The University of Zambia
Department of Mathematics and Statistics
MAT 3110 - Engineering Mathematics II

Tutorial Sheet 10 - Line Integrals **October, 2024**

1. Evaluate $\int_C (xy + y + z)\,\mathrm{d}s$ where C is the curve along $\mathbf{r}(t) = 2t\,\mathbf{i} + t\,\mathbf{j} + (2-2t)\,\mathbf{k}$, $0 \le t \le 1$.

2. Evaluate $\int_C \sqrt{x^2+y^2}\mathrm{d}s$ where C is the curve along $\mathbf{r}(t) = (4\cos t)\,\mathbf{i} + (4\sin t)\,\mathbf{j} + 3t\,\mathbf{k}$, $-2\pi \le t \le 2\pi$.

3. Evaluate $\int_C (x + y + z)\,\mathrm{d}s$ where C is the straight-line segment from $(1, 2, 3)$ to $(0, -1, 1)$.

4. Evaluate $\int_C \left(\frac{\sqrt{3}}{x^2+y^2+z^2}\right)\mathrm{d}s$ where C is the curve $\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}$, $1 \le t \le \infty$.

5. Find the work done by the following vector fields $\vec{\mathbf{F}}$ from $(0, 0, 0)$ to $(1, 1, 1)$ over each of the following paths.

   - The staight-line path $C_1 : \mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}$, $0 \le t \le 1$.
   - The curved path $C_2 : \mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^4\,\mathbf{k}$, $0 \le t \le 1$.
   - The path $C_3 \cup C_4$ consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the line segment from $(1, 1, 0)$ to $(1, 1, 1)$,

   (a) $\vec{\mathbf{F}}(x, y, z) = 3y\,\mathbf{i} + 2x\,\mathbf{j} + 4z\,\mathbf{k}$
   (b) $\vec{\mathbf{F}}(x, y, z) = \left(\frac{1}{x^2+1}\right)\mathbf{j}$
   (c) $\vec{\mathbf{F}}(x, y, z) = \sqrt{z}\,\mathbf{i} - 2x\,\mathbf{j} + \sqrt{y}\,\mathbf{k}$
   (d) $\vec{\mathbf{F}}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$
   (e) $\vec{\mathbf{F}}(x, y, z) = (3x^2 - 3x)\,\mathbf{i} + 3z\,\mathbf{j} + \mathbf{k}$
   (f) $\vec{\mathbf{F}}(x, y, z) = (y + z)\,\mathbf{i} + (z + x)\,\mathbf{j} + (x + y)\,\mathbf{k}$

6. Find the work done by the following vector field over the given curves in the direction of increasing t

   (a) $\vec{\mathbf{F}}(x, y, z) = xy\,\mathbf{i} + y\,\mathbf{j} - yz\,\mathbf{k}$, $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k}$, $0 \le t \le 1$.
   (b) $\vec{\mathbf{F}}(x, y, z) = 2y\,\mathbf{i} + 3x\,\mathbf{j} + (x + y)\,\mathbf{k}$, $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + \left(\frac{t}{6}\right)\mathbf{k}$, $0 \le t \le 2\pi$.
   (c) $\vec{\mathbf{F}}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$, $\mathbf{r}(t) = (\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + t\,\mathbf{k}$, $0 \le t \le 2\pi$.
   (d) $\vec{\mathbf{F}}(x, y, z) = 6z\,\mathbf{i} + y^2\,\mathbf{j} + 12x\,\mathbf{k}$, $\mathbf{r}(t) = (\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + \left(\frac{t}{6}\right)\mathbf{k}$, $0 \le t \le 2\pi$.

7. Evaluate $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\mathbf{r}$ where $\vec{\mathbf{F}}(x, y) = xy\,\mathbf{i} + (x + y)\,\mathbf{j}$ and C is the curve along $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

8. Evaluate $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\mathbf{r}$ where $\vec{\mathbf{F}}(x, y) = (x - y)\,\mathbf{i} + (x + y)\,\mathbf{j}$ and C is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ with counterclockwise direction.

9. Evaluate $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\mathbf{r}$ where $\vec{\mathbf{F}}(x, y) = x^2\,\mathbf{i} - y\,\mathbf{j}$ and C is the curve along $x = y^2$ from $(4, 2)$ to $(1, -1)$.

10. Evaluate $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\mathbf{r}$ where $\vec{\mathbf{F}}(x, y) = y\,\mathbf{i} - x\,\mathbf{j}$ and C is the curve along the unit circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$ in the counterclockwise direction.

