} -2\, dx + xdy$$
$$= \int_0^4 xdx + i\int_0^4 -2dx$$
$$= 8-8i.$$

Thus,

$$\int_{C_2} \overline{z}\, dz = \int_{C_{2a}} \overline{z}\, dz + \int_{C_{2b}} \overline{z}\, dz = 2+8-8i = 10-8i.$$

2. Along C_1, $y=\dfrac{x}{2}$ and letting $t=y$ implies that $x=2t \Rightarrow z=2t+t\,i.$ The point $z=0$ and $z=2+i$ correspond to $t=0$ and $t=1.$

$$\therefore \int_{C_1} z^2 dz = \int_{C_1} \left(2t+t\,i\right)^2 d\left(2t+t\,i\right) = \int_{C_1} 3t^2(2dt) - 4t^2dt + i\int_{C_1} 4t^2(2dt) + 3t^2dt$$
$$= \int_0^1 2t^2dt + i\int_0^1 11t^2dt$$
$$= \frac{2}{3}t^3\Big|_0^1 + i\ \frac{11}{3}t^3\Big|_0^1$$
$$= \frac{2}{3} + \frac{11}{3}i$$

Along C_{2a}, $y=0,\ 0\le x\le 2 \Rightarrow z=x,\ 0\le x\le 2$.

$$\therefore \int_{C_{2a}} z^2\, dz = \int_0^2 x^2 dx = \frac{8}{3}$$

Along C_{2b}, $x=2,\ 0\le y\le 1 \Rightarrow z=2+iy.$

$$\therefore \int_{C_{2b}} z^2\, dz = \int_{C_{22}} (4-y^2+4yi)\, d\left(2+iy\right) = \int_{C_{2a}} \left(4-y^2\right)dx - 4ydy\ +\ i\int_{C_{2a}} 4y\, dx + \left(4-y^2\right)dy$$
$$= \int_0^1 -4y\, dy + i\int_0^1 \left(4-y^2\right)dy$$
$$= -2 + \frac{11}{3}i.$$

Thus,

$$\int_{C_2} z^2\, dz = \int_{C_{2a}} z^2\, dz + \int_{C_{2b}} z^2\, dz = \frac{8}{3} - 2 + \frac{11}{3}i = \frac{2}{3} + \frac{11}{3}i.$$

△

Note that the integrand for question 1 of example 8.4.2 contains a nonanalytic point along and inside the region enclosed by two curves and that integration along two different paths gives different results. Note also that since the integrand for question 2 is entire, the result is the same along different paths. In general, if the integrand is nonanalytic, then integration depends on the path chosen and if the integrand is analytic, then integration can be evaluated along any path.

Theorem 8.4.1

Let $f(z)$ be analytic in a simply connected domain D. Then, if γ is any arc lying entirely in D with initial point z_0 and terminal point z, then the antiderivative

$$F(z) = \int_\gamma f(\xi)\,d\xi = \int_{z_0}^{z} f(\xi)\,d\xi$$

is a single-valued analytic function of z independent of γ and such that $F'(z) = f(z)$.

□

Theorem 8.4.2

Let $f(z)$ be analytic in a simply connected domain D and $F(z)$ be an antiderivative of $f(z)$. Then, for any two points z_0 and z_1 in D

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0).$$

□

Example 8.4.3

Evaluate each of the following:

(a) $\int_1^{2+3i} \sinh 3z\,dz$ (b) $\int_{1-i}^{1+3i} e^{-z}\,dz$

Solutions:

(a) The integrand $f(z) = \sinh 3z$ is analytic in the finite $z-$plane.

$$\int_1^{2+3i} \sinh 3z\,dz = \frac{1}{3}\cosh 3z\Big|_1^{2+3i} = \frac{1}{3}\left[\cosh(6+9i) - \cosh 3\right] = \frac{1}{3}\left(\cosh 6\cos 9 - \cosh 3\right) + \frac{1}{3}i\sinh 6\sin 9.$$

