21. METHODS OF DIFFERENTIATION AND APPLICATIONS OF DERIVATIVES

1. INTRODUCTION

The rate of change of one dependent quantity with respect to another dependent quantity has great importance. E.g. the rate of change of displacement of a particle with respect to time is called its velocity and the rate of change of velocity is called its acceleration. The rate of change of a quantity 'y' with respect to another quantity 'x' is known as the derivative or differentiable coefficient of 'y' with respect to 'x.'

According to the first principle of calculus, if y = f(x) is the derivative function, then the derivative of f(x) with respect to x is given by:

$$f'(x) = \frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Note: y', y_1, Dy can also be used to denote the derivative of y with respect to x. Differentiation is the process of finding the derivative of a function. $\Rightarrow \sin\beta > 0;\ \cos\alpha < 0$

2. DERIVATIVES OF SOME STANDARD FUNCTIONS

Different types of differentiation formulae

(a) $\frac{d}{dx}(\text{constant}) = 0$

(b) $\frac{d}{dx}(e^x) = e^x$

(c) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(d) $\frac{d}{dx}(\sin x) = \cos x$

(e) $\frac{d}{dx}(\tan x) = \sec^2 x$

(f) $\frac{d}{dx}(x^n) = nx^{n-1}$

(g) $\frac{d}{dx}(a^x) = a^x \log_e a$

(h) $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}$

(i) $\frac{d}{dx}(\cos x) = -\sin x$

(j) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(k) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(l) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

(m) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}, x \in R$

(n) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}} |x| > 1$

(o) $\frac{d}{dx}(\sinh x) = \cosh x$

(p) $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$

(q) $\frac{d}{dx}(\text{sech}x) = -\text{sech}x \tanh x$

(r) $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}, \forall x \in R$

(s) $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}, x \pm 1$

(t) $\frac{d}{dx}(\text{sech}^{-1}x) = -\frac{1}{|x|\sqrt{1-x^2}}, |x| < 1$

(u) $\frac{d}{dx}(e^{ax}\sin bx) = e^{ax}(a \sin bx + b \cos bx) = \sqrt{a^2+b^2}\, e^{ax}\sin(bx + \tan^{-1} b/a)$

(v) $\frac{d}{dx}(e^{ax}\cos bx) = e^{ax}(a \cos bx - b \sin bx) = \sqrt{a^2+b^2}\, e^{ax}\cos(bx + \tan^{-1} b/a)$

(w) $\frac{d}{dx}|x| = \frac{x}{|x|}\ (x \neq 0)$

(x) $\frac{d}{dx}[x] = 0, \forall x \in -I$ (where [.] denotes greatest integer function)

(y) $\frac{d}{dx}\{x\} = 1, \forall x \in R$ (where { . } denotes fractional part function)

(z) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(aa) $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

(ab) $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}, \forall x \in R$

(ac) $\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}} |x| > 1$

(ad) $\frac{d}{dx}(\cosh x) = \sinh x$

(ae) $\frac{d}{dx}(\coth x) = -\text{cosech}^2 x$

(af) $\frac{d}{dx}(\text{cosech}x) = -\text{cosech}x \coth x$

(ag) $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}, |x| > 1$

(ah) $\frac{d}{dx}(\coth^{-1}x) = \frac{1}{x^2-1}, x \neq \pm 1$

(ai) $\frac{d}{dx}(\text{cosech}^{-1}x) = \frac{-1}{|x|\sqrt{x^2+1}}, \forall x \in R$

(aj) $\frac{d}{dx}\log|x| = \frac{1}{x}, (x \neq 0)$

MASTERJEE CONCEPTS

If the function is continuous, you do not have to apply the first principle method to check differentiability. You can go directly for dy/dx and check whether dy/dx exists on both the left and right sides and are equal. If dy/dx does not exist for either one side or both the sides or if both the derivatives exist, but are not equal or finite, then the function is not differentiable.

E.g. Let y = sin(x) be a continuous function. Check differentiability at $x = \pi/2$.On checking for dy/dx = cos(x) on both the right and left sides, it is found to be equal and finite. Hence, y = sin(x) is differentiable at $x = \pi/2$.

MASTERJEE CONCEPTS

Misconception:

(i) In dy/dx, dy or dx does not exist individually.

(ii) $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ only if both $\frac{dy}{dx}$ and $\frac{dx}{dy}$ exist.

