Euler Equations

$$ax^2y'' + bxy' + cy = 0$$ around $x_0 = 0$, called Euler Equation

a solution to Euler equations can written in the form

$$y = x^r, \quad y' = rx^{r-1}, \quad y'' = r(r-1)x^{r-2}$$

Now plug into the differential equation

$$ax^2 \times r(r-1)x^{r-2} + bx \times rx^{r-1} + cx^r = 0$$

$$ar(r-1)x^r + brx^r + cx^r = 0$$

$$(ar(r-1) + br + c)x^r = 0$$ assuming $x^r \neq 0$

then $(ar(r-1) + br + c) = 0$

$$ar(r-1) + br + c = 0$$ → quadratic form were r can be solved.

After solving for r, there will be 3 cases

→ Real, Distinct Roots

→ Double Roots

→ Complex Roots

Real, Distinct Roots

General Form

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$$

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Double Roots

- In this case it can be shown that the second solution will be

$$y_2(x) = x^r \ln x$$

General solution

$$y(x) = c_1 x^r + c_2 x^r \ln x = x^r (c_1 + c_2 \ln x)$$

Complex Roots

$$r_{1,2} = \lambda \pm \mu i$$

taking the 1st root we'll get the following solution

$$x^{\lambda + \mu i}$$

to eliminate complex solutions, recall that $x^r = e^{\ln x^r} = e^{r \ln x}$

$$x^{\lambda + \mu i} = e^{(\lambda + \mu i)\ln x}$$

$$= e^{\lambda \ln x} \times e^{\mu i \ln x}$$

$$= e^{\ln x^\lambda}(\cos(\mu \ln x) + i \sin(\mu \ln x))$$

$$= x^\lambda [\cos(\mu \ln x) + i \sin(\mu \ln x)]$$

general solution

$$y(x) = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x).$$

$$y(x) = x^\lambda (c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x))$$

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Examples (MAT3110 TS-2, 2024) Q5