(b) Note that $f(z) = e^{-z}$ is entire. Thus,

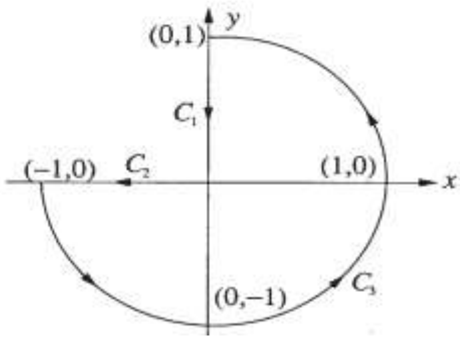
$$\int_{1-i}^{1+3i} e^{-z}\,dz = -e^{-z}\Big|_{1-i}^{1+3i} = -\left(e^{-(1+3i)} - e^{-(1-i)}\right) = e^{-1}\left[\left(\cos 1 - \cos 3\right) + i\left(\sin 1 + \sin 3\right)\right]$$

△

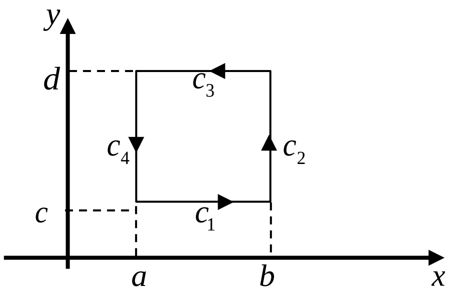
We now consider line integrals involving closed contours. These line integrals will be denoted by $\oint$.

<u>Example 8.4.4</u>

1. Evaluate $\oint_C z\,dz$ along C given below:



2. Evaluate $\oint_C \sin z\,dz$ along C given below:



3. Evaluate $\oint_C \dfrac{1}{z - z_0}\,dz,$ where C is a directed contour $|z - z_0| = R,\ R > 0.$

<u>Solutions</u>:

1. Along C_1, $z = iy$

$$\int_{C_1} z\,dz = \int_1^0 -y\,dy = \frac{1}{2}$$

Along C_2, $z = x$

$$\int_{C_2} z\,dz = \int_0^{-1} x\,dx = \frac{1}{2}$$

Along $C_3,\ z=e^{i\theta},\ -\pi\le\theta\le\frac{\pi}{2}$

$$\int_{C_3} z\,dz=\int_{-\pi}^{\frac{\pi}{2}} e^{i\theta}\,d\left(e^{i\theta}\right)=\int_{-\pi}^{\frac{\pi}{2}} ie^{i\theta}.e^{i\theta}\,d\theta=i\int_{-\pi}^{\frac{\pi}{2}} e^{2\theta i}\,d\theta=\frac{i}{2i}e^{2\theta i}\Bigg|_{-\pi}^{\frac{\pi}{2}}=-1$$

$$\therefore\oint z\,dz=\int_{C_1} z\,dz+\int_{C_2} z\,dz+\int_{C_3} z\,dz=\frac{1}{2}+\frac{1}{2}-1=0.$$

2. Recall that if $z=x+iy,$ then $\sin z=\sin x\cosh y+i\cos x\sinh y.$

   Along $C_1, a\le x\le b,\ y=c$

   $$\int_{C_1}\sin z\,dz=\int_a^b\sin x\cosh c\,dx+i\int_a^b\cos x\sinh c\,dx=\cosh c\left(\cos a-\cos b\right)+i\sinh c\left(\sin b-\sin a\right)$$

   Along $C_2,\ x=b,\ c\le y\le d,$

   $$\int_{C_2}\sin z\,dz=\int_c^d-\cos b\sinh y\,dy+i\int_c^d\sin b\cosh y\,dy=\cos b\left(\cosh c-\cosh d\right)+i\sin b\left(\sinh d-\sinh c\right)$$

   Since C_3 is taken in the opposite direction of C_1 and C_4 is taken in the opposite direction of $C_2,$ we have that

   $$\int_{C_3}\sin z\,dz=-\int_{C_1}\sin z\,dz\quad\text{and}\quad\int_{C_4}\sin z\,dz=-\int_{C_2}\sin z\,dz.$$