Rohit Kumar (JEE 2012, AIR 79)

3. PRODUCT RULE

(a) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(b) $\frac{d}{dx}(uvw) = uv\frac{d(w)}{dx} + uw\frac{d(v)}{dx} + vw\frac{d(u)}{dx}$

4. DIVISION RULE

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, where $v \neq 0$ (known as the quotient rule)

5. CHAIN RULE

$\frac{d}{dx}(f(g(x))) = f'(g(x)) \,.\, g'(x)$ or $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

Note: (a) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$, on condition that both $f'(x)$ and $g'(x)$exist

(b) $\frac{d}{dx}(k\, f(x)) = k\frac{d}{dx}(f(x))$, where k is any constant

Illustration 1: If $y = \frac{x^4 + x^2 + 1}{x^2 + x + 1}$; $\frac{dy}{dx} = px + q$, find p and q. **(JEE MAIN)**

Sol: Differentiate and compare.

$$y = \frac{(x^2+1)^2 - x^2}{x^2+x+1} = \frac{\left(\left(x^2+1\right)+x\right)\left(x^2+1-x\right)}{x^2+x+1} = x^2 + 1 - x \Rightarrow \frac{dy}{dx} = 2x - 1 \Rightarrow p = 2 \text{ and } q = -1$$

$$y = \frac{((x^1+1)+x)(x^2+1-x)}{x^2+x+1}$$

Illustration 2: If $y = \frac{x^3 + 2^x}{e^x}$, then find $\frac{dy}{dx}$. **(JEE MAIN)**

Sol: Differentiate

$$y' = \frac{e^x(3x^2 + 2^x \ln 2) - (x^3 + 2^x)e^x}{e^{2x}} = \frac{(3x^2 + 2^x \ln 2) - (x^3 + 2^x)}{e^x}$$

Illustration 3: If $y = \frac{\tan^{-1}x - \cot^{-1}x}{\tan^{-1}x + \cot^{-1}x}$, find $\left(\frac{dy}{dx}\right)_{x=1}$. **(JEE MAIN)**

Sol: Differentiate and put x = 1.

$y = \frac{2}{\pi}\left(\tan^{-1}x - \cot^{-1}x\right)$ ...... $\left(\tan^{-1} + \cot^{-1}x = \frac{\pi}{2}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\pi(1+x^2)} + \frac{2}{\pi(1+x^2)} = \frac{4}{\pi(1+x^2)} \Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = \frac{4}{2\times\pi} = \frac{2}{\pi}$

Illustration 4: Differentiate the following functions with respect to x: **(JEE MAIN)**

(i) $\sqrt{3x+2} + \frac{1}{\sqrt{2x^2+4}}$ (ii) $e^{\sec^2 x} + 3\cos^{-1}x$ (iii) $\log_7(\log x)$

Sol: (i) Let $y = \sqrt{3x+2} + \frac{1}{\sqrt{2x^2+4}} = (3x+2)^{1/2} + (2x^2+4)^{-1/2}$

$\frac{dy}{dx} = \frac{1}{2}(3x+2)^{\frac{1}{2}-1}\frac{d}{dx}(3x+2) + \left(-\frac{1}{2}\right)(2x^2+4)^{\frac{-1}{2}-1}\frac{d}{dx}(2x^2+4)$

$= \frac{1}{2}(3x+2)^{-\frac{1}{2}}.(3) - \left(\frac{1}{2}\right)(2x^2+4)^{-\frac{3}{2}}.4x = \frac{3}{2\sqrt{3x+2}} - \frac{2x}{(2x^2+4)^{3/2}}$

(ii) Let $y = e^{\sec^2 x} + 3\cos^{-1}x$

$\frac{dy}{dx} = e^{\sec^2 x}.\frac{d}{dx}(\sec^2 x) + 3\left(-\frac{1}{\sqrt{1-x^2}}\right) = e^{\sec^2 x}.\left(2\sec x\frac{d}{dx}(\sec x)\right) + 3\left(-\frac{1}{\sqrt{1-x^2}}\right)$

$= 2\sec x(\sec x\tan x)\, e^{\sec^2 x} + 3\left(-\frac{1}{\sqrt{1-x^2}}\right) = 2\sec^2 x\tan x\, e^{\sec^2 x} - 3\left(+\frac{1}{\sqrt{1-x^2}}\right)$

(iii) Let $y = \log_7(\log x) = \frac{\log(\log x)}{\log 7}$ (using change of base formula)

$\frac{dy}{dx} = \frac{1}{\log 7}\frac{d}{dx}(\log(\log x)) = \frac{1}{\log 7}\frac{1}{\log x}.\frac{d}{dx}(\log x) = \frac{1}{x\log 7\log x}$

Illustration 5: Find $\frac{dy}{dx}$, if $y = 3\tan x + 5\log_a x + \sqrt{x} - 3e^x + \frac{1}{x}$. **(JEE MAIN)**

Sol: Chain rule.

