The University of Zambia

School of Natural Sciences

Department of Mathematics & Statistics

2021/2022 Academic Year Final Examinations

MAT 4119 - Engineering Mathematics III

Time allowed : **Three (3)** hours **Full marks : 100**

Instructions:

- Indicate your **computer number** on all answer booklets.
- There are seven (7) questions in this examination. Attempt **any five (5)** questions. All questions carry equal marks.
- The use of non-programmable calculators is allowed.
- **Full credit** will only be given when **all necessary work** is shown.

This question paper consists of 5 pages.

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1. **(a)** Define each of the following errors:

   (i) Truncation error [2]

   (ii) Relative error [2]

   (b) Estimate $\left(\frac{1}{25}\right)^{\frac{1}{5}}$ to 2 significant figures using the Taylor series

   $$x^{\sqrt{x}} = 1+(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{8}(x-1)^3+\frac{1}{24}(x-1)^4-\frac{1}{128}(x-1)^5+\dots$$

   and taking 0.525306 as the true value. [7]

   (c) The data below shows the results of a tensile test of a steel specimen, where y is elongation in thousands of inches that resulted when the tensile force was x thousands of pounds:

x	0.8	1.6	3.1	4.4	6.3
y	2.8	4.9	6.5	8.1	8.8

   Fit an exponential function to the data. [9]

Turn Over/...

⋆ 2. (a) Given the function

$$f(x) = x^5 + 4x + 2,$$

(i) show that f has a root in the interval $[-1, 0]$. [2]

(ii) Determine the number of iterations necessary to approximate the root of f by bisection method with accuracy within 10^{-5} in the interval $[-1, 0]$. [4]

(b) An engineer needs $4800\,m^3$, $5810\,m^3$ and $5690\,m^3$ of sand, fine gravel and coarse gravel, respectively, at a construction site. There are three sources where these materials can be obtained and the composition of the material from these sources is given below:

Source	*%Sand*	*%Fine Gravel*	*%Coarse Gravel*
1	52	30	18
2	20	50	30
3	25	20	55

(i) Form a system of linear equations for this problem. [1]

(ii) Solve the system in part (i) by performing three iterations of the Gauss-Seidel method to determine the number of cubic metres that must be hauled from each source in order to meet the engineer's needs, starting with $X^{(0)} = (0, 0, 0)^t$. [6]

(c) Given that

$$\cosh\left(\sqrt{x}\right) = 1 + \frac{x^2}{8} + \frac{x^4}{384} + \cdots,$$

the Maclaurin polynomial

$$P_4(x) = 1 + \frac{x^2}{8} + \frac{x^4}{384}$$

can be used to approximate

$$\int_0^1 \cosh\left(\sqrt{x}\right)\,dx.$$

Find an upper bound of the error involved in making such an approximation.

[7]

Turn Over/...

3. (a) The equation

$$3^x \cos x - \frac{1}{2} = 0$$

can be written in the form $x = g(x)$, where $g(x) = \arccos\left(\frac{3^{-x}}{2}\right)$.

(i) Show that $g(x)$ converges to a unique fixed-point in the interval $[0, \frac{\pi}{2}]$. [5]

(ii) Starting with $p_0 = 1.5$, perform two iterations to approximate the solution to the equation. [2]

(b) Given that $f(z) = e^z$, where $z = x + iy,\ x, y \in \mathbb{R}$, is defined on

$$S = \{z \in \mathbb{C} : -1 \leq x \leq 1,\ 0 \leq y \leq \pi\},$$

(i) find the image of f. [4]

(ii) Hence sketch the image of f. [2]

(c) Determine an upper bound of the error involved in using the difference formula to differentiate

$$f(x) = \sin\left(\sqrt{x}\right)$$

at $x_0 = 2.1$ with $h = 0.05$. [7]

4. (a) A rod subject to an axial load will be deformed, resulting in a stress-strain curve, where stress, s, is in kips per square inch $\left(10^3\ lb/in^2\right)$ and strain, e, is dimensionless. The table below shows values of e and s:

e	0.02	0.05	0.10	0.15
s	40.0	37.5	43.0	52.0

Use Lagrange interpolating polynomial to estimate s when $e = 0.12$. [7]