ⓐ $x^2y'' - 6y = 0$

$x^2[r(r-1)x^{r-2}] - 6[x^r] = 0$

$r(r-1)x^r - 6x^r = 0$

$[r^2 - r - 6]x^r = 0$

$r^2 - r - 6 = 0$

$r^2 - 3r + 2r - 6 = 0$

$\rightarrow r(r-3)2(r-3) = 0$

$r_1 = -2 \qquad r_2 = 3$

$r_1 = -2 \qquad r_2 = 3$

Real, Distinct Roots

$y(x) = C_1 x^{r_1} + C_2 x^{r_2}$

$y(x) = C_1 x^{-2} + C_2 x^3$

$y(x) = \dfrac{C_1}{x^2} + C_2 x^3$

ⓑ $x^2y'' + 4xy' = 0$

$x^2[r(r-1)x^{r-2}] + 4x[rx^{r-1}] = 0$

$r(r-1)x^r + 4rx^r = 0$

$[r(r-1) + 4r]x^r = 0$

$r^2 - r + 4r = 0$

$r^2 + 3r = 0$

$r(r+3) = 0$

$r_1 = 0 \quad r_2 = -3$

$y(x) = C_1 x^0 + C_2 x^{-3}$

$y(x) = C_1 + \dfrac{C_2}{x^3}$

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c) $x^2y'' - 2xy' + 2y = 0$

$x^2[r(r-1)x^{r-2}] - 2x[rx^{r-1}] + 2x^r = 0$

$r(r-1)x^r - 2rx^r + 2x^r = 0$

$[r(r-1) - 2r + 2]x^r = 0$

$r(r-1) - 2r + 2 = 0$

$r^2 - r - 2r + 2 = 0$

$\rightarrow r^2 - 3r + 2 = 0$

$r^2 - 2r - r + 2 = 0$

$r(r-2) - (r-2) = 0$

$r_1 = 1 \qquad r_2 = 2$

$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$

$y(x) = \underline{c_1 x + c_2 x^2}$

d) $x^2y'' + 9xy' + 16y = 0$

$x^2[r(r-1)x^{r-2}] + 9x[rx^{r-1}] + 16[x^r] = 0$

$r(r-1)x^r + 9rx^r + 16x^r = 0$

$[r(r-1) + 9r + 16]x^r = 0$

$r^2 + 8r + 16 = 0$

$r^2 + 4r + 4r + 16 = 0$

$r(r+4)\,4(r+4) = 0$

$r_1 = r_2 = -4$

double roots

$y(x) = c_1 x^{-4} + c_2 x^{-4}\ln(x)$

$y(x) = \underline{\dfrac{c_1 + c_2 \ln(x)}{x^4}}$

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$$x^2y'' + 3xy' + 5y = 0$$

$$x^2\left[r(r-1)x^{r-2}\right] + 3x\left[rx^{r-1}\right] + 5x^r = 0$$

$$r(r-1)x^r + 3rx^r + 5x^r = 0$$

$$\left[r(r-1) + 3r + 5\right]x^r = 0$$

$$r^2 + 2r + 5 = 0$$

$$\left.\begin{array}{l} r_1 = -1 - 2i \\ r_2 = -1 + 2i \end{array}\right\} \text{complex roots}$$

general form.

$$y(x) = x^{\lambda}\left[C_1\cos(\mu\ln(x)) + C_2\sin(\mu\ln x)\right]$$

$$r_{1,2} = -1 \pm 2i$$

$$\lambda = -1 \qquad \mu = 3$$

$$y(x) = x^{-1}\left[C_1\cos(3\ln(x)) + C_2\sin(3\ln(x))\right]$$

(b) $x^2y'' + xy' + y = 0$

$$x^2\left[r(r-1)x^{r-2}\right] + x\left[rx^{r-1}\right] + x^r = 0$$

$$r(r-1)x^r + rx^r + x^r = 0$$

$$\left[r(r-1) + r + 1\right]x^r = 0 \qquad r_{1,2} = \pm i$$

$$r^2 - r + r + 1 = 0$$

$$y(x) = C_1\cos[\ln(x)] + C_2\sin[\ln(x)]$$

$$r^2 = -1$$

$$r = \pm i \quad \text{complex roots}$$

$$\lambda = 0$$

$$\mu = 1$$

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(e) $x^2y'' + xy' - y = 0$

$$x^2[r(r-1)x^{r-2}] + x[rx^{r-1}] - x^r = 0$$

$$r(r-1)x^r + rx^r - x^r = 0$$

$$[r(r-1) + r - 1]x^r = 0$$

$$r(r-1) + r - 1 = 0$$

$$r^2 - r + r - 1$$

$$r^2 = 1$$

$$r_1 = -1 \quad r_2 = 1$$

$$y(x) = C_1x^{r_1} + C_2x^{r_2}$$

$$y(x) = C_1x^{-1} + C_2x$$

$$y(x) = \frac{C_1}{x} + C_2x$$

(f) $x^2y'' + 3xy' + y = 0$

$$x^2[r(r-1)x^{r-2}] + 3x[rx^{r-1}] + x^r = 0$$

$$r(r-1)x^r + 3rx^r + x^r = 0$$

$$[r(r-1) + 3r + 1]x^r = 0$$

$$r(r-1) + 3r + 1 = 0$$

$$r^2 + 2r + 1 = 0$$

$$r^2 + r + r + 1$$

$$r(r+1)(r+1)$$

$$r_1 = r_2 = -1$$

double roots

$$y(x) = C_1x^{r} + C_2x^{r}\ln(x)$$

$$y(x) = C_1x^{-1} + C_2x^{-1}\ln(x)$$

$$y(x) = \frac{C_1 + C_2\ln(x)}{x}$$

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Examples (MAT3110 TS-2, 2024) Q6

(a) $x^2y'' - 4xy' + 4y = 0, \quad y(1) = 4, \quad y'(1) = 13$

$$x^2[r(r-1)x^{r-2}] - 4x[rx^{r-1}] + 4x^r = 0$$

$$r(r-1)x^r - 4rx^r + 4x^r = 0$$

$$[r(r-1) - 4r + 4]x^r = 0$$

$$r(r-1) - 4r + 4 = 0 \quad \rightarrow \text{Assuming } x^r \neq 0$$

$$r^2 - 5r + 4 = 0 \qquad r_1 = 1 \qquad r_2 = 4 \rightarrow \text{Real, Distinct Roots}$$

$$y(x) = c_1x^{r_1} + c_2x^{r_2} \text{ --- } \alpha \qquad y'(x) = r_1c_1x^{r_1-1} + r_2c_2x^{r_2-1} \text{ --- } \beta$$