11. Which of the following vector fields are conservative and which ones are not?
    (a) $\vec{\mathbf{F}}(x, y, z) = yz\ \mathbf{i} + xz\ \mathbf{j} + xy\ \mathbf{k}$
    (b) $\vec{\mathbf{F}}(x, y, z) = (y \sin z)\ \mathbf{i} + (x \sin z)\ \mathbf{j} + (xy \cos z)\ \mathbf{k}$
    (c) $\vec{\mathbf{F}}(x, y, z) = y\ \mathbf{i} + (x + z)\ \mathbf{j} - y\ \mathbf{k}$
    (d) $\vec{\mathbf{F}}(x, y) = -y\ \mathbf{i} + x\ \mathbf{j}$
    (e) $\vec{\mathbf{F}}(x, y, z) = (z + y)\ \mathbf{i} + z\ \mathbf{j} + (y + x)\ \mathbf{k}$
    (f) $\vec{\mathbf{F}}(x, y, z) = (e^x \cos y)\ \mathbf{i} - (e^x \sin y)\ \mathbf{j} + z\ \mathbf{k}$

12. Show that the following vector fields are conservative and find a potential function f for the vector field $\vec{\mathbf{F}}$.
    (a) $\vec{\mathbf{F}}(x, y, z) = (\ln x + \sec^2(x + y))\ \mathbf{i} + \left(\sec^2(x + y) + \frac{y}{y^2+z^2}\right)\ \mathbf{j} + \frac{z}{y^2+z^2}\ \mathbf{k}$
    (b) $\vec{\mathbf{F}}(x, y, z) = \frac{y}{1+x^2y^2}\ \mathbf{i} + \left(\frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}}\right)\ \mathbf{j} + \left(\frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z}\right)\ \mathbf{k}$

13. How are the constant a, b, and c related if the following vector field is conservative.

$$\vec{\mathbf{F}}(x, y, z) = \left(ay^2 + 2czx\right)\ \mathbf{i} + y\,(bx + cz)\ \mathbf{j} + \left(ay^2 + cx^2\right)\ \mathbf{k}$$

14. For what values of b and c will the vector field

$$\vec{\mathbf{F}}(x, y, z) = \left(y^2 + 2czx\right)\ \mathbf{i} + y\,(bx + cz)\ \mathbf{j} + \left(y^2 + cx^2\right)\ \mathbf{k}$$

be a conservative vector field.

15. Verify Green's theorem for each of the given vector fields given below by evaluating both sides of the equation in the statement of Green's theorem. Take the domain of integration to be the unit disk $x^2+y^2 \leq 1$ and its bounding circle $\mathbf{r}(t) = (\cos t)\ \mathbf{i}+(\sin t)\,\mathbf{j}$, $0 \leq t \leq 2\pi$.
    (a) $\vec{\mathbf{F}}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$
    (b) $\vec{\mathbf{F}}(x, y) = y\,\mathbf{i}$
    (c) $\vec{\mathbf{F}}(x, y) = 2x\,\mathbf{i} - 3y\,\mathbf{j}$
    (d) $\vec{\mathbf{F}}(x, y) = -x^2y\,\mathbf{i} + xy^2\,\mathbf{j}$

16. Use Green's theorem to evaluate the line integral

$$\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\mathbf{r}$$

given the following vector fields $\vec{\mathbf{F}}$ and curves C.

    (a) $\vec{\mathbf{F}}(x, y) = (x - y)\ \mathbf{i} + (y - x)\ \mathbf{j}$ and C is the square bounded by $x = 0$, $x = 1$, $y = 0$ and $y = 1$.

    (b) $\vec{\mathbf{F}}(x, y) = (x^2 + 4y)\ \mathbf{i} + (x + y^2)\ \mathbf{j}$ and C is the square bounded by $x = 0$, $x = 1$, $y = 0$ and $y = 1$.

(c) $\vec{\mathbf{F}}(x, y) = (y^2 - x^2)\ \mathbf{i} + (x^2 + y^2)\ \mathbf{j}$ and C is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

(d) $\vec{\mathbf{F}}(x, y) = (x + y)\ \mathbf{i} - (x^2 + y^2)\ \mathbf{j}$ and C is the triangle bounded by $y = 0$, $x = 1$, and $y = x$.

(e) $\vec{\mathbf{F}}(x, y) = \left(\tan^{-1}\left(\frac{y}{x}\right)\right)\ \mathbf{i} + \ln\left(x^2 + y^2\right)\ \mathbf{j}$ and C is the square bounded by $x = 0$, $x = 1$, $y = 0$ and $y = 1$.

End of Tutorial Sheet