The University of Zambia
Department of Mathematics and Statistics
MAT 3110 - Engineering Mathematics II

Tutorial Sheet 6 - Double Integrals in Polar Coordinates **August, 2024**

1. Change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

(a)
$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \mathrm{d}y\mathrm{d}x$$

(b)
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \mathrm{d}y\mathrm{d}x$$

(c)
$$\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} \left(x^2+y^2\right)\,\mathrm{d}x\mathrm{d}y$$

(d)
$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(x^2+y^2\right)\,\mathrm{d}x\mathrm{d}y$$

(e)
$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \mathrm{d}y\mathrm{d}x, \quad a\in\mathbb{R}, a>0$$

(f)
$$\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} \left(x^2+y^2\right)\,\mathrm{d}x\mathrm{d}y$$

(g)
$$\int_{0}^{6}\int_{0}^{y} x\,\mathrm{d}x\mathrm{d}y$$

(h)
$$\int_{0}^{2}\int_{0}^{x} y\,\mathrm{d}y\mathrm{d}x$$

(i)
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,\mathrm{d}y\mathrm{d}x$$

(j)
$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0} \frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}\,\mathrm{d}x\mathrm{d}y$$

(k)
$$\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln x)^2-y^2}} e^{\sqrt{x^2+y^2}}\,\mathrm{d}x\mathrm{d}y$$

(l)
$$\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} e^{-\left(x^2+y^2\right)}\,\mathrm{d}y\mathrm{d}x$$

(m)
$$\int_{0}^{2}\int_{0}^{1-(x-1)^2} \frac{x+y}{x^2+y^2}\,\mathrm{d}y\mathrm{d}x$$

(n)
$$\int_{0}^{2}\int_{-\sqrt{1-(y-1)^2}}^{0} xy^2\,\mathrm{d}x\mathrm{d}y$$

(o)
$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)\,\mathrm{d}x\mathrm{d}y$$

(p)
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\left(1+x^2+y^2\right)^2}\,\mathrm{d}y\mathrm{d}x$$

2. Evaluate $\displaystyle\iint_D \mathbf{e}^{1-x^2-y^2}\,\mathrm{d}A$ where D is the region in the 4th quadrant between $x^2+y^2=16$ and $x^2+y^2=36$.

3. Evaluate $\displaystyle\iint_D 6xy+4x^2\,\mathrm{d}A$ where D is the portion of $x^2+y^2=9$ in the 4th quadrant.

4. Integrate $f(x, y) = \dfrac{\ln(x^2 + y^2)}{\sqrt{x^2 + y^2}}$ over the region $1 \le x^2 + y^2 \le e$.

5. Integrate $f(x, y) = \dfrac{\ln(x^2 + y^2)}{\sqrt{x^2 + y^2}}$ over the region $1 \le x^2 + y^2 \le e^2$.

6. The region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ is the base of a solid right cylinder. The top of the cylinder lies in the plane $z = x$. Find the cylinder's volume.

7. The region enclosed by the lemniscate $r^2 = 2\cos\theta$ is the base of a solid cylinder whose top is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.

8. The usual way to evaluate the improper integral

$$I = \int_0^\infty e^{-x^2}\, \mathrm{d}x$$

is to first to calculate its square:

$$I^2 = \left(\int_0^\infty e^{-x^2}\, \mathrm{d}x\right)\left(\int_0^\infty e^{-y^2}\, \mathrm{d}y\right) = \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)}\, \mathrm{d}x\mathrm{d}y$$

Evaluate the above integral using polar coordinates and find the value of I.

9. Evaluate the integral $\displaystyle\int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2}\, \mathrm{d}x\mathrm{d}y$

10. Integrate the function $f(x, y) = \dfrac{1}{1 - x^2 - y^2}$ over the disk $x^2 + y^2 \le \frac{3}{4}$.

11. Suppose that the area of a region in the polar coordinate plane is given by the double integral

$$A = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{\csc\theta}^{2\sin\theta} r\, \mathrm{d}r\mathrm{d}\theta.$$

Sketch the region and find its area.

12. Use a double integral to determine the volume of the solid that is inside both the cylinder $x^2 + y^2 = 9$ and the sphere $x^2 + y^2 + z^2 = 16$.

13. Use a double integral to determine the volume of the solid that is bounded by $z = 12 - 3x^2 - 3y^2$ and $z = x^2 + y^2 - 8$.

——— *End of Tutorial Sheet* ———