Homework 1, Math 244, Spring 2017

Last name, first name, ID number : Score:

1. (5 points) Evaluate the double integral $\iint_R xye^{xy^2}dA$ on the region $R: 0 \le x \le 2,\ 0 \le y \le 1$.

   Solution:

$$\begin{aligned}\iint_R xye^{xy^2}dA &= \int_0^2\int_0^1 xye^{xy^2}dydx \\ &= \int_0^2\int_0^x \frac{1}{2}e^u dudx \qquad \text{(sub. } u = xy^2\text{)} \\ &= \frac{1}{2}\int_0^2 e^x - 1dx \\ &= \frac{1}{2}(e^2 - 3)\end{aligned}$$

2. (5 points) Find the volume V of the solid cut from the square column $|x| + |y| \le 1$ by the planes $z = 0$ and $3x + z = 3$.

   Solution: The region of integration R can be written $R = R_1 \cup R_2$ where

$$R_1 = \begin{cases} 0 \le y \le 1 \\ y - 1 \le x \le 1 - y \end{cases} \quad \text{and } R_2 = \begin{cases} -1 \le y \le 0 \\ -y - 1 \le x \le y + 1 \end{cases}$$

Since the solid is symmetric about the xz-plane, we have

$$\begin{aligned}V &= \iint_R 3 - 3xdA \\ &= \iint_{R_1} 3 - 3xdA + \iint_{R_2} 3 - 3xdA \\ &= 2\iint_{R_1} 3 - 3xdA \\ &= 2\int_0^1\int_{y-1}^{1-y} 3 - 3xdxdy \\ &= 2\int_0^1 3x - \frac{3}{2}\Big|_{y-1}^{1-y}dy \\ &= 12\int_0^1 1(1 - y)dy \\ &= 6\end{aligned}$$

3. (5 points) The integral $\int_{-1}^{2}\int_{y^2}^{y+2} dxdy$ gives the area of a region in the xy-plane. Sketch the region, label each bounding curve with its equation and give the coordinates of the points where the curves intersect. Then find the area A of the region.

Solution:

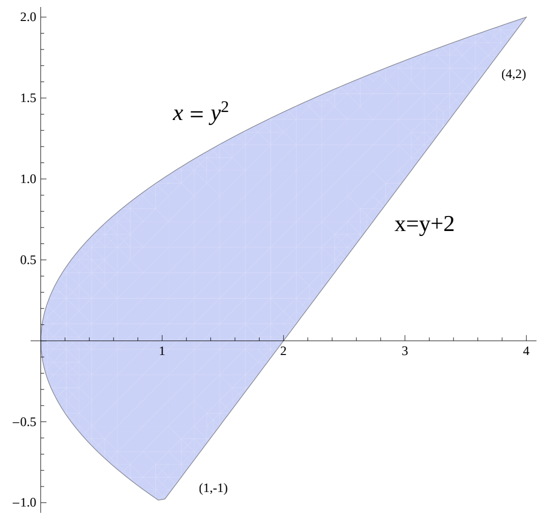

Figure 1: Region of integration

$$\begin{aligned}
A &= \int_{-1}^{2}\int_{y^2}^{y+2} dxdy \\
&= \int_{-1}^{2} x\Big|_{y^2}^{y+2} dy \\
&= \int_{-1}^{2} y+2-y^2 dy \\
&= \frac{y^2}{2}+2y-\frac{y^3}{3}\Big|_{-1}^{2} \\
&= \frac{9}{2}.
\end{aligned}$$

4. (5 points) The region enclosed by the the lemniscate $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume V.

Solution: The region of integration is depicted in the figure below:

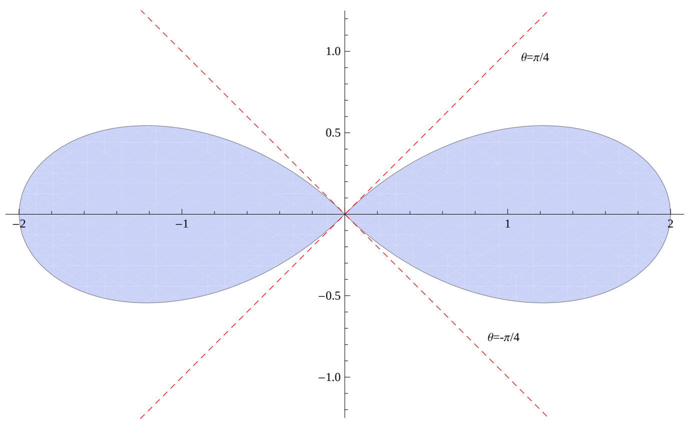

Figure 2: The region enclosed by the the lemniscate $r^2 = 2\cos 2\theta$.