We have $y = 3\tan x + 5\log_a x + \sqrt{x} - 3e^x + \frac{1}{x}$

Thus, $\frac{dy}{dx} = \frac{d}{dx}\left(3\tan x + 5\log_a x + \sqrt{x} - 3e^x + \frac{1}{x}\right)$

$= \frac{d}{dx}(3\tan x) + \frac{d}{dx}(5\log_a x) + \frac{d}{dx}(\sqrt{x}) - \frac{d}{dx}(3e^x) + \frac{d}{dx}\left(\frac{1}{x}\right) = 3\sec^2 x + \frac{5}{x}(\log_e a)^{-1} + \frac{1}{2}x^{-\frac{1}{2}} - 3e^x - x^{-2}.$

Illustration 6: Let f, g and h be differentiable functions. If f(0) = 1, g(0) = 2, h(0) = 3 and the derivative pairwise products at x = 0 are (fg)′ (0) = 6, (gh)′(0) =4 and (hf)′ (0) = 5, then compute the value of (fgh)′ (0).

(JEE ADVANCED)

Sol: Product rule

$(fgh)' = f'gh + fhg' + fgh'$ ....(i)

$(fg)'(0) = 6 \Rightarrow (fg' + gf')(0) = 6$

$(gh)'(0) = 4 \Rightarrow (gh' + hg')(0) = 4$

$(hf)'(0) = 5 \Rightarrow (hf' + fh')(0) = 5$

$$(fgh)' = \frac{1}{2}(2f'gh + 2fg'h + 2fgh') = \frac{1}{2}(f'gh + f'gh + fg'h + fg'h + fgh' + fgh')$$

$$= \frac{1}{2}[h(f'g + fg') + g(f'h + fh') + f(g'h + gh')] = \frac{1}{2}[h(fg)' + g(fh)' + f(gh)']$$

$$\Rightarrow (fgh)'(0) = \frac{1}{2}[(3)(6) + (2)(5) + (1)(4)] = \frac{1}{2}[18 + 10 + 4] = 16$$

6. TRIGONOMETRIC TRANSFORMATIONS

In case of inverse trigonometric functions, it becomes very easy to differentiate a function by using trigonometric transformations.Given below are some important results on trigonometric and inverse trigonometric functions.

(a) $\sin 2x = 2\sin x\cos x = \dfrac{2\tan x}{1+\tan^2 x}$

(b) $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

(c) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(n) $\sin 3x = 3\sin x - 4\sin^3 x$

(d) $\cos 3x = 4\cos^3 x - 3\cos x$

(o) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

(e) $\sin^{-1} x + \cos^{-1} x = \tan^{-1} x + \cot^{-1} x = \sec^{-1} x + \text{cosec}^{-1} x = \dfrac{\pi}{2}$

(f) $\sin^{-1}(-x) = -\sin^{-1} x$

(p) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(d) $\tan^{-1}(-x) = -\tan^{-1} x$

(q) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$

(h) $\cot^{-1}(-x) = \pi - \cot^{-1} x$

(r) $\sec^{-1}(-x) = \pi - \sec^{-1} x$

(i) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$

(s) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$

(j) $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left|\dfrac{x \pm y}{1 \mp xy}\right|$

(t) $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$ (Be aware of ranges for 'x')

(k) $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$

(u) $2\tan^{-1} x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(l) $\dfrac{\pi}{4} - \tan^{-1} x = \tan^{-1}\left(\dfrac{1-x}{1+x}\right)$

(v) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$

(m) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$

(w) $3\tan^{-1} x = \tan^{-1}\left|\dfrac{3x - x^3}{1 - 3x^2}\right|$

MASTERJEE CONCEPTS

Some useful substitutions in finding derivatives are given below.

Sl. No.	Function	Substitution
(i)	$\sqrt{a^2-x^2}$	x = a sinθ or a cosθ
(ii)	$\sqrt{x^2+a^2}$	x = a tan θ or a cotθ
(iii)	$\sqrt{x^2-a^2}$	x = a sec θ or a cosecθ
(iv)	$\sqrt{\frac{a-x}{a+x}}$	x = a cos 2θ
(v)	$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2 \cos 2\theta$
(vi)	$\sqrt{ax-x^2}$	$x = a \sin^2\theta$
(vii)	$\sqrt{\frac{x}{a+x}}$	$x = a \tan^2\theta$
(viii)	$\sqrt{\frac{x}{a-x}}$	$x = a \sin^2\theta$
(ix)	$\sqrt{(x-a)(x-b)}$	$x = a \sec^2\theta - b \tan^2\theta$
(x)	$\sqrt{(x-a)(b-x)}$	$x = a \cos^2\theta + b \sin^2\theta$

Rohit Kumar (JEE 2012, AIR 79)

Illustration 7: If $y = \cot^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)$, find $\frac{dy}{dx}$. **(JEE MAIN)**

Sol: Substitute a suitable trigonometric function in place of x and simplify.

