The University of Zambia
Department of Mathematics and Statistics
MAT 3110 - Engineering Mathematics II

Tutorial Sheet 9 - Triple Integrals in Spherical Coordinates **Sept, 2024**

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1. Evaluate the following spherical coordinate integrals.

   (a) $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2\sin\phi \; d\rho\, d\phi\, d\theta$ (b) $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2} (\rho\cos\phi)\,\rho^2\sin\phi \; d\rho\, d\phi\, d\theta$

   (c) $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{1-\cos\phi}{2}} \rho^2\sin\phi \; d\rho\, d\phi\, d\theta$ (d) $\displaystyle\int_0^{\frac{3\pi}{2}}\int_0^{\pi}\int_0^{1} 5\rho^3\sin^3\phi \; d\rho\, d\phi\, d\theta$

   (e) $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec\phi}^{2} 3\rho^2\sin\phi \; d\rho\, d\phi\, d\theta$ (f) $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sec\phi} (\rho\cos\phi)\,\rho^2\sin\phi \; d\rho\, d\phi\, d\theta$

2. Evaluate the following spherical coordinate integrals.

   (a) $\displaystyle\int_0^{2}\int_{-\pi}^{0}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \rho^3\sin(2\phi) \; d\phi\, d\theta\, d\rho$ (b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi} \rho^2\sin\phi \; d\theta\, d\rho\, d\phi$

   (c) $\displaystyle\int_0^{1}\int_0^{\pi}\int_0^{\frac{\pi}{4}} 12\rho\sin^3\phi \; d\phi\, d\theta\, d\rho$ (d) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\csc\phi}^{2} 5\rho^4\sin^3\phi \; d\rho\, d\theta\, d\phi$

3. Let D be the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the plane $z = 1$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

   (a) $d\rho\, d\phi\, d\theta$ (b) $d\phi\, d\rho\, d\theta$

4. In each of the following, find the spherical coordinate limits for the integral that calculates the volume of the given solid and then evaluate the integral.

   (a) The solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2$, $z \geq 0$.

   (b) The solid bounded below by the sphere $rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2+y^2}$.

   (c) The solid bounded below by the xy-plane, on the sides by the sphere $rho = 2$, and above by the cone $\phi = \frac{\pi}{3}$.

5. Let D be region in the first octant that is bounded below by the cone $\phi = \frac{\pi}{4}$ and above by the sphere $\rho = 3$. Express the volume of D as an triple integral in

   (a) cylindrical coordinates

   (b) spherical coordinates

   Find the volume of the region D.

In the following problems, decide which coordinates systems is appropriate to use to find the volumes of the the following solids.

6. The solid that is between the sphere $\rho = 2$ and the cones $\phi = \frac{\pi}{3}$ and $\phi = \frac{2\pi}{3}$.

7. The solid that is between the sphere $\rho = 3$ and the half planes $\theta = 0$ and $\theta = \frac{\pi}{6}$.

8. The solid that is the smaller region of the regions bounded by the sphere $\rho = 2$ and the plane $z = 1$.

9. The solid bound below by the plane $z = 0$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

10. The solid bounded below by the paraboloid $z = x^2+y^2$, laterally by the cylinder $x^2+y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$.

11. The solid that is bounded laterally by cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$, bounded below by the cone $z = -\sqrt{x^2 + y^2}$, and bounded above by the cone $z = \sqrt{x^2 + y^2}$.

12. The solid that lies inside the sphere $x^2+y^2+z^2 = 2$ and outside the cylinder $x^2+y^2 = 1$.

13. The solid enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $x + y + z = 4$.

14. The solid that is bounded above by the paraboloid $z = 5 - x^2 - y^2$, and below by the paraboloid $z = 4x^2 + 4y^2$.

15. The solid that is bounded above by the paraboloid $z = 9-x^2-y^2$, below by the xy-plane, and lying outside the cylinder $x^2 + y^2 = 1$.

16. The solid that is inside the cylinder $x^2 + y^2 = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

17. The solid bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

End of Tutorial Sheet