Since the region of integration is symmetric about the origin and about the x-axis, we have

$$\begin{aligned}
V &= 4\int_0^{\frac{\Pi}{4}}\int_0^{\sqrt{2\cos 2\theta}} \sqrt{2 - r^2}\, r\,dr\,d\theta \\
&= -\frac{4}{3}\int_0^{\frac{\Pi}{4}} (2 - r^2)^{\frac{3}{2}}\Big|_0^{\sqrt{2\cos 2\theta}} d\theta \\
&= -\frac{4}{3}\int_0^{\frac{\Pi}{4}} (2 - 2\cos 2\theta)^{\frac{3}{2}} - 2^{\frac{3}{2}}\,d\theta \\
&= -\frac{8\sqrt{2}}{3}\int_0^{\frac{\Pi}{4}} (2\sin^2\theta)^{\frac{3}{2}} - 1\,d\theta \\
&= -\frac{8\sqrt{2}}{3}\int_0^{\frac{\Pi}{4}} 2\sqrt{2}(1 - \cos^2\theta)\sin(\theta)\,d\theta + \frac{2\sqrt{2}\pi}{3} \\
&= \frac{32}{3}\int_0^{\frac{\sqrt{2}}{2}} 1 - u^2\,du + \frac{2\sqrt{2}\pi}{3} \qquad (\text{sub. } u = \cos\theta) \\
&= \frac{(40 + 6\pi)\sqrt{2} - 64}{9}
\end{aligned}$$

5. (5 points) Find the volume V of the region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$.

Solution: The region of integration R can be written

$$R = \begin{cases} -2 \leq x \leq 2 \\ -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2} \\ 0 \leq z \leq 3 - x. \end{cases}$$

Hence the volume V is given by

$$\begin{aligned} V &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{-\sqrt{4-x^2}} \int_{0}^{3-x} dzdydx \\ &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{-\sqrt{4-x^2}} 3 - x \; dydx \\ &= 2\int_{-2}^{2} \sqrt{4-x^2}(3-x)dx \\ &= 6\int_{-2}^{2} \sqrt{4-x^2}dx - 2\int_{-2}^{2} x\sqrt{4-x^2}dx. \end{aligned}$$

The second integral is the integral of an odd function over the interval $[-2, 2]$ (centered at 0) so it equals 0. To calculate the first integral, we can use the trigonometric substitution $x = 2\sin u$, which leads to

$$\begin{aligned} V &= 6\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^2 udu \\ &= 12\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + \cos 2udu \\ &= 12\Big(u + \frac{\sin 2u}{2}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big) \\ &= 12\pi \end{aligned}$$

Remark: we can also use the cylindrical coordinates (r, θ, z) which lead to a simpler integral. Indeed, we have $V = \int_0^{2\pi} \int_0^2 \int_0^{3-r\cos\theta} rdzdrd\theta = \cdots = 12\pi$

6. (5 points) Find the center of mass and moment of inertia about the x-axis of the thin plate bounded by the curves $x = y^2$ and $x = 2y - y^2$ if the density at the point (x, y) is $\delta(x, y) = y + 1$.

<u>Solution</u>: The thin plate is depicted in the figure below:

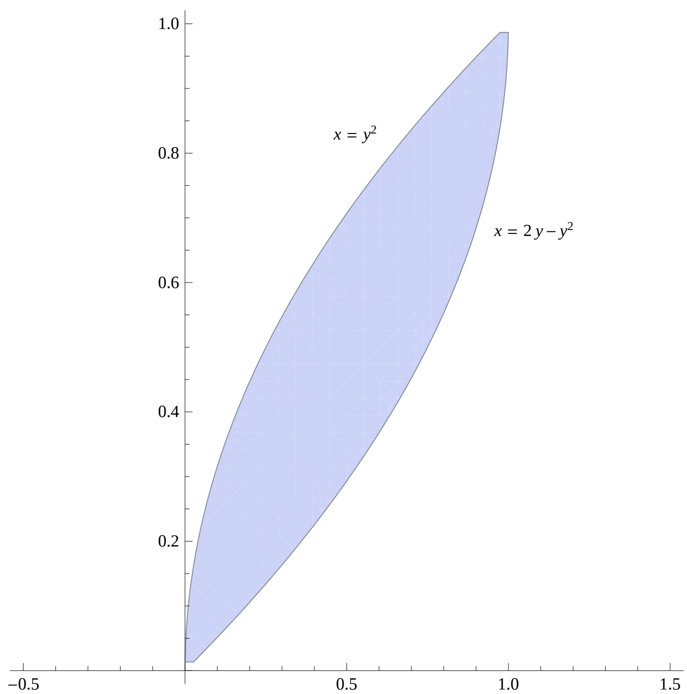

Figure 3: Thin plate bounded by $x = y^2$ and $x = 2y - y^2$.

Therefore, the mass of the thin plate is

$$M = \int_0^1 \int_{y^2}^{2y-y^2} y + 1 dxdy = \frac{1}{2}$$

and the first moments are given by

$$M_x = \int_0^1 \int_{y^2}^{2y-y^2} x(y + 1)dxdy = \frac{4}{15}$$

and

$$M_y = \int_0^1 \int_{y^2}^{2y-y^2} y(y + 1)dxdy = \frac{4}{15}$$

so the coordinates of ther center of mass are $(\frac{M_y}{M}, \frac{M_x}{M}) = (\frac{8}{5}, \frac{8}{5})$. The moment of inertia about the x-axis is

$$I_x = \int_0^1 \int_{y^2}^{2y-y^2} y^2(y + 1)dxdy = \frac{1}{6}.$$