$y(1) = 4$ $\qquad$ $y'(1) = 13$

$$4 = c_1 + c_2 \text{ ---- (i)} \qquad 13 = c_1 + 4c_2 \text{ ---- (ii)}$$

Solving (i) & (ii)

$$4 = c_1 + c_2 \qquad c_2 = 3 \quad c_1 = 1$$
$$13 = c_1 + 4c_2$$

$\therefore$ Gen solution

$$y(x) = c_1x + c_2x^4$$

$$y(x) = x + 3x^4$$

b) $4x^2y'' - 4xy' - y = 0 \qquad y(4) = 2 \qquad y'(4) = 1/4$

$4x^2[r(r-1)x^{r-2}] - 4x[rx^{r-1}] - x^r = 0$

$4r(r-1)x^r - 4rx^r - x^r = 0 \Rightarrow [4r(r-1) - 4r - 1]x^r = 0$

$4r^2 - 8r - 1 = 0 \qquad r_1 = \frac{8 - 4\sqrt{5}}{8}, \quad r_2 = \frac{8 + 4\sqrt{5}}{8}$

$r_1 = -0.118 \qquad r_2 = 2.118 \longrightarrow$ Real, Distinct Roots

$y(x) = C_1 x^{r_1} + C_2 x^{r_2}$ --- α $\qquad y'(x) = r_1 C_1 x^{r_1 - 1} + r_2 C_2 x^{r_2 - 1}$ ---- β

Finding C_1 & C_2

$y(4) = 2 \qquad\qquad y'(4) = \frac{1}{4}$

$2 = C_1(4)^{-0.118} + C_2(4)^{2.118}$

$\frac{1}{4} = (-0.118)C_1(4)^{-0.118-1} + (2.118)C_2(4)^{2.118-1}$

$0.25 = (-0.118)C_1(4)^{-1.118} + (2.118)C_2(4)^{1.118}$

$2 = 0.849C_1 + 18.844C_2$ ---① $\qquad 0.25 = -0.025C_1 + 9.978C_2$ ----②

Solving ① & ②

$$\left(\begin{array}{l} 2 = 0.849C_1 + 18.844C_2 \\ 0.25 = -0.025C_1 + 9.978C_2 \end{array}\right) \begin{array}{l} 0.025 \Rightarrow 0.05 = 0.021C_1 + 0.471C_2 \\ 0.849 \Rightarrow 0.212 = -0.021C_1 + 8.471C_2 \end{array} \Big/ +$$

$C_2 = 0.029$

$C_1 = 1.731$

$\therefore y(x) = 1.731x^{-0.118} + 0.029x^{2.118}$

$y(x) = 1.731x^{(1-\sqrt{5}/2)} + 0.029x^{(1+\sqrt{5}/2)}$

$$\underline{\underline{y(x) = 1.731x^{\left(1-\frac{\sqrt{5}}{2}\right)} + 0.029x^{\left(1+\frac{\sqrt{5}}{2}\right)}}}$$

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3) $x^2y'' - 5xy' + 8y = 0, \quad y(1) = 5 \quad y'(1) = 18$

$$x^2[r(r-1)x^{r-2}] - 5x[rx^{r-1}] + 8x^r = 0$$

$$r(r-1)x^r - 5rx^r + 8x^r = 0$$

$$[r^2 - r - 5r + 8]x^r = 0 \Rightarrow r^2 - 6r + 8 = 0$$

$r^2 - 6r + 8 = 0 \qquad r(r-2) - 4(r-2) = 0$

$r^2 - 2r - 4r + 8 = 0 \qquad r_1 = 4 \quad r_2 = 2 \rightarrow$ Real, Distinct Roots

$$y(x) = c_1x^{r_1} + c_2x^{r_2} \text{---} \alpha, \quad y'(x) = r_1c_1x^{r_1-1} + r_2c_2x^{r_2-1} \text{---}$$

$y(1) = 5 \qquad\qquad y'(1)$

$5 = c_1 + c_2$ ----① $\qquad 18 = 4c_1 + 2c_2$ ----②

$$\left.\begin{pmatrix} 5 = c_1 + c_2 \\ 18 = 4c_1 + 2c_2 \end{pmatrix}\begin{matrix} 2 \\ 1 \end{matrix}\right\} \Rightarrow \begin{matrix} 10 = 2c_1 + 2c_2 \\ 18 = 4c_1 + 2c_2 \end{matrix} \Big/ -$$

$-8 = -2c_1 \qquad c_2 = 5 - c_1$

$c_1 = 4 \qquad c_2 = 5 - 4$

$c_2 = 1$

$$\therefore \quad y(x) = 4x^4 + x^2$$

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