Integrals in polar form

Reminder

$r = \sqrt{x^2+y^2}$ $\qquad x = r\cos\theta \qquad \theta = \tan^{-1}\left[\frac{y}{x}\right]$

$r^2 = x^2 + y^2 \qquad y = r\sin\theta$

$$\iint_R \phi(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{f(\theta)} \phi(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta$$

$$\iint_R \phi(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{f(\theta)} \overbrace{\phi(r\cos(\theta), r\sin(\theta))}^{f(r,\theta)}\, r\,dr\,d\theta$$

$$dx\,dy = r\,dr\,d\theta$$

lower limits

in this case $\begin{cases} \theta = \beta \text{ and } \theta = \alpha \\ r = f(\theta) \text{ and } r = g(\theta) \end{cases}$

Upper limits

$$\iint_R f(x,y)\,dA = \int\int f(r,\theta)\,r\,dr\,d\theta$$

$x = r\cos\theta \qquad y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

using the Jacobian

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= \underline{\underline{r}}$$

$$dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = r\,dr\,d\theta$$

$$\hookrightarrow dx\,dy \,/\, dy\,dx \Rightarrow r\,dr\,d\theta$$

Take Note:

- radius (r) is simply the furthest distance from the origin.

- It is important to understand the region over which ~~of~~ you are integrating.

Examples

Change the cartesian integral into an equivalent polar integral.

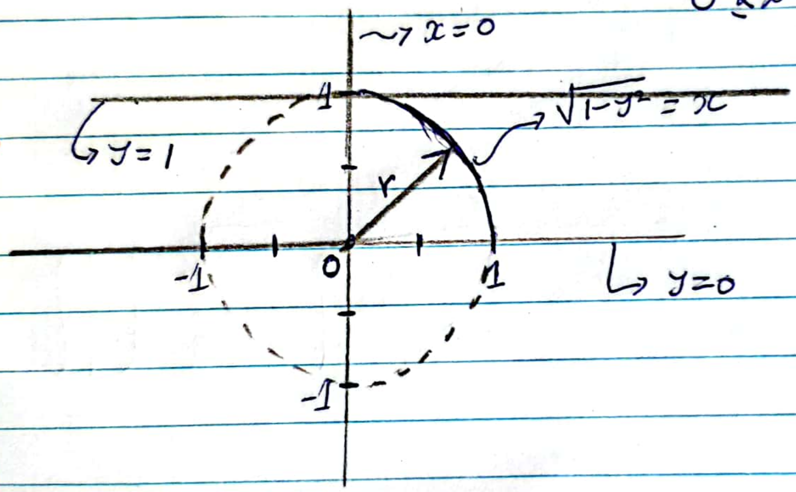
(a) $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy$

Reminder

$x^2+y^2 = r^2$ $\quad$ $x = r\cos\theta$ $\quad$ $\tan\theta = \left[\frac{y}{x}\right]$

$dx\,dy = r\,dr\,d\theta$ $\quad$ $y = r\sin\theta$ $\quad$ $\theta = \tan^{-1}\left[\frac{y}{x}\right]$

1st sketch the region of integration.



$0 \leq x \leq \underbrace{\sqrt{1-y^2}}$

circle of radius 1

$x^2+y^2 = 1$

$0 \leq y \leq 1$

clearly

r starts from zero(0) which is the origin and extends furthest from the origin up to 1

$\therefore \quad 0 \leq r \leq 1$

Clearly

θ ranges from zero in the x axis up to $\frac{\pi}{2}$ in the y axis

$\therefore \quad 0 \leq \theta \leq \frac{\pi}{2}$

hence $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^1 r^3\, dr\, d\theta$