Putting x = tan θ, we have

$$y = \cot^{-1}\left(\frac{\sec\theta+1}{\tan\theta}\right) = \cot^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \cot^{-1}(\cot\theta/2) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{1+x^2}$$

MASTERJEE CONCEPTS

To differentiate a complex function, put x in some trigonometric form so that the function can be easily differentiated and then put back x in the form of an inverse trigonometric function.

E.g. Find the derivatives of $\sec^{-1}[1/(2x^2-1)]$ with respect to $\sqrt{1-x^2}$ at x = 1/2.

Sol. Putting $x = \cos\theta$, we get

$$u = \sec^{-1}\frac{1}{2\cos^2\theta-1} = \sec^{-1}(\sec 2\theta) = 2\theta \text{ and } y = \sqrt{1-x^2} = \sin\theta$$

$$\therefore\ u = 2\sin^{-1}y \Rightarrow \frac{du}{dy} = \frac{2}{\sqrt{1-y^2}} = \frac{2}{\sqrt{x^2}} \qquad \text{Thus, } \left.\frac{du}{dy}\right|_{x=1/2} = 4$$

Ravi Vooda (JEE 2009, AIR 71)

Illustration 8: If $y = \sqrt{(a-x)(x-b)} - (a-b)\tan^{-1}\sqrt{\left(\frac{a-x}{x-b}\right)}$, find $\frac{dy}{dx}$. **(JEE ADVANCED)**

Sol: Use Substitution to simplify the given expression and then differentiate.

Let $x = a\cos^2\theta + b\sin^2\theta$

$\therefore\ a - x = a - a\cos^2\theta - b\sin^2\theta = (a-b)\sin^2\theta$ ... (i)

$x - b = a\cos^2\theta + b\sin^2\theta - b = (a-b)\cos^2\theta$ ... (ii)

$\therefore\ y = (a-b)\sin\theta\cos\theta - (a-b)\tan^{-1}(\tan\theta)$

$$y = \frac{(a-b)}{2}\sin 2\theta - (a-b)\,\theta$$

$$\text{Then, } \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{(a-b)\cos 2\theta-(a-b)}{(b-a)\sin 2\theta} = \frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta = \sqrt{\left(\frac{a-x}{x-b}\right)} \quad \text{[From (i) and (ii)]}$$

7. LOGARITHMIC DIFFERENTIATION

If differentiation of an expression is done after taking log on both the sides, then it is known as logarithmic differentiation. This method is used when a given expression is in one of the following forms:

(a) $(f(x))^{g(x)}$

Let $y = (f(x))^{g(x)}$

Taking logarithm of both the sides, we get $\log y = g(x)\log f(x)$

Differentiating with respect to x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = g(x)\cdot\frac{1}{f(x)}\cdot f'(x) + \log f(x)\cdot g'(x) \Rightarrow \frac{dy}{dx} = y\left(\frac{g(x)}{f(x)}f'(x)+\log f(x).g'(x)\right)$$

$$\Rightarrow \frac{dy}{dx} = (f(x))^{g(x)}\left(\frac{g(x)}{f(x)}f'(x)+\log f(x).g'(x)\right)$$

Short method: The derivative of $[f(x)]^{g(x)}$can be directly written as:

$$\frac{d}{dx}(f(x))^{g(x)} = f(x)^{g(x)}\left(\frac{d}{dx}\{g(x)\log f(x)\}\right)$$

(b) Product of three or more functions

If y = f(x).g(x).h(x), then y′ = $f(x).g(x).h(x).\left(\frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}+\frac{h'(x)}{h(x)}\right)$

Illustration 9: If $f(x) = \prod_{n=1}^{100}(x-n)^{n(101-n)}$, find f′(101)/f(101). **(JEE ADVANCED)**

Sol: Use logarithms followed by differentiation.

$$f(x) = \prod_{n=1}^{100}(x-n)^{n(101-n)}$$

$$\ln f(x) = \sum_{n=1}^{100} n(101-n)\ln(x-n) \Rightarrow \frac{f'(x)}{f(x)} = \sum_{n=1}^{100}\frac{n(101-n)}{x-n}$$

$$\Rightarrow \frac{f'(101)}{f(101)} = \sum_{n=1}^{100}\frac{n(101-n)}{101-n} = \frac{100x101}{2} = 5050$$

