SYMMETRY

1) Find the coordinates of the centre and the radius of the circle whose equation is
$(x-\frac{1}{2})^2+(y+5)^2=15$

$x-h=x-\frac{1}{2}$, $y-k=y+5$, $r^2=15$
$h=\frac{1}{2}$ $k=-5$ $r=\sqrt{15}$

$(\frac{1}{2},-5)$ $r=\sqrt{15}$

General Form

$Ax^2+Ay^2+Dx+Ey+F=0 \quad A\neq 0$

Examples

1) $2x^2+2y^2-12x+16y-31=0$

Sol

$2x^2+2y^2-12x+16y-31=0$

$2x^2-12x+2y^2+16y=31$

$2(x^2-6x)+2(y^2+8y)=31$

$(x^2-6x)+(y^2+8y)=\frac{31}{2}$

completing the square

$(x^2-6x+9)+(y^2+8y+16)=\frac{31}{2}+9+16$

$(x^2-3x-3x+9)+(y^2+4y+4y+16)=\frac{81}{2}$

$(x(x-3)-3(x-3))+(y(y+4)+4(y+4))=\frac{81}{2}$

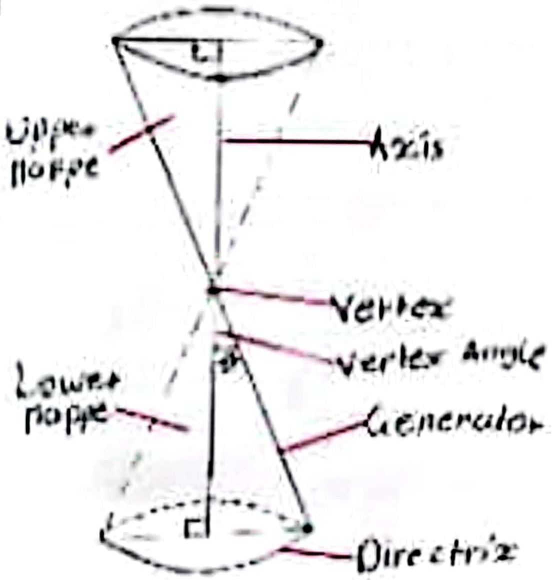
$(x-3)^2+(y+4)^2=\frac{81}{2}$ Standard Form

CONIC SECTIONS

- The intersection of a plane and a double-napped cone produces conic sections namely;

i) Circle (plane perpendicular to cone axis)
ii) Parabola (plane parallel to side of cone)
iii) Ellipse (plane inclined at an angle)
iv) Hyperbola (plane parallel to cone axis)

Double-Napped Cone



1) CIRCLE

Conic section formed when the plane is perpendicular to cone axis

⇒ A circle is the set of points in a plane that are at a given distance from a fixed point called the center and the distance is called the radius (r).

Center (h, k), any point on the circumference of the circle is (x, y)

distance formula

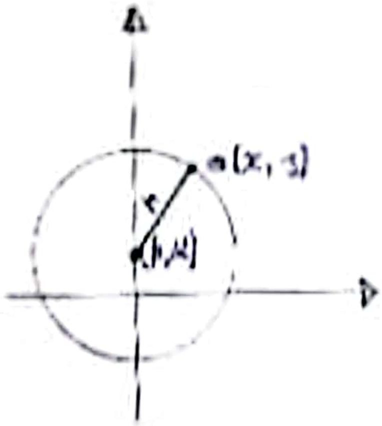
Radius (r)

$$\boxed{r^2 = (x-h)^2 + (y-k)^2}$$ Standard form of equation of circle

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

Note

The center (h, k) can start from the origin such that $h = 0$, $k = 0$ $(0,0)$ or can be at any point on the xy plane



Example

Q) Write the standard form of the equation of the circle with center at $(2, -5)$ and radius 3

Sol

$h = 2$, $k = -5$, $r = 3$

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-2)^2 + (y-(-5))^2 = 3^2$$

$$(x-2)^2 + (y+5)^2 = 9$$

Find the equation of the circle satisfying the given information

1) passes through (2,3) (3,2) and (-4,3)

