The University of Zambia

[redacted]

(c) Find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of

$$x \sin x - 1 = 0$$

on the interval $[0, 2]$ while using the bisection method. [6

2. (a) Use the fixed-point iteration to find the unique root of the equation

$$x^3 + 4x^2 - 10 = 0, \quad \text{using} \quad g(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

in the interval $[1, 2]$ with $p_0 = 1.5$ as the initial approximation. [8

(b) Given the function

$$f(x) = (x-1)(e^{x-1} - 1)$$

(i) Find the multiplicity of $x = 1$ as a zero of $f(x)$. [2

(ii) Hence, use the Modified Newton-Raphson's method to approximate the same zero starting with $x_0 = 0$. [4

(c) Show that the mapping $f(z) = z^2$ maps the strip

$$S = \{z : 1 \leq Re(Z) \leq 3\}$$

onto a set of parabolas and describe the parabolas. [6

3. (a) For the nonlinear system

$$x_1 = g_1(x_1, x_2) = \frac{x_1^2 + x_2^2 + 8}{10}, \quad x_2 = g_2(x_1, x_2) = \frac{x_1 x_2^2 + x_1 + 8}{10}:$$

Show that $\mathbf{G} = (g_1, g_2)^t$ mapping

$$D = \{\mathbf{X} = (x_1, x_2)^t : 0 \leq x_i \leq 1.5, i = 1, 2\}$$

into $\mathbb{R}^2$ has a unique fixed point in D. [8]

(b) Apply functional iteration to approximate the unique solution in (a) with accuracy of 2 significant figures using Newton's method or the fixed point method . [8]

(c) Find

$$\lim_{z \to -\pi i} e^{\frac{z^2+\pi^2}{z+\pi i}}.$$

[4]

4. (a) Given the data $(1,0),(2,3),(3,2)$:

   (i) Fit a quadratic spline to the function $f(x)$ defined by this set of points. [4]

   (ii) Use the quadratic spline found in (i) to approximate $f(2.5)$. [2]

(b) Use the appropriate Newton's interpolation method (forward- or backward-) to find the annual premium at the age of 38, using the data given in the table below:

Age in years	24	28	32	36	40
Annual premium	28.06	30.19	32.75	34.94	40

[10]

(c) Find all numbers z such that $(z+1)^3 = 2+2i$. [4]

5. (a) Approximate $f'(0.4)$ using the approximation formula

$f(x) = \cos \pi x$

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right] + \frac{h^2}{3}f^{(3)}(c),$$

and the values of $f(x)$ at

$$x = 0.25, 0.30, 0.35, 0.4, 0.45, 0.5, \quad h = 0.05,$$

and where $x_0 < c < x_0 + 2h$. [8]

(b) Find a bound for the error in (a) above. [6]

(c) Using the Composite Trapezoidal rule, determine the value of n and h required to approximate

$$\int_0^2 e^x \sin 3x \, dx$$

to within 10^{-4} accuracy. [6]

6. (a) Use Taylor's method of order two to approximate the solution to

$$y' = \frac{1}{t}(y^2 + y), \quad 1 \leq t \leq 3, \quad y(1) = -2, \quad h = 0.5.$$

[10]

(b) Find the image of the semi strip

$$S = \{z = x + iy, \ \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}, \quad y \geq 0\}$$

under the mapping $f(z) = \sin z$. [~~10~~ 6]

(c) Determine whether the function

$$f(z) = \begin{cases} \frac{z-i}{z^2+1}, & \text{if } z \neq i \\ \frac{-i}{2}, & \text{if } z = i. \end{cases}$$

is continuous at the point $z = i$. [4]

End

SELECTED NUMERICAL FORMULAE

- Secant Method:

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1} - f(p_{n-2})}$$

- Modified Newton-Raphson's Method:

$$p_n = p_{n-1} - \frac{f(p_{n-1})f'(p_{n-1})}{[f'(p_{n-1})]^2 - f(p_{n-1})f''(p_{n-1})}$$

- Newton Forward-Difference Formula:

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} s(s-1)\ldots(s-k+1)h^k f[x_0, x_1, \ldots x_k]$$