Illustration 10: Find the derivative of $(\sin x)^{\cos x}$. **(JEE MAIN)**

Sol: Take logarithms on both sides and differentiate.

$$\frac{d}{dx}(\sin x)^{\cos x} = (\sin x)^{\cos x}\left[\frac{d}{dx}\{\cos x\log(\sin x)\}\right] = (\sin x)^{\cos x}[\cos x . \cot x - \sin x . \log \sin x]$$

Illustration 11: Find the derivative of x^x with respect to x. **(JEE MAIN)**

Sol: Use logarithms to find the derivative.

Let $y = x^x$ or $y = x^x$

log y = x log $\quad$ x = $e^{x \ln x}$

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \log x \quad \text{(or)} \frac{dy}{dx} = \frac{d}{dx} e^{x\ln x} = e^{x \ln x}\frac{d}{dx}(x \ln x)$$

$$\frac{dy}{dx} = x^x(1 + \log x) \quad \text{(or)} \quad = e^{x \ln x}\left\{x.\frac{1}{x} + \ln x 1\right\}$$

$$\therefore \frac{d}{dx}x^x = x^x\left(1+\log_e x\right) \quad \text{(or)} \quad = x^x(1 + \ln x)$$

Hence $\frac{d}{dx}(x^x) = x^x(1 + \ln x)$

Illustration 12: Differentiate $x^{\sin x}$ with respect to x. **(JEE MAIN)**

Sol: Similar to the previous illustration.

First method: Let $y = x^{\sin x}$

$\therefore$ log y = log x^{sinx} = sinx log x

Differentiating we get, $\frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x}$ + cos x log x

$\therefore \frac{dy}{dx} = \left(\frac{\sin x}{x} + \cos x \log x\right)x^{\sin x}$

Second Method: y = x^{sinx} = $e^{\sin x \log x}$

Therefore, $\frac{dy}{dx} = e^{\sin x \log x}\left[\frac{\sin x}{x} + \cos x \log x\right] = \left(\frac{\sin x}{x} + \cos x \log x\right)x^{\sin x}$

Illustration 13: Differentiate $e^{\cos^{-1}(x+1)}$ with respect to x. **(JEE ADVANCED)**

Sol: Similar to the previous illustration.

Let y = $e^{\cos^{-1}(x+1)}$ Then, $\frac{dy}{dx} = e^{\cos^{-1}(x+1)} \cdot \frac{dy}{dx}[\cos^{-1}(x+1)]$

$= e^{\cos^{-1}(x+1)} \cdot \frac{-1}{\sqrt{1-(x+1)^2}} \cdot \frac{d}{dx}(x+1) = \frac{-1}{\sqrt{1-(x+1)^2}} e^{\cos^{-1}(x+1)}$

Illustration 14: Differentiate x^{sinx}, x > 0, with respect to x. **(JEE ADVANCED)**

Sol: Let y = $x^{\sin x}$

Taking logarithm on both the sides, we get log y = sin xlog x

Therefore, $\frac{1}{y} \cdot \frac{dy}{dx} = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x)$

$\frac{1}{y} \cdot \frac{dy}{dx} = (\sin x)\frac{1}{x} + \log x \cos x \frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \log x\right] = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$

Illustration 15: Find f'(x), if f(x) = $(\sin x)^{\sin x}$, for all 0 < x < π. **(JEE ADVANCED)**

Sol: Similar to the previous illustration.

log y = log$(\sin x)^{\sin x}$ = sin x log (sin x)

Then, $\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\sin x \log(\sin x)) = \cos x \log(\sin x) + \sin x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x)$

= cos x log (sin x) + cos x = (1 + log (sin x)) cos x

$\Rightarrow \frac{dy}{dx}$ = y((1 + log (sinx)) cos x) = (1 + log (sinx)) sin x^{sinx} cosx

Illustration 16: Differentiate $x^{\cos^{-1}x}$ with respect to x. **(JEE MAIN)**

Sol: (i) Let y = $x^{\cos^{-1}x}$

Then, y = $e^{\cos^{-1}x . \ln x}$

Differentiating both the sides with respect to x, we get

$\frac{dy}{dx} = e^{\cos^{-1}x.\log x}\ \frac{d}{dx}(\cos^{-1}x\ .\ \log x)\ \Rightarrow \frac{dy}{dx}\ =\ x^{\cos^{-1}x}\left\{\log x.\frac{d}{dx}(\cos^{-1}x)+\cos^{-1}x.\frac{d}{dx}(\log x)\right\}$