Solution

Method 1

Gen. form

$$x^2+y^2+Dx+Ey+F=0$$

replace x & y for each point in the General equation

$(2,3)$

$$2^2+3^2+2D+3E+F=0$$

$$4+9+2D+3E+F=0$$

$$2D+3E+F=-13 \quad \dots (i)$$

$(3,2)$

$$3^2+2^2+3D+2E+F=0$$

$$9+4+3D+2E+F=0$$

$$3D+2E+F=-13 \quad \dots (ii)$$

$(-4,3)$

$$(-4)^2+3^2+(-4)D+3E+F=0$$

$$16+9-4D+3E+F=0$$

$$-4D+3E+F=-25 \quad \dots (iii)$$

From (iii) Let $F=4D-25-3E \quad \dots (iv)$

replace (iv) in (i) and (ii)

in (i)

$$2D+\cancel{3E}+4D-25-\cancel{3E}=-13$$

$$6D=-13+25$$

$$\underline{D=2}$$

In (ii) $3D+2E+4D-25-3E=-13$

$$7D-E=-13+25$$

$$7D-E=12$$

$$7(2)-E=12$$

$$\underline{\underline{E=2}}$$

In either (i), (ii) or (iii) replace the values of D and E to find F

In (i)

$D=2$

$E=2$

$F=?$

$$2D+3E+F=-13$$

$$2(2)+3(2)+F=-13$$

$$F=-13-10$$

$$F=-23$$

replace D, E & F in general equation

$D=2$, $E=2$ & $F=-23$

$$x^2+y^2+Dx+Ey+F=0$$

$$x^2+y^2+2x+2y-23=0 \quad \text{Equation}$$

In standard form

$$x^2+2x+y^2+2y-23=0$$

$$x^2+2x+y^2+2y=23$$

complete the square

$$(x^2+2x+1)+(y^2+2y+1)=23+1+1$$

$$(x^2+x+x+1)+(y^2+y+y+1)=25$$

$$\underline{(x+1)^2+(y+1)^2=25}$$

Method 2 (graphing the points)

i) Find the center & radius

ii) The center will always lie on the perpendicular bisector (Geometric Principle)

iii) The perpendicular & bisector is half of the chord

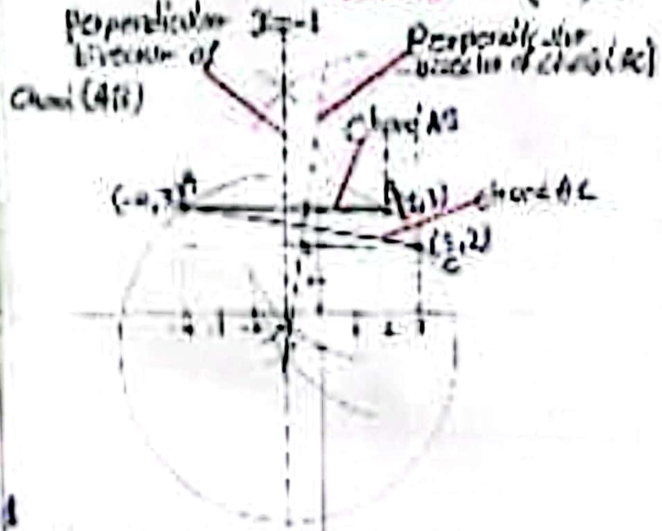
iv) Check if you've a vertical or horizontal chord from points given.