   Therefore,

   $$\begin{aligned}\oint_C\sin z\,dz&=\int_{C_1}\sin z\,dz+\int_{C_2}\sin z\,dz+\int_{C_3}\sin z\,dz+\int_{C_4}\sin z\,dz\\&=\int_{C_1}\sin z\,dz+\int_{C_2}\sin z\,dz-\int_{C_3}\sin z\,dz-\int_{C_4}\sin z\,dz\\&=0.\end{aligned}$$

3. Parametrising $\left|z-z_0\right|<R,$ we get

   $$z-z_0=\mathrm{Re}^{i\theta}\Rightarrow z=z_0+\mathrm{Re}^{i\theta},\ 0\le\theta\le2\pi$$

   $$\therefore\oint_C\frac{1}{z-z_0}dz=\int_0^{2\pi}\frac{i\,\mathrm{Re}^{i\theta}}{z_0+\mathrm{Re}^{i\theta}-z_0}d\theta=i\int_0^{2\pi}d\theta=2\pi i.$$

   △

8.4.2. *The Cauchy-Goursat Theorem*

The Cauchy-Goursat theorem helps to determine the value of the integral, in some instances, without resorting to the elementary evaluations.

Definition 8.4.4

The point z_0 is said to be a singular point or singularity of $f(z)$ if $f(z)$ is not analytic at z_0, but is analytic in at least part of every neighbourhood of z_0.

z_0 is called an isolated singularity if $f(z)$ is analytic in every neighbourhood of z_0, except at z_0.

◊

Example 8.4.5

Determine the singular points of each of the following:

(a) $f(z)=\dfrac{2(z+1)}{(z^2+3)^2(z^2-1)}$ (b) $f(z)=\dfrac{1}{\sin\left(\frac{1}{z}\right)}$

Solutions:

(a) $f(z)$ has isolated singularities at $(z^2+3)^2(z^2-1)=0$, that is, at $z=\pm\sqrt{3}\,i$ and $z=\pm 1$.

(b) $f(z)$ has isolated singularity at $\sin\left(\frac{1}{z}\right)=0 \Rightarrow \dfrac{1}{z}=n\pi$ or $z=\dfrac{1}{n\pi}$, $n\in\mathbb{Z}\,/\,\{0\}$, and also at $z=0$, which is an accumulation point of $z=\dfrac{1}{n\pi}$.

△

Theorem 8.4.3 (Green's Theorem for a Simple Region)

Let Ω be a simple region in the $xy-$plane with a boundary Υ that is traversed such that the area of Ω lies to the left as Υ is traversed in the positive sense. Then, if P and Q together with their derivatives are continuous is Ω and on Υ, then

$$\iint_\Omega\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\int_\Upsilon P\,dx+Q\,dy$$

□

Using Theorem 8.4.3, we notice that if $f(z)=u(x,y)+i\,v(x,y)$, in a simply connected domain D, then

$$\oint_C f(z)\;dz=\oint_C u(x,y)\,dx-v(x,y)\,dy+i\oint_C v(x,y)\,dx+u(x,y)\,dy,$$

where C is a smooth closed curve. Assuming that $f(z)$ is analytic and that $f'(z)$ is continuous in D, then

$$\begin{aligned}\oint_C f(z)\;dz&=-\iint_D\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)dxdy+i\iint_D\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dxdy\\&=0,\end{aligned}$$

Since $u(x, y)$ and $v(x, y)$ satisfy Cauchy-Riemann equations. Thus, we have the following theorem:

Theorem 8.4.4 *(Cauchy-Goursat Theorem)*

If $f(z)$ is analytic in a simply connected domain D and also on its boundary C which is a smooth closed curve, then

$$\oint_C f(z)\,dz = 0.$$

□

Example 8.4.6

We can use the Cauchy-Goursat theorem to evaluate integrals for question 1 and 2 of example 8.4.3. Note that in each case, $f(z)$ is entire and so

$$\int_C z\,dz = 0 \text{ and } \int_C \sin z\,dz = 0\,.$$

But question 3 had $f(z) = \dfrac{1}{z - z_0}$, which has a singularity at $z = z_0$ in the circle $|z - z_0 = R|$. Thus,

$$\oint_C \frac{1}{z - z_0}\,dz = 2\pi i \neq 0.$$

△

Example 8.4.7

Evaluate each of the following:

1. $\displaystyle\oint_C \frac{(4-2i)z^2 + (2-5i)z + 3 - 2i}{(z^2+1)(z+2)}\,dz,$

   where C is the directed contour (i) $|z - i| = 1$ (ii) $|z + 2| = 1$.