$\Rightarrow \frac{dy}{dx}\ =\ x^{\cos^{-1}x}\left(\frac{-\log x}{\sqrt{1-x^2}}+\frac{\cos^{-1}x}{x}\right)$

(ii) Let $(\sin x)^{\cos^{-1}x}$

Then, y = $e^{\cos^{-1}x.\log\sin x}$

Differentiating both the sides with respect to x, we get

$\frac{dy}{dx} = e^{\cos^{-1}x.\log\sin x}\ \frac{d}{dx}(\cos^{-1}x\ .\ \log\sin x)$

$\Rightarrow \frac{dy}{dx}\ =\ (\sin x)^{\cos^{-1}x}\left\{\cos^{-1}x.\frac{d}{dx}(\log\sin x)+\log\sin x\frac{d}{dx}(\cos^{-1}x)\right\}$

$\Rightarrow \frac{dy}{dx}\ =\ (\sin x)^{\cos^{-1}}x\left\{\cos^{-1}.\frac{1}{\sin x}\cos x+\log\sin x\left(\frac{-1}{\sqrt{1-x^2}}\right)\right\}$

$\Rightarrow \frac{dy}{dx}\ =\ (\sin x)^{\cos^{-1}x}\left\{\cos^{-1}x.\ \cot x-\frac{\log\sin x}{\sqrt{1-x^2}}\right\}$

Illustration 17: Find $\frac{dy}{dx}$, if y = $(\sin x)^{\sin x^{\sin x^{\cdots\cdots\infty}}}$ **(JEE ADVANCED)**

Sol: Write the given expression as y = $(\sin x)^y$ and proceed.

We have y = $(\sin x)^y$, Therefore, log y = y log sin x

Differentiating both the sides with respect to x, we get

$\frac{1}{y}\frac{dy}{dx}\ = y\frac{d}{dx}(\log\sin x) + (\log\sin x)\ \frac{dy}{dx}\ = y\ \frac{\cos x}{\sin x}\ + \log\sin x\ \frac{dy}{dx}$

$\Rightarrow \left(\frac{1}{y}-\log\sin x\right)\frac{dy}{dx}\ = y\cot x \text{ or } \frac{dy}{dx}\ =\ \frac{y^2\cot x}{1-y\log\sin x} = \frac{y^2\cot x}{1-\log y}$

8. DIFFERENTIATION OF IMPLICIT FUNCTION

If in an equation, both x and y occur together, i.e. f(x, y) = 0, and the equation cannot be solved for either x or y, then x (or y) is called the implicit function of y (or x).

E.g. $x^3 + y^3 + 3axy + c = 0$, $x^y + y^x = a^b$, etc.

Working rule for finding the derivative

First method:

(a) Every term of f(x, y) = 0 should be differentiated with respect to x.

(b) The value of dy/dx should be obtained by rearranging the terms.

Second method:

If f(x, y) = constant, then $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$, where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are the partial differential coefficients of f(x, y) with respect to x and y, respectively.

Note: Partial differential coefficient of f(x, y) with respect to x can be defined as the ordinary differential coefficient of f(x, y) with respect to x keeping y constant.

E.g. $z = x^2y \Rightarrow \dfrac{\partial z}{\partial y} = x^2, \dfrac{\partial x}{\partial x} = 2xy$

Illustration 18: If y = $\dfrac{\sin x}{1+\dfrac{\cos x}{1+\dfrac{\sin x}{1+\dfrac{\cos x}{1+....\text{to } \infty}}}}$, then prove that $\dfrac{dy}{dx} = \dfrac{(1+y)\cos x + y\sin x}{1+2y+\cos x - \sin x}$. **(JEE ADVANCED)**

Sol: Write the R.H.S. in terms of x and y. Then differentiate the equation on both sides.

We have, $y = \dfrac{\sin x}{1+\left((\cos x)/(1+y)\right)} = \dfrac{(1+y)\sin x}{1+y+\cos x} \Rightarrow y + y^2 + y\cos x = (1+y)\sin x$

On differentiating both the sides with respect to x, we get

$$\frac{dy}{dx} + 2y\frac{dy}{dx} + \frac{dy}{dx}\cos x - y\sin x = \frac{dy}{dx}\sin x + (1+y)\cos x$$

$$\Rightarrow \frac{dy}{dx}\{1 + 2y + \cos x - \sin x\} = (1+y)\cos x + y\sin x \Rightarrow \frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1+2y+\cos x-\sin x}$$