graph

$(2,3)$, $(3,2)$, $(-4,3)$
B, C, A

midpoint AB $= \left(\frac{2-4}{2}, \frac{3+3}{2}\right)$

$\boxed{x=-1}$ $(-1,3)$



midpoint (AC) $= \left(\frac{3-4}{2}, \frac{2+3}{2}\right)$

$= \left(-\frac{1}{2}, \frac{5}{2}\right)$

Gradient of AC

$m = \frac{y_2-y_1}{x_2-x_1} = \frac{3-2}{-4-3}$

$= -\frac{1}{7}$

From Gradient of AC, find gradient of perpendicular bisector of AC

$m_1 m_2 = -1$

$-\frac{1}{7} m_2 = -1$ $\quad \underline{m_2 = 7}$

Equation of perpendicular bisector of AC $\quad y - y_1 = m(x - x_1)$

$y_1 = \frac{5}{2}$ & $x_1 = -\frac{1}{2}$

$y - \frac{5}{2} = 7(x - (-\frac{1}{2}))$

$y - \frac{5}{2} = 7x + \frac{7}{2}$

$y = 7x + \frac{7}{2} + \frac{5}{2}$

$y = 7x + \frac{12}{2}$

From equation of the perpendicular bisector of AC, find y coordinate of the center as the intersection of the two perpendicular bisectors form the center.

& the x coordinate of the center is -1

$y = 7(-1) + \frac{12}{2}$ $\quad$ Hence $\boxed{\text{center } (-1,-1)}$

$y = \frac{-14+12}{2}$

$\boxed{y=-1}$

From center coordinate find radius using either A, B or C

$r = \sqrt{(x-h)^2 + (y-k)^2}$ $\quad h=-1, \; k=-1$

$r = \sqrt{(-4+1)^2 + (3+1)^2}$ $\quad A(-4, 3)$

$r = \sqrt{9+16}$

$r = 5$

General form of Equation

$(x-h)^2 + (y-k)^2 = r^2$

$(x+1)^2 + (y+1)^2 = (5)^2$

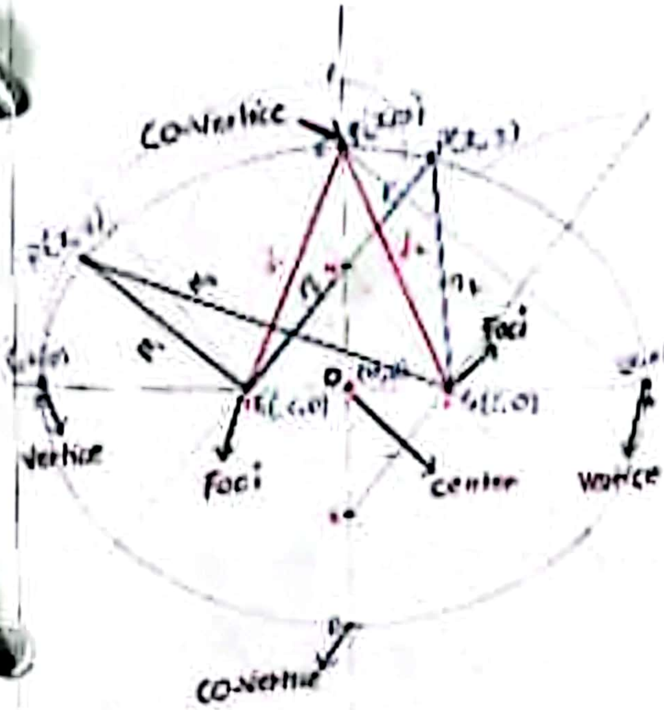
$\underline{(x+1)^2 + (y+1)^2 = 25}$

The Ellipse

- Is the set of all points the sum of whose distance from two fixed points is a constant.
- The angle between the plane and the axis is greater than the vertex angle.
- Locus of a point which moves so that the ratio of its distance from a fixed point to its distance from a fixed line is equal to $e<1$.

The fixed point is F_1 or F_2

The fixed line is called a directrix



$AB = \sqrt{(a-(-a))^2 + (0-0)^2}$

$= \sqrt{(a+a)^2}$

$= \sqrt{(2a)^2}$

$AB = 2a$

$\boxed{m_1 + m_2 = d_1 + d_2 = n_1 + n_2 = 2a}$

NOTE

$(m_1 + m_2 = d_1 + d_2 = n_1 + n_2) \neq F_1F_2$

$OC = b$ $\quad d_1 = d_2 = 2a$

$OA = a$ $\quad d_1 = a$

$OF_1 | OF_2 = c$ $\quad d_2 = a$

Standard Equation of an Ellipse

$|F_1P| + |PF_2| = 2a$

$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$

$x^2c^2 = a^2x^2 + a^2c^2 + a^2y^2 - a^4$

$x^2c^2 - a^2x^2 = a^2c^2 + a^2y^2 - a^4$

$\frac{x^2(c^2-a^2)}{(c^2-a^2)} = \frac{a^2(c^2-a^2)}{(c^2-a^2)} + \frac{a^2y^2}{(c^2-a^2)}$