- Newton Backward-Difference Formula:

$$P_n(x) = P_n(x_n + sh) = f[x_n] + \sum_{k=1}^{n} s(s+1)\ldots(s+k-1)h^k f[x_k, x_{n-1}, \ldots, x_0]$$

- Chebyshev polynomial of degree n over the interval $[-1, 1]$:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

where $T_0(x) = 1$ and $T_1(x) = x$ and

$$x^2 = \tfrac{1}{2}[T_0(x) + T_2(x)]$$
$$x^3 = \tfrac{1}{4}[3T_1(x) + T_3(x)]$$
$$x^4 = \tfrac{1}{8}[3T_0(x) + 4T_2(x) + T_4(x)]$$
$$x^5 = \tfrac{1}{16}[10T_1(x) + 5T_3(x) + T_5(x)]$$
$$x^6 = \tfrac{1}{32}[10T_0(x) + 15T_2(x) + 6T_4(x) + T_6(x)]$$
$$\vdots$$

- Three-Point Formulas:

$$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)] + \frac{h^2}{3}f^{(3)}(\xi_0)$$

$$f'(x_0) = \frac{1}{2h}[f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6}f^{(3)}(\xi_1)$$

- Five-Point Formulas:

$$f'(x_0) = \frac{1}{12h}[-25f(x_0) + 48f(x_0 + h) - 36f(x_0 + 2h) + 16f(x_0 + 3h) - 3f(x_0 + 4h)] + \frac{h^4}{5}f^{(5)}(\xi)$$

$$f'(x_0) = \frac{1}{12h}[f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)] + \frac{h^4}{30}f^{(5)}(\xi)$$

- Second Derivative Midpoint Formula:

$$f''(x_0) = \frac{1}{h^2}[f(x_0 - h) - 2f(x_0) + f(x_0 + h)] - \frac{h^2}{12}f^{(4)}(\xi)$$

- The $O(h^{2j})$ approximation to $f'(x_0)$:

$$N_j(h) = \frac{4^{j-1}N_{j-1}(\frac{h}{2}) - N_{j-1}(h)}{4^{j-1} - 1}, \; j = 2, 3, 4, \ldots$$

- Difference-equations for Runge-Kutta methods of order four:

$$\begin{aligned}
w_0 &= \alpha \\
k_1 &= hf(t_i, w_i) \\
k_2 &= hf(t_i + \tfrac{h}{2}, w_i + \tfrac{1}{2}k_1) \\
k_3 &= hf(t_i + \tfrac{h}{2}, w_i + \tfrac{1}{2}k_2) \\
k_4 &= hf(t_{i+1}, w_i + k_3) \\
w_{i+1} &= w_i + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$

and

$$\begin{aligned}
&W_{1,j} = \alpha_1, \; W_{2,j} = \alpha_2, \; \ldots, W_{m,j} = \alpha_m \\
&K_{1,i} = hf_i(t_j, W_{1,j}, W_{2,j}, \ldots, W_{m,j}) \\
&K_{2,i} = hf_i(t_j + \tfrac{h}{2}, W_{1,j} + \tfrac{1}{2}K_{1,1}, W_{2,j} + \tfrac{1}{2}K_{1,2}, \ldots, W_{m,j} + \tfrac{1}{2}K_{1,m}) \\
&K_{3,i} = hf_i(t_j + \tfrac{h}{2}, W_{1,j} + \tfrac{1}{2}K_{2,1}, W_{2,j} + \tfrac{1}{2}K_{2,2}, \ldots, W_{m,j} + \tfrac{1}{2}K_{2,m}) \\
&K_{4,i} = hf_i(t_j + h, W_{1,j} + K_{3,1}, W_{2,j} + K_{3,2}, \ldots, W_{m,j} + K_{3,m}), \\
&\qquad \text{for } i = 1, 2, \ldots, m \\
&W_{i,j+1} = W_{i,j} + \tfrac{1}{6}(K_{1,i} + 2K_{2,i} + 2K_{3,i} + K_{4,i})
\end{aligned}$$