Illustration 19: If f(x) = $x + \dfrac{1}{2x+\dfrac{1}{2x+\dfrac{1}{2x+......\infty}}}$, then compute the value of f (100) . f′(100). **(JEE MAIN)**

Sol: Same as above $y - x = \dfrac{1}{2x+\dfrac{1}{2x+\dfrac{1}{2x+......\infty}}}$

$$\Rightarrow \quad y - x = \frac{1}{2x+y-x} \quad \Rightarrow \quad (y-x)(x+y) = 1 \quad \Rightarrow \quad y^2 - x^2 = 1$$

$$\Rightarrow \quad (f(x))^2 = 1 + x^2 \Rightarrow \quad 2(f(x)) \times f'(x) = 2x$$

$$\Rightarrow \quad f(100) \,.\, f'(100) = 100$$

Illustration 20: If $y = \left((\ln x)^{\ln x}\right)^{(\ln x)^{(\ln x)^{\infty}}}$, then find $\dfrac{dy}{dx}$. **(JEE MAIN)**

Sol: Same as above

ln y = y ln(lnx)

$$\frac{1}{y} \times y' = \frac{y}{x\ln x} + \ln(\ln x).y' \qquad \Rightarrow y'\left(\frac{1}{y} - \ln(\ln x)\right) = \frac{y}{x\ln x} \Rightarrow \quad y'\left(\frac{1-y\ln(\ln x)}{y}\right) = \frac{y}{x\ln x}$$

$$\Rightarrow y' = \frac{y^2}{(x\ln x)(1-\ln(\ln x)y)}$$

Illustration 21: If $\log(x^2 + y^2) = 2\tan^{-1}\left(\dfrac{y}{x}\right)$, then prove that $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$. **(JEE ADVANCED)**

Sol: Differentiating both the sides of the given relation with respect to x, We get, $\frac{d}{dx}[\log(x^2+y^2)] = 2\frac{d}{dx}\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$

$$\Rightarrow \frac{1}{x^2+y^2}\cdot\frac{d}{dx}(x^2+y^2) = 2\cdot\frac{1}{1+(y/x)^2}\cdot\frac{d}{dx}\left(\frac{y}{x}\right) \Rightarrow \frac{1}{x^2+y^2}\left\{2x+2y\frac{dy}{dx}\right\} = 2.\frac{x^2}{x^2+y^2}\left\{\frac{x\frac{dy}{dx}-y.1}{x^2}\right\}$$

$$\Rightarrow 2.\left\{x+y\frac{dy}{dx}\right\} = 2\left\{x\frac{dy}{dx}-y\right\} \Rightarrow x+y\frac{dy}{dx} = x\frac{dy}{dx}-y \Rightarrow \frac{dy}{dx}(y-x) = -(x+y) \Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$

9. DIFFERENTIATION OF PARAMETRIC FORM

x and y are sometimes given as functions of a single variable, E.g. $x = \phi(t)$ and $y = \psi(t)$ are two functions, where t is a variable. Then in such cases, x and y are called parametric functions or parametric equations and t is called the parameter. To find $\frac{dy}{dx}$ in parametric functions, the relationship between x and y should be obtained by eliminating the parameter t and then it should be differentiated with respect to x. However, it is not convenient to eliminate the parameter every time. Therefore, $\frac{dy}{dx}$ can also be found by using the formula $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

E.g. If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, then find $\frac{dy}{dx}$.

Sol: Given that $\frac{dx}{d\theta} = a(1+\cos\theta)$, $\frac{dy}{d\theta} = a(\sin\theta)$

Therefore, $x = 0(\theta+\sin\theta)$. $y = a(1-\cos\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2}$$

Note: It may be noted here that $\frac{dy}{dx}$ can be expressed in terms of the parameter only without directly involving the main variables x and y.

Illustration 22: Find $\frac{dy}{dx}$, if $x = a\cos\theta$ and $y = a\sin\theta$. **(JEE MAIN)**

Sol: Differentiate the two equations w.r.t. θ and eliminate θ.

Given that $x = a\cos\theta$ and $y = a\sin\theta$

Therefore, $\frac{dx}{d\theta} = -a\sin\theta$, $\frac{dy}{d\theta} = a\cos\theta$.

Hence, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$.