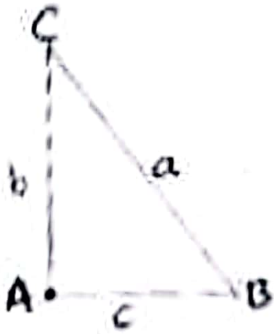
$x^2 = a^2 + \frac{a^2y^2}{c^2-a^2}$

$x^2 - \frac{a^2y^2}{c^2-a^2} = a^2$

$x^2 + \frac{\frac{a^2y^2}{a^2-c^2}}{a^2} = \frac{a^2}{a^2}$

$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$

$a^2 - c^2 = b^2$



$b^2 = a^2 - c^2$

$\therefore$ Equation of an ellipse is

- center of the ellipse $(0,0)$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with x-axis as the major axis

NOTE $a^2 > b^2$

$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$ with y-axis as the major axis

NOTE $a^2 \quad b^2$

Finding the vertices, foci & co-vertices when centre is at (h,k)

When $a>b$

=> Vertices

$V_1(a+h,k)$ $V_2(-a+h,k)$

=> Co-vertices

$C.V_1(h,b+k)$ $C.V_2(h,-b+k)$

=> Foci

$F_1(h+c,k)$ $F_2(h-c,k)$

When $a<b$

=> Vertices

$V_1(h,k+b)$ $V_2(h,k-b)$

=> Co-vertices

$C.V_1(h+a,k)$ $C.V_2(h-a,k)$

=> Foci

$F_1(h,c+k)$ $F_2(h,k-c)$

Eccentricity

=> The eccentricity of an ellipse describes the shape of an ellipse, the degree of ovalness of an ellipse is described by the eccentricity.

- The closer the eccentricity value is to zero(0) the more oval its shape is
- When eccentricity is zero(0) the shape is a circle
- The closer the eccentricity value is to 1 the more flat its shape is.

$$\text{Eccentricity}(e) = \frac{c}{a}$$ When x-axis is the major axis

$$\text{Eccentricity}(e) = \frac{c}{b}$$ When y-axis is the major axis

Directrix

When the x-axis is the major axis

$$\text{Directrix} = x = \frac{a^2}{c} = \frac{a}{e}$$

When the y-axis is the major axis

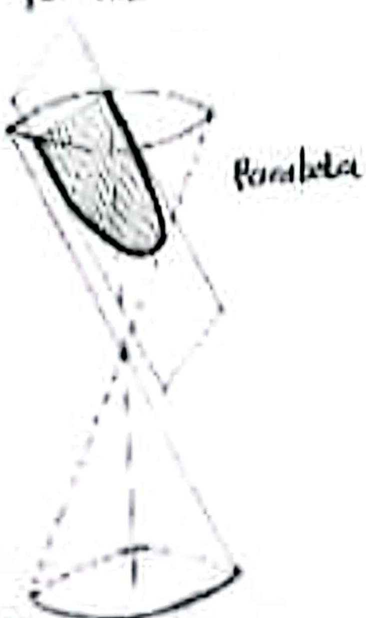
$$\text{Directrix} = y = \frac{b^2}{c} = \frac{b}{e}$$

ii) PARABOLA

A parabola is the set of all points that are equidistant from a fixed line called the **directrix** and a fixed point called the **focus**.

The midpoint between the focus and the directrix is the **vertex** and the line passing through the focus and the vertex is the **axis**.

