Midterm 1, Math 244, Spring 2017

Last name, first name, ID number : Score: /30

1. (5 points) Evaluate the double integral $\iint_R xy\cos y dA$ over the region $R: -1 \le x \le 1,\ 0 \le y \le \pi$.

   Solution:

$$\begin{aligned}
\iint_R xy\cos y dA &= \int_{-1}^{1}\int_0^{\pi} xy\cos y dy dx \\
&= \int_{-1}^{1} \Big(y\sin y\Big|_0^{\pi} - \int_0^{\pi}\sin y\Big)x dx \ \text{(Integration By Part)} \\
&= \int_{-1}^{1} \Big(\cos y\Big|_0^{\pi}\Big)x dx \\
&= \int_{-1}^{1} -2x dx \\
&= -x^2\Big|_{-1}^{1} \\
&= 0.
\end{aligned}$$

2. (6 points) Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top is bounded by the plane $z = x + 4$.

   Solution: the volume V is given by

$$\begin{aligned}
V &= \int_{-4}^{1}\int_{3x}^{4-x^2} x + 4 \ dy dx \\
&= \int_{-4}^{1} (x+4)y\Big|_{3x}^{4-x^2} \ dx \\
&= \int_{-4}^{1} -x^3 - 7x^2 - 8x + 16 \ dx \\
&= -\frac{x^4}{4} - \frac{7x^3}{3} - 4x^2 + 16x\Big|_{-4}^{1} \\
&= \frac{625}{12}.
\end{aligned}$$

3. (5 points) Sketch the region bounded by the curves $y = \ln x$, $y = 2\ln x$ and the lines $x = 1$ and $x = e$. Then express the region's area as a double integral and evaluate the integral.

   Solution:

$$\begin{aligned}
A &= \int_1^e \int_{\ln x}^{2\ln x} dy dx \\
&= \int_1^e \ln x dx \\
&= x\ln x - x\Big|_1^e \\
&= 1.
\end{aligned}$$

4. (6 points) Find the volume of the noncircular right cylinder whose base lies <u>inside</u> the cardioid $r = 1 + \cos\theta$ and <u>outside</u> the circle $r = 1$ and top lies in the plane $z = x$.

Hint: You can use the following antiderivatives

$$\int \cos^2\theta d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C,$$
$$\int \cos^3\theta d\theta = \sin\theta - \frac{\sin^3\theta}{3} + C,$$
$$\int \cos^4\theta d\theta = \frac{1}{4}\Big(\frac{3\theta}{2} + \sin 2\theta + \frac{\sin 4\theta}{4}\Big) + C.$$

Solution: Since the base of the cylindar is symmetric about the $x-$axis, its volume V is given by

$$\begin{aligned}
V &= 2\Big(\int_0^{\frac{\pi}{2}}\int_1^{1+\cos\theta} r^2\cos\theta dr d\theta\Big)\\
&= 2\Big(\int_0^{\frac{\pi}{2}} \frac{r^3\cos\theta}{3}\Big|_1^{1+\cos\theta} d\theta\Big)\\
&= 2\Big(\int_0^{\frac{\pi}{2}} \cos^2\theta + \cos^3\theta + \frac{\cos^4\theta}{3} d\theta\Big)\\
&= 2\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_0^{\frac{\pi}{2}} + \sin\theta - \frac{\sin^3\theta}{3}\Big|_0^{\frac{\pi}{2}} + \frac{1}{4}\Big(\frac{3\theta}{2} + \sin 2\theta + \frac{\sin 4\theta}{4}\Big|_0^{\frac{\pi}{2}}\Big)\right)\\
&= 2\Big(\frac{\pi}{4} + \frac{2}{3} + \frac{\pi}{16}\Big)\\
&= \frac{4}{3} + \frac{5\pi}{8}.
\end{aligned}$$

5. (5 points) Find the average value of the function $f(x, y, z) = x^2 + 9$ over the cube in the first octant bounded by the planes $x = 2$, $y = 2$ and $z = 2$.

Solution: the volume of the cube is $2^3 = 8$. Moreover, the triple integral of the f over the cube is

$$\begin{aligned}
\int_{\text{cube}} f(x, y, z) dV &= \int_0^2\int_0^2\int_0^2 x^2 + 9\ dxdydz\\
&= \int_0^2\int_0^2 \frac{x^3}{3} + 9x\Big|_0^2\ dydz\\
&= \int_0^2\int_0^2 \frac{62}{3}\ dydz\\
&= 4.\frac{62}{3}.
\end{aligned}$$

Therefore, the average value of f over the cube is given by

$$\text{Average} = \frac{\int_{\text{cube}} f(x, y, z)dV}{\text{Volume}} = \frac{4.\frac{62}{3}}{8} = \frac{31}{3}.$$

6. (6 points) Find the mass of the solid bounded by the planes $x + z = 1$, $x - z = -1$, $y = 0$ and the surface $y = \sqrt{z}$. The density is $\delta(x, y, z) = 2y + 5$.
Reminder: If $\delta(x, y, z)$ is the density of an object occupying a region D in space, the mass of the object is $\int\int\int_D \delta(x, y, z)dV$.
Hint: Integrate in the order $dydzdx$.

Solution: Since the solid is symmetric about the $yz-$plane and the density only depends on y, its mass M is

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\begin{aligned}
M &= 2\Big(\int_0^1\int_0^{1-x}\int_0^{\sqrt{z}} 2y+5\ dydzdx\Big) \\
&= 2\Big(\int_0^1\int_0^{1-x} y^2+5y\Big|_0^{\sqrt{z}}\ dydzdx\Big) \\
&= 2\Big(\int_0^1\int_0^{1-x} z+5\sqrt{z}\ dzdx\Big) \\
&= 2\Big(\int_0^1 \frac{z^2}{2}+\frac{10}{3}z^{\frac{3}{2}}\Big|_0^{1-x}\ dx\Big) \\
&= 2\Big(\int_0^1 \frac{(1-x)^2}{2}+\frac{10}{3}(1-x)^{\frac{3}{2}}\ dx\Big) \\
&= 2\Big(-\frac{(1-x)^3}{6}-\frac{4}{3}(1-x)^{\frac{5}{2}}\Big|_0^1\Big) \\
&= 3.
\end{aligned}
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