2. $\displaystyle\oint_C \frac{e^{-z}}{z - \frac{\pi}{2}}\,dz,$ where C is a unit circle centred at the origin.

Solutions:

1. The singularities if $f(z)$ are $z = \pm i$ and $z = -2$. Decomposing $f(z)$ into partial fractions, we get

$$f(z) = \frac{-i}{z - -i} + \frac{1-i}{z+i} + \frac{3}{z+2}.$$
$$\therefore \oint_C f(z)\,dz = -i\oint_C \frac{1}{z-i}\,dz + (1-i)\oint_C \frac{1}{z+i}\,dz + 3\oint_C \frac{1}{z+2}\,dz.$$

(i) Since $z=-i$ and $z=-2$ lie outside the circle $|z-i|=1,$ we have that

$$\oint_C \frac{1}{z+i}\,dz = 0 = \oint_C \frac{1}{z+2}\,dz.$$

$$\therefore \oint_C f(z)\,dz = -i\left(2\pi i\right) = 2\pi.$$

(ii) Only $z=-2$ lies inside the circle $|z+2|=1.$ Thus,

$$\oint_C f(z)\,dz = -i(0) + (1-i)(0) + 3\left(2\pi i\right) = 6\pi i.$$

2. Note that the only singularity $z=\dfrac{\pi}{2}$ lies outside $|z|=1.$

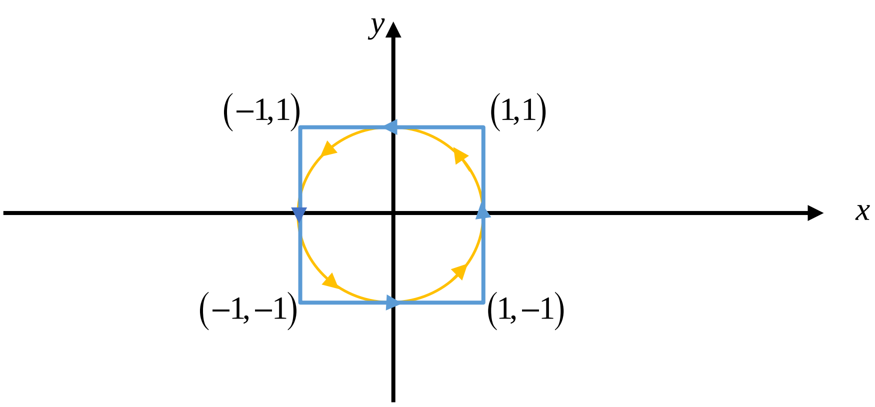
$$\oint_C \frac{e^{-z}}{z-\frac{\pi}{2}}\,dz = 0.$$

$\triangle$

<u>The Principle of Deformation of Contour</u>: The value of a line integral of an analytic function around any simple closed contour remains unchanged if we deform the contour in such a manner that we do not pass over a nonanalytic point.

<u>Example 8.4.8</u>

Consider the integral of $f(z)=\dfrac{1}{z}$ around the contour consisting of a square centred at the origin with vertices $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$. Note that direct integration of $\oint_C \dfrac{dz}{z}$ is very cumbersome. We can deform the original contour into a circle centred at the origin with radius 1.



In this case, we get the same answer as long as the deformed contour encloses the singularity of $f(z)$. Thus,

$$\oint_C \frac{dz}{z} = \oint_C \frac{dz}{z-0} = 2\pi i.$$

△

8.4.3. *The Cauchy's Integral Formula*

Suppose that g is defined as

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0}, & z \neq z_0 \\ f'(z), & z = z_0 \end{cases}$$

where $z_0 \in D$ and not on C. Then, g is continuous in D and the point z_0 is a removable singularity. Also, g is analytic in D except at $z = z_0$. Thus, by Cauchy-Goursat theorem

$$\begin{aligned} &\oint_C g(z)\,dz = 0. \\ \Rightarrow &\oint_C \frac{f(z) - f(z_0)}{z - z_0}\,dz = 0 \\ \Rightarrow &\oint_C \frac{f(z)}{z - z_0}\,dz = \oint_C \frac{f(z_0)}{z - z_0}\,dz = f(z_0)\oint_C \frac{1}{z - z_0}\,dz. \end{aligned}$$