The angle between the plane and the axis is equal to the vertex angle, making the plane to be parallel with the generator and hence resulting into a conic section called a parabola.



i) $x^2 = 4Py$

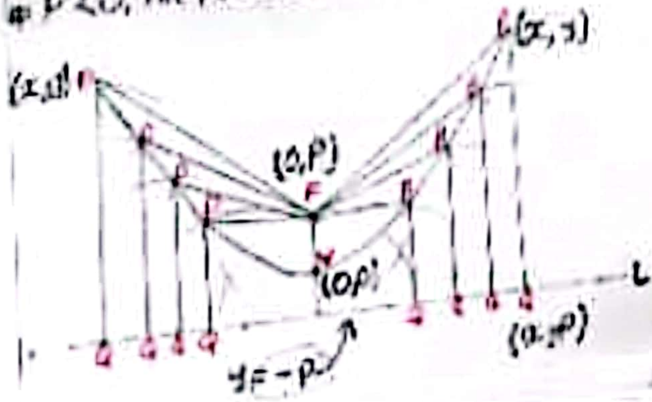
$P > 0$, the parabola opens upward

$P < 0$, the parabola opens downwards

ii) $y^2 = 4Px$

$P > 0$, the parabola opens to the right

$P < 0$, the parabola opens to the left



F = focus $(0, P)$

V = vertex $(0, 0)$

$$\overline{PF} = \overline{PQ}$$

$$\sqrt{(x-0)^2 + (y-P)^2} = \sqrt{(x-x)^2 + (y+P)^2}$$

$$\left(\sqrt{x^2 + (y-P)^2}\right)^2 = \left(\sqrt{(x-x)^2 + (y+P)^2}\right)^2$$

$$x^2 + y^2 - 2yP + P^2 = y^2 + 2yP + P^2$$

$$-2yP = -x^2 + 2yP$$

$$x^2 = 2yP + 2yP$$

$$x^2 = 4yP$$

$\therefore$ $x^2 = 4yP$ is the equation of a parabola whose vertex is at the origin.

⇒ Note that substituting $-x$ for x leaves the equation unchanged, verifying symmetry with respect to the y-axis.

$y^2 = 4Px$

⇒ Substituting $-y$ for y leaves the equation unchanged, verifying symmetry with respect to the x-axis.

Vertex at (h, k)

The form of the equation depends on whether the axis of the parabola is parallel to the x-axis or to the y-axis

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Standard form Of The Equation Of the Parabola with vertex (h,k)

