Eigenvalues & Eigenvectors

If I multiply an $n \times n$ matrix by an $n \times 1$ vector, I will get ~~back~~ a new $n \times 1$ vector back

$$A\vec{\eta} = \vec{y}$$

What I want to know is if it is possible for the following to happen. Instead of just getting a brand new vector out of the multiplication is it possible instead to get the following

$$A\bar{\eta} = \lambda\bar{\eta} \text{ ------- } (1)$$

lambda

eta

in other words, is it possible for at least certain λ and $\bar{\eta}$ to have multiplication be the same as just multiplying the vector by a constant.

$\lambda \rightarrow$ eigenvalue of A

$\eta \rightarrow$ eigenvector of A

to find the eigenvalue and eigenvector

if $\bar{\eta} = \vec{0}$ then (1) is going to be true for any value of λ

$\bar{\eta} \neq \vec{0}$

$$A\bar{\eta} - \lambda\bar{\eta} = \vec{0}$$

$$A\bar{\eta} - \lambda I_n\bar{\eta} = \vec{0}$$

$$(A - \lambda \underbrace{I_n}_{\boxed{\text{Identity Matrix}}})\bar{\eta} = 0$$

NOTE

r can as well be used in place of lambda (λ)

Fact

If A is an $n \times n$ matrix then $\det(A - \lambda I) = 0$ is an n^{th} degree polynomial.
This polynomial is called the characteristic polynomial.

if $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the complete list of eigenvalues for A (including all repeated eigenvalues) then,

1. if λ occurs only once in the list then we call λ simple
2. if λ occurs $k > 1$ times in the list then we say that λ has multiplicity k.

3. if $\lambda_1, \lambda_2, \ldots, \lambda_k$ $(k \leq n)$ are the simple eigenvalues in the list with corresponding eigenvectors $\vec{\eta}^{(1)}, \vec{\eta}^{(2)}, \ldots, \vec{\eta}^{(k)}$ then the eigenvectors are all linearly independent.

Extra

A matrix has its own eigenvalue and a unique eigen vector relative to the eigenvalue.

For simplicity

instead of transforming a matrix my multiplying it by its unique eigenvector, I can multiply the eigenvector by the eigenvalue. See below

$$A\eta_n = \lambda_n \eta_n$$

matrix — A; eigenvalue — λ_n; eigenvector — η_n

Examples (TG 2 MAT3110, Q9)

a) $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$ let $A = \begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$ $\det \begin{bmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{bmatrix}$

$(5-\lambda)(-6-\lambda) + 18 = 0$

$\lambda^2 - 30 - 5\lambda + 6\lambda + 18 = 0 \rightarrow \lambda^2 + \lambda - 12 = 0$

$\lambda^2 + 4\lambda - 3\lambda - 12 = 0 \Rightarrow \lambda(\lambda+4) - 3(\lambda+4)$

$\lambda = 3, \lambda = -4$ eigenvalues

b. to determine eigenvectors

Remember $(A - \lambda I)\eta = 0$

matrix — A; eigenvalue — λ; Identity matrix — I; eigenvector — η

For $\lambda = 3$

$$\begin{bmatrix} 5-3 & -2 \\ 9 & -6-3 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{eigenvector in form of a column matrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 \\ 9 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 - 2x_2 = 0$ ----- (1)

$9x_1 - 9x_2 = 0$ ----- (2)

Using either eqn's (1 or 2)

$x_1 = x_2$

say $x_2 = 1$

$x_1 = 1$

$\therefore \eta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Check Proof on Next Page

Proof

$A\eta = \lambda\eta$

Data

$$A = \begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$$

$$\lambda = 3$$

$$\eta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{2\times 1} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2×1

$$\begin{bmatrix} 5 \cancel{*} + (-2)(1) \\ 9\times 1 + (-6)(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} 5-2 \\ 9-6 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

hence shown that $A\eta_n = \lambda_n \eta_n$

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More Examples (MAT3110, TS-2, 2024)

b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\quad \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$

$(1-\lambda)(4-\lambda) - 4 = 0$

$4 - \lambda - 4\lambda + \lambda^2 - 4 = 0$ $\rightarrow$ $\lambda^2 - 5\lambda = 0$ $\quad \lambda_1 = 0$

$\lambda(\lambda - 5) = 0$ $\quad \lambda_2 = 5$

For $\lambda_1 = 0$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 + 2x_2 = 0$ ---- (I)

$2x_1 + 4x_2 = 0$ ---- (II)

using either eqn's (I or II)

$x_1 = -2x_2$

Say $x_2 = 1$

$x_1 = -2$ $\quad \therefore \eta_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

For $\lambda_2 = 5$

$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -4x_1 + 2x_2 = 0$ ---- (I)

$2x_1 - x_2 = 0$ ---- (II)

using eqn (I)

$x_2 = 2x_1$

Say $x_1 = 1$

$x_2 = 2$ $\quad \therefore \eta_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(c)

$$A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \quad \det \begin{bmatrix} -\lambda & 3 \\ -3 & -\lambda \end{bmatrix}$$

$\lambda^2 + 9 = 0$ $\quad \lambda = \pm\sqrt{-9}$ $\quad \lambda_1 = 3i$ $\quad \lambda_2 = -3i$

$\lambda^2 = -9$ $\quad \lambda = \pm 3i$

for

$\lambda_1 = 3i$

$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3ix_1 + 3x_2 = 0$ ------ (i)

$-3x_1 - 3ix_2 = 0$ ----- (ii)

using either eqn(i)

$-3ix_1 + 3x_2 = 0 \Rightarrow 3x_2 = 3ix_1$

$x_2 = ix_1$

Say $x_1 = 1$

$x_2 = i$

$$\eta_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

for $\lambda_2 = -3i$

$$\begin{bmatrix} 3i & 3 \\ -3 & 3i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3ix_1 + 3x_2 = 0$ ---- (i)

$-3x_1 + 3ix_2 = 0$ ---- (ii)

using either eqn(i) $\quad x_2 = -ix_1$ $\quad$ Say $x_1 = 1$

$x_2 = -i$

$$\eta_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

(d)

$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\quad \det \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix}$

$(1-\lambda)(3-\lambda) = 0 \rightarrow \lambda^2 - 4\lambda + 3 = 0$

$3 - \lambda - 3\lambda + \lambda^2 = 0 \quad \lambda^2 - \lambda - 3\lambda + 3 = 0$

$\lambda(\lambda - 1) - 3(\lambda - 1) \quad \lambda_1 = \underline{\underline{1}}, \; \lambda_2 = \underline{\underline{3}}$

For $\lambda_1 = 1$

$\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 = 0$, $x_2 = 0$ $\quad \therefore \eta_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For $\lambda_2 = 3$

$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -2x_1 + 2x_2 = 0$, $x_1 = x_2$ $\quad$ say $x_1 = 1$, $x_2 = 1$ $\quad \therefore \eta_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

__Proof__

$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 + 2 \\ 0 + 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

True

(e)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \det\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$$

$$\lambda^2 = 0 \qquad \lambda = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0$$

$$x_2 = 0 \qquad \eta_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$