The University of Zambia
Department of Mathematics and Statistics
MAT 4119 - Engineering Mathematics III
Tutorial Sheet 2

April 2023

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1. For each of the following functions f, show that there is a root to the given equation $f(x) = 0$ on the given interval. Hence, approximate the root to 5 significant figures.
   (a) $f(x) = (x - 0.3)(x - 0.5)$ on $[0.31, 1.00]$
   (b) $f(x) = 3x + \sin x - e^x$ on $[0, 1]$
   (c) $f(x) = 2x - \log_{10} x - 7$ on $[3, 4]$
   (d) $f(x) = x - 2^{-x}$ on $[0, 1]$.
   (e) $f(x) = 4x^2 - 1 - e^{\frac{x^2}{2}}$ on $[1, 3]$

2. (a) By sketching
   $$y = \cos x \text{ and } y = x^3 - 1$$
   on the same coordinate system, show that $f(x) = \cos x - x^3 + 1 = 0$ in some interval.
   (b) Using part (a), find the interval where the root lies.
   (c) Use busection method to find an approximate value of the solution to $f(x) = 0$ in the interval found in part (b).

3. (a) By sketching the graphs of
   $$y = 4 \sin x \text{ and } y = e^x$$
   on the same coordinate system, find the number of solutions to the equation
   $$f(x) = 4e^{-x} \sin x - 1 = 0$$
   in the interval $[0, \pi]$.
   (b) Show that one zero of $f(x) = 0$ lies on the interval $[0, 0.5]$.
   (c) Hence, use bisection method to perform five iterations to approximate the root of $f(x) = 0$ in the interval $[0, 0.5]$.

4. Find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of each of the following equations in the given interval while using the bisection method:
   (a) $x^3 + x - 4 = 0$, in $[1, 4]$
   (b) $x^3 - x - 1 = 0$ in $[1, 2]$
   (c) $2x^6 - 5x^4 + 2 = 0$ in $[0, 1]$
   (d) $x \sin x - 1 = 0$ in $[0, 2]$.

5. For each of the following, the equation $f(x) = 0$ has been written in the form $x = g(x)$. Determine which ones converge to a fixed-point in the specified interval:
   (a) $g(x) = \dfrac{5}{x^2} + 2$ on $[2.5, 3]$.
   (b) $g(x) = \pi + 0.5 \sin\left(\dfrac{x}{2}\right)$ on $[0, 2\pi]$.
   (c) $g(x) = 2^{-x}$ on $\left[\dfrac{1}{3}, 1\right]$.
   (d) $g(x) = \dfrac{2 + x^2 - e^x}{3}$ on $[0, 1]$.

6. Determine the root of $f(x) = 2x^3 - 11.7x^2 + 17.7x - 5$ using fixed-point iteration with 3 iterations and $x_0 = 3$.

7. Most functions can be rearranged in several ways to give $x = g(x)$ with which to begin the fixed-point method. For $f(x) = e^x - 2x^2 = 0$, one $g(x)$ is $x = \pm\sqrt{\frac{e^x}{2}}$.
   (a) Show that this converges to the root near 1.5 if the positive value is used and to the root near -0.5 if the negative value is used.
   (b) There is a third root near 2.6. Show that we do not converge to this root even though values near the root such as $x_0 = 2.5$ or $p_0 = 2.7$ are used to begin the iteration.
   (c) Find another rearrangement that does converge correctly to the third root.

8. Here are three different $g(x)$ functions. All are rearrangements of the same $f(x)$.

$$g(x) = \frac{4 + 2x^3}{x^2} - 2x, \qquad g(x) = \sqrt{\frac{4}{x}}, \qquad g(x) = \frac{16 + x^3}{5x^2}.$$

   (a) What is $f(x)$?
   (b) Are there starting values for which one or more of these converge or diverge?

9. Use the Newton-Raphson method and the Secant method to find solutions accurate to within 10^{-4} for each of the following problems, starting with the given x_0.
   1. $x^3 + 3x^2 - 1 = 0, \quad x_0 = -3.$
   2. $4e^{-x}\sin x - 1 = 0, \quad x_0 = 0.336.$
   3. $\ln(x-1) + \cos(x-1) = 0, \quad x_0 = 1.3.$
   4. $2\sin x - 2^{\frac{x}{4}} - 1 = 0, \quad x_0 = -5.$

10. Newton-Raphson method is to be applied for approximating a root of the nonlinear equation $x^4 - x - 10 = 0$.
    (a) How many solutions of the nonlinear equation are there in $[1, \infty)$? Are they simple?
    (b) Find an interval $[1, b]$ that contains the smallest positive solution of the nonlinear equation.
    (c) Compute five iterations of Newton-Raphson method, for each of the initial guesses $x_0 = 1, \quad x_0 = 2, \quad x_0 = 100$. What are your observations?

11. Find the multiplicity of each given zero of $f(x) = 0$. Hence, use the Modified Newton-Raphson's method to approximate the same zero starting with the given x_0.
    (a) $f(x) = (x+1)^3, \ x = -1, \ x_0 = 0.$ (b) $f(x) = (x-1)(e^{x-1} - 1), \ x = 1, \ x_0 = 0.$
    (c) $f(x) = \left(x - \frac{\pi}{2}\right)^2 (\cot x - 1), \ x = \frac{\pi}{4}, \ x_0 = \frac{\pi}{6}.$