* $\{(x-h)^2 = 4p(y-k)\}$ $\{\text{axis} \Rightarrow x = h\}$

$\Rightarrow$ Focus $(h, k+p)$ Directrix $y = k-p$

* $\{(y-k)^2 = 4p(x-h)\}$ $\{\text{axis} \Rightarrow y = k\}$

$\Rightarrow$ Focus $(h+p, k)$ Directrix $x = h-p$

$\Rightarrow$ The graph of a parabola whose equation is

$$(x-h)^2 = 4p(y-k)$$

opens upward if $p>0$

opens downward if $p<0$

$\Rightarrow$ The graph of a parabola whose equation is

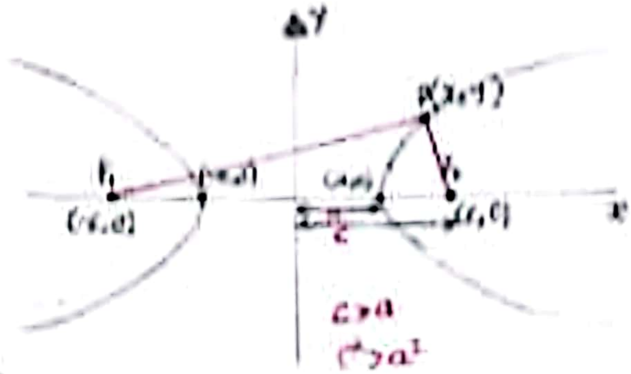
$$(y-k)^2 = 4p(x-h)$$

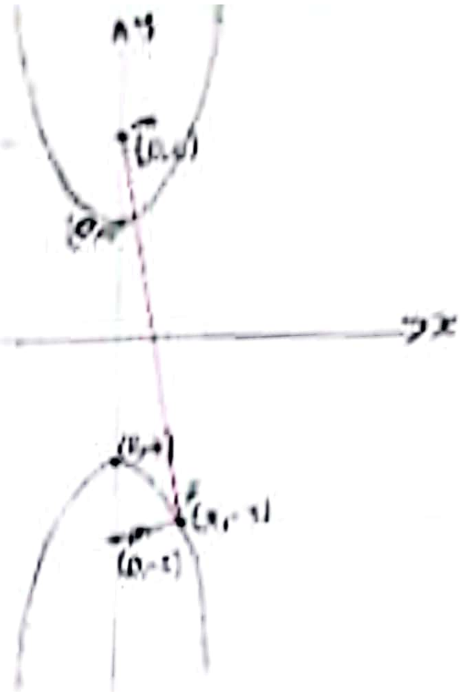
opens to the right if $p>0$

opens to the left if $p<0$

The Hyperbola

* a set of all points the difference of whose distances from two fixed points is a positive constant.
The two fixed points are called the foci of the hyperbola (F_1 & F_2)





$$|PF_1 - PF_2| = 2a$$

$F_1(-c,0)$
$F_2(c,0)$
$P(x,y)$

$$\left(\sqrt{(x+c)^2+y^2}\right)-\left(\sqrt{(x-c)^2+y^2}\right) = 2a$$

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1 \qquad c^2>a^2 \quad c^2-a^2>0$$

We let $b^2=c^2-a^2$
We then have

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$$

Asymptote

$$\boxed{y=\pm\frac{b}{a}x}$$

Foci $(\pm c,0)$
Vertices $(\pm a,0)$

$$|F_1P - PF_2| = 2a$$

$F_1(0,c)$
$F_2(0,-c)$
$P(x,y)$

$$\left(\sqrt{x^2+(y+c)^2}\right)-\left(\sqrt{x^2+(y-c)^2}\right) = 2a$$

$c>0$
$c^2>a^2$
$c^2-a^2>0$

$$\frac{y^2}{a^2}-\frac{x^2}{c^2-a^2}=1$$

We let $b^2=c^2-a^2$
We then have

$$\boxed{\frac{y^2}{a^2}-\frac{x^2}{b^2}=1}$$

Asymptote

$$\boxed{y=\pm\frac{a}{b}x}$$

Eccentricity

* the ratio of the distance of a point on the hyperbola from the focus and the directrix

$e=\frac{\text{Foci}}{\text{Vertice}}=\frac{c}{a}$ when x is the prime axis

$\cancel{e=\frac{\text{Foci}}{\text{Vertice}}=\frac{c}{b}}$ / or

When is at (h,k)

Directrix

⇒ It is straight line that is perpendicular to the prime axis that is used to generate a curve, it is a line from which the hyperbola curve ...

⇒

when x is the prime axis

$$x = \frac{a^2}{c} = \frac{a}{e}$$

when y is the prime axis

$$y = \frac{a^2}{c} = \frac{a}{e}$$

When center is at (h,k)

When symmetric about the x-axis

x^2

Equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Focus $(h \pm c, k)$

Vertices $(h \pm a, k)$

Asymptotes

$$y-k = \pm \frac{b}{a}(x-h)$$

when symmetric about the y-axis

equation

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Focus $(h, k \pm c)$ Vertices $(h, k \pm a)$

Asymptotes

$$y-k = \pm \frac{a}{b}(x-h)$$

Parametric Equations

- An equation where the variables (usually x and y) are expressed in terms of a third parameter usually expressed as t.