Illustration 23: If $x = a\sec^2\theta$ and $y = a\tan^3\theta$, where $q \in R$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{8}$. **(JEE MAIN)**

Sol: Differentiation of Parametric form.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\,a\tan^2\theta\times\sec^2\theta}{2a\sec\theta\times\sec\theta\tan\theta} = \frac{3}{2}\tan\theta;\quad \text{At } \theta = \frac{\pi}{8}, \frac{dy}{dx} = \frac{3}{2}\left(\sqrt{2}-1\right)$$

Illustration 24: If x = cosecθ – sinθ and y = $cosec^n\theta - sin^n\theta$, then find $\frac{dy}{dx}$. **(JEE ADVANCED)**

Sol: Differentiation of Parametric form.

$$x = \text{cosec}\,\theta - \sin\theta$$

$$\Rightarrow \quad x^2 + 4 = (\text{cosec}\,\theta - \sin\theta)^2 + 4 = (\text{cosec}\,\theta + \sin\theta)^2 \quad \text{... (i)}$$

$$\text{and} \quad y^2 + 4 = (\text{cosec}^n\theta - \sin^n\theta)^2 + 4 = (\text{cosec}^n\theta + \sin^n\theta)^2 \quad \text{... (ii)}$$

$$\text{Now, } \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{n(\text{cosec}^{n-1}\theta)(-\text{cosec}\theta\cot\theta) - n\sin^{n-1}\theta\cos\theta}{-\text{cosec}\theta\cot\theta - \cos\theta}$$

$$= \frac{n(\text{cosec}^n\theta\cot\theta + \sin^{n-1}\theta\cos\theta}{(\text{cosec}\theta\cot\theta + \cos\theta)} = \frac{n\cot\theta(\text{cosec}^n\theta + \sin^n\theta)}{\cot\theta(\text{cosec}\theta + \sin\theta)} = \frac{n(\text{cosec}^n\theta + \sin^n\theta)}{(\text{cosec}\theta + \sin\theta)}$$

Illustration 25: Find $\frac{dy}{dx}$ if x = $at^2$and y = 2at. **(JEE MAIN)**

Sol: Given that x = at^2, y = 2at

Therefore, $\frac{dx}{dt}$ = 2at and $\frac{dy}{dt}$ = 2a.

$$\text{Hence, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}.$$

Illustration 26: If x = $\cos^3 t$ and y = $\sin^3 t$, then find $\frac{dy}{dx}$, for $\left(t \in \left(0, \frac{\pi}{2}\right)\right)$. **(JEE MAIN)**

Sol: $\frac{dx}{dt} = -3\cos^2 t \sin t \; (\neq 0) \Rightarrow \frac{dy}{dt} = 3\sin^2 t\cos t \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\sin^2 t\cos t}{-3\cos^2 t\sin t} = -\tan t$

Illustration 27: If y = sec 4x and t = tan x, then prove that $\frac{dy}{dt} = \frac{16t(1-t^4)}{(1-6t^2+t^4)^2}$. **(JEE ADVANCED)**

Sol: Write y in terms of t and differentiate.

$$y = \frac{1}{\cos 4x} = \frac{1+\tan^2 2x}{1-\tan^2 2x} = \frac{1+\left(2t/(1-t^2)\right)^2}{1-\left(2t/(1-t^2)\right)^2}$$

$$y = \frac{(1-t^2)^2+4t^2}{(1-t^2)^2-4t^2} = \frac{1+t^4-2t^2+4t^2}{1+t^4-2t^2-4t^2} = \frac{1+t^4+2t^2}{1+t^4-6t^2}; \qquad \frac{dy}{dt} = \frac{16t(1-t^4)}{(1-6t^2+t^4)^2}$$

Illustration 28: If x = $\frac{1+\ln t}{t^2}$ and y = $\frac{3+2\ln t}{t}$, then show that $\frac{ydy}{dx} = 2x\left(\frac{dy}{dx}\right)^2 + 1$ **(JEE MAIN)**.

Sol: Differentiation of Parametric form.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dt} = \frac{t\left(0+2(1/t)\right)-(3+2\ln t)1}{t^2} = \frac{2-3-2\ln t}{t^2} = -\left(\frac{1+2\ln t}{t^2}\right)$$

$$\frac{dx}{dt} = \frac{t^2\left(0+(1/t)\right)-(1+\ln t)2t}{t^4} = \frac{t-2t-2t\ln t}{t^4} = \frac{1-2-2\ln t}{t^3} = -\left(\frac{1+2\ln t}{t^3}\right)$$

$$\Rightarrow \frac{dy}{dx} = t \quad \Rightarrow 2x\left(\frac{dy}{dx}\right)^2 + 1 = 2.\frac{1+\ln t}{t^2}.t^2 + 1 = 3+2\ln t = yt = y\frac{dy}{dx}$$

10. DIFFERENTIATING WITH RESPECT TOANOTHER FUNCTION

Suppose u = f(x) and v = g(x) are two functions of x. To find the derivative of f(x) with respect to g(x), i.e