(b) Starting with $X^{(0)} = (x_0, y_0)^t = (0, 0)^t$, carry out a single iteration of the Newton's method to solve the following system of non-linear equations:

$$x + y - \arcsin x = 0$$
$$y + \arccos y = x$$

[6]

(c) Use the Runge-Kutta method of order four with $h = 0.1$ to solve the Initial-Value Problem (IVP)

$$y' = ty^{\frac{1}{3}}, \quad 1 \leq t \leq 5, \quad y(1) = 1$$

at $t = 1.1$. [7]

Turn Over/...

5. **(a)** Show that

$$S(x) = \begin{cases} 1+x, & x \in [0,3] \\ 4+(x-3)+(x-3)^3, & x \in [3,4]. \end{cases}$$

is a clamped cubic spline that can be fitted to the data points $(0,1)$, $(3,4)$ and $(4,6)$. [7]

(b) The vapour pressure of water from temperatures of 40^oC to 72^oC is given in the table below:

$T(^oC)$	$P(mmHg)$
40	55.3
48	83.7
56	123.8
64	179.2
72	254.5

The slope of the vapour pressure curve at a specified temperature gives an estimate of the evaporation rate. Find the most accurate estimation of the evaporation rate at $T = 72^oC$. [4]

(c) **(i)** Evaluate the integral

$$\int_C |z|^2\, dz,$$

where C is a straight line from $z = 4$ to $z = 3i$. [5]

(ii) Evaluate the integral

$$\oint_C \frac{e^{-z}}{z^{2n+1}}\, dz, \quad n \geq 1, \quad n \in \mathbb{N},$$

where C is a directed contour $|z| = 1$. [4]

6. **(a)** Show that the Initial-Value Problem (IVP)

$$\frac{dy}{dt} = ty^{\frac{1}{3}}, \quad y(1) = 1$$

is well-posed on $D = \{(t,y) : 1 \leq t \leq 5, \ \ 1 \leq y \leq 27\}$. [4]

(b) Solve the IVP in part (a) using Taylor's method of order two at $t = 1.2$ with $h = 0.1$. [5]

Turn Over/...

(c) (i) Given that $z = x + iy, \quad x, y \in \mathbb{R}$, show that

$$f(z) = \frac{x+1}{(x+1)^2 + y^2} - \frac{y}{(x+1)^2 + y^2} i$$

is analytic everywhere on $\mathbb{C}$ except at $z = -1$. [7]

(ii) Show that $f'(z) = \frac{-1}{(z+1)^2}$. [4]

7. (a) (i) By sketching $x^2 - y^2 = 2$ and $y - \sqrt{2x} = 0$ on the same coordinate system, show that there is a root of the equation

$$\sqrt{x^2 - 2} - \sqrt{2x} = 0$$

in the interval $[2, 3]$. [2]

(ii) Starting with $p_0 = 3$, perform two iterations using Newton-Raphson method to approximate the root. [6]

(b) Taylor's theorem can be used to express the three-point midpoint formula as

$$f'(x_0) = \frac{1}{2h}[f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6} f^{(3)}(x_0) - \frac{h^4}{120} f^{(5)}(x_0) - \ldots.$$

Find an approximation of order $\mathcal{O}(h^4)$ for $f'(1.2)$ with $h = 0.1$ when

$$f(x) = x^{\sqrt{2^x}} \ln\left(\sqrt{2^{3x}}\right).$$ [8]

(c) The total mass, M, entering and leaving a reactor over a specified time period is given by

$$M = \int_{t_1}^{t_2} Qc \, dt,$$

where Q (m^3/min) is the flow rate, $c(mg/m^3)$ is the concentration and t_1 and t_2 is the initial and final time, respectively. If the flow rate is constant, i.e. $Q = 4m^3/min$, find the total mass based on the following measurements:

t(min)	0	10	20	30	40
$c(mg/m^3)$	10	35	55	52	37

[4]

END OF EXAMINATION!