Assuming that z_0 lies inside C gives

$$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0).$$

Thus, we have the following theorem:

<u>*Theorem 8.4.5*</u> *(Cauchy Integral Formula)*

Let $f(z)$ be analytic in a domain D and let C be a smooth closed contour in D taken in the positive sense. Let z_0 be a point in D not on C. Then,

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz \qquad \textbf{(8.11)}$$

□

Example 8.4.9

Evaluate each of the following:

1. $\oint_C \frac{z^2+3z-1}{(z-1)(z+1-i)}dz$, where C is $\left|z+\frac{1}{2}-\frac{3}{2}i\right|=1$.

2. $\frac{1}{2\pi i}\oint_C \frac{\cos(\pi z)}{z^2-1}dz$, where C is a rectangle with corners at $\pm i$ and $2\pm i$.

Solutions:

1. Note that only $z_0=-1+i$ lies inside C. Letting

$$f(z)=\frac{z^2+3z-1}{z-1}$$

implies that $f(-1+i)=\frac{(-1+i)^2+3(-1+i)-1}{-1+i-1}=\frac{9}{5}+\frac{2}{5}i.$

$$\therefore \quad \oint_C \frac{z^2+3z-1}{(z-1)(z+1-i)}dz=2\pi i\, f(-1+i)=2\pi i\left(\frac{9}{5}+\frac{2}{5}i\right)=-\frac{4\pi}{5}+\frac{18\pi}{5}i.$$

2. Note that only $z_0=1$ lies inside C.

$$\therefore \quad \frac{1}{2\pi i}\oint_C \frac{\cos(\pi z)}{z^2-1}dz=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-1}dz,$$

where

$$f(z)=\frac{\cos(\pi z)}{z+1}\Rightarrow f(1)=\frac{\cos(\pi)}{1+1}=-\frac{1}{2}.$$

$$\therefore \quad \frac{1}{2\pi i}\oint_C \frac{\cos(\pi z)}{z^2-1}dz=\frac{1}{2\pi i}(2\pi i)\, f(1)=-\frac{1}{2}.$$

△

By taking n derivatives of **(8.11)**, we can extend the Cauchy's Integral Formula as the next theorem shows:

Theorem 8.4.6

Let $f(z)$ be analytic in a domain D and let C be a smooth closed contour within D taken in the positive sense. Then, for all points z_0 interior C,

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz, \quad n=0,1,2,....$$

□

Example 8.4.10

Evaluate

(a) $\oint_C \frac{e^z}{(z-1)^2(z-3)}\,dz$ (b) $\oint_C \frac{z^2}{(z-1)^4}\,dz$ (c) $\oint_C \frac{z^3}{(z+i)^3}\,dz$

on $|z| = 2.$

Solutions:

(a) $\oint_C \frac{e^z}{(z-1)^2(z-3)}\,dz = \oint_C \frac{f(z)}{(z-1)^2}\,dz,$ where $f(z) = \frac{e^z}{z-3}.$

$$\therefore \oint_C \frac{f(z)}{(z-1)^2}\,dz = \frac{2\pi i}{1!}\,f'(1) = 2\pi i\left(\frac{e^1(1-3)-e^1}{(1-3)^2}\right) = -\frac{3\pi e^1}{2}\,i.$$

(b) Since only $z_0 = 1$ lies inside $C,$ letting $f(z) = z^2$ implies that

$$\oint_C \frac{z^2}{(z-1)^4}\,dz = \frac{2\pi i}{3!}\,f'''(1) = 0.$$

(c) Let $f(z) = z^3.$ Then, $f'(z) = 3z^2$ and $f''(z) = 6z.$ Since $z_0 = -i,$ we have

$$\oint_C \frac{z^3}{(z+i)^3}\,dz = \frac{2\pi i}{2!}\,f''(-i) = i\pi\big(6(-i)\big) = -6\pi.$$

△

THE END!