The object's coordinates are expressed as a function of t and then write x and y as

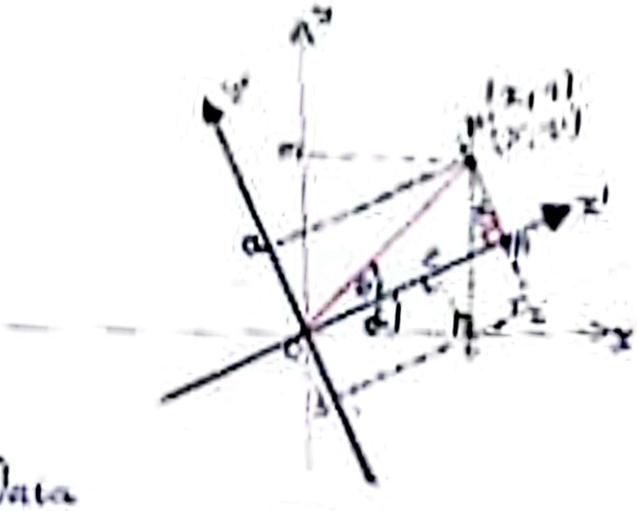
$$x = f(t) = 24\sqrt{2}\,t$$

$$y = g(t) = -16t^2 + 24\sqrt{2}\,t$$

From $y = -\frac{x^2}{72} + x$

ROTATION OF AXES

In the general equation for conic sections $Ax^2+Cy^2+Dx+Ey+F=0$, the xy term did not appear because the axes were parallel to the coordinate axes. To eliminate the xy term, the coordinate axes are rotated counterclockwise about the origin through an angle α.



$$\boxed{\begin{aligned} x &= r\cos(\theta+\alpha) \\ x &= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \end{aligned}}$$

$$\boxed{\begin{aligned} y &= r\sin(\theta+\alpha) \\ y &= r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{aligned}}$$

$x' = r\cos\theta \quad y' = r\sin\theta$

$$\boxed{x = x'\cos\alpha - y'\sin\alpha \; \ldots \; (i)}$$

$$\boxed{y = y'\cos\alpha + x'\sin\alpha \; \ldots \; (ii)}$$

Data

$mb = oc = x$

$no = bc = y$

$ab = oc = (dF + Fc)$

$cd = bc$

$oe = dF$

$eF = nc$

$$\boxed{\begin{aligned} oe &= oF\cos\alpha \\ bc &= y\cos\alpha \end{aligned}} \qquad \boxed{\begin{aligned} oe &= oF\cos\alpha \\ oe &= x\cos\alpha \end{aligned}}$$

$$\boxed{\begin{aligned} Fc &= cF\sin\alpha \\ Fc &= y\sin\alpha \end{aligned}} \qquad \boxed{\begin{aligned} eF &= oF\sin\alpha \\ eF &= x\sin\alpha \end{aligned}}$$

To find equations of x' & y'

$x' = ob = dF + Fc$

$dF = oe = x\cos\alpha$

$Fc = y\sin\alpha$

$x' = dF + Fc$

$$\boxed{x' = x\cos\alpha + y\sin\alpha \; \ldots \; (iii)}$$

$y' = bn = (bc - nc)$

$bc = y\cos\alpha$

$nc = eF = x\sin\alpha$

$y' = bc - nc$

$$\boxed{y' = y\cos\alpha - x\sin\alpha}$$

<u>Half Angle Identities</u> →

$$\sin\theta = \pm\sqrt{\frac{1}{2}(1-\cos 2\theta)}$$

$$\cos\theta = \pm\sqrt{\frac{1}{2}(1+\cos 2\theta)}$$

THE DISCRIMINANT TEST

⇒ It is used for identifying the conic section without rotating the axes.

discriminant

$$\boxed{B^2 - 4AC = B'^2 - 4A'C'}$$

If :

$B^2 - 4AC = 0$ Parabola

$B^2 - 4AC > 0$ Hyperbola

$B^2 - 4AC < 0$ Ellipse

Example

Latus Rectum

⇒ The latus rectum of the parabola is a line segment passing through its focus and perpendicular to its axis.

→ It equals to four times the focal length.

$\underline{\underline{4P}}$

Circle

$$\begin{cases} x = h + r\cos(t) \\ y = k + r\sin(t) \end{cases}$$

Ellipse

$$\begin{cases} x = h + r_1\cos(t) \\ y = k + r_2\sin(t) \end{cases}$$

Hyperbola

Horizontal

$$\rightarrow \begin{cases} x = h + r_1\sec(t) \\ y = k + r_2\tan(t) \end{cases}$$

or

$$\rightarrow \begin{cases} x = h + r_1\csc(t) \\ y = k + r_2\cot(t) \end{cases}$$

Vertical

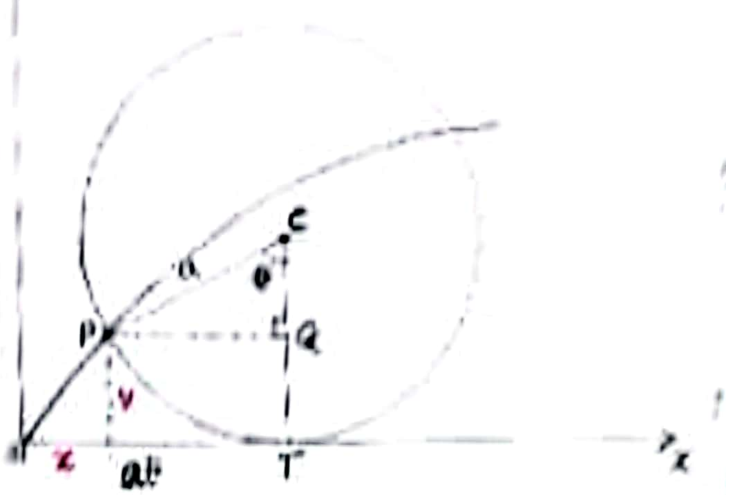
$$\rightarrow \begin{cases} x = h + r_1\tan(t) \\ y = k + r_2\sec(t) \end{cases}$$

or

$$\rightarrow \begin{cases} x = h + r_1\cot(t) \\ y = k + r_2\csc(t) \end{cases}$$

$\boxed{\theta \text{ is in radians}}$

$PC = TC = a \qquad TP = OT = a\theta$



$CQ = a\cos\theta \qquad PQ = a\sin\theta$

$y = TC - CQ \qquad x = OT - PQ$

$x = a\theta - a\sin\theta$

$y = a - a\cos\theta$

coordinates

$\boxed{y = a - a\cos\theta} \quad \boxed{x = a\theta - a\sin\theta}$

~~Example~~

Standard Parametrisations

Circle $x^2 + y^2 = a^2$

$x = a\cos t \quad y = a\sin t \quad 0 \le t \le 2\pi$

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x = a\cos t, \ y = b\sin t \quad 0 \le t \le 2\pi$

Cycloid generated by a circle of radius a:

$x = a(t - \sin t), \ y = a(1 - \cos t)$

$t \ge 0$

Higher Order Derivatives

Intro.

When $y = f(x)$, the diff. coefficient of y w.r.t x is $\frac{dy}{dx}$ or y' or $f'(x)$

FPS: $y = ax^n \quad \frac{dy}{dx} = nax^{n-1}$

y - [illegible] $- f(x)$

$\frac{dy}{dx}$ = [illegible]

Exercise

1. $f(x) = 3x^2 + 1$

$f'(x) = 6x$

(2) $f(x) = 2 - 4x^2 + \frac{2}{x} + x^{\frac{5}{2}}$

$= 2 - 4x^2,\ 2x^{-1},\ x^{\frac{5}{2}}$

$= -8x - 2x^{-2} + \frac{5}{2}x^{\frac{3}{2}}$

$f'(x) = -8x - \frac{2}{x^2} + \frac{5}{2}x^{\frac{3}{2}}$

If the expression $\frac{dy}{dx}$ or $f'(x)$ is diff. again we get the 2nd derivative written as $\frac{d^2y}{dx^2}$ or $f''(x)$

Methods Of Differentiation

1. When $y = uv = f(x)g(x)$, then

$$\boxed{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{or} \quad \frac{dy}{dx} = uv' + vu'}$$ Product Rule

2. When $y = \frac{u}{v} = \frac{f(x)}{g(x)}$, then

$$\boxed{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

or

$$\boxed{\frac{dy}{dx} = \frac{vu' - uv'}{v^2}}$$ Quotient Rule

3. If y is a function of u we may write $y = f(u)$ where u is a function of x, so that $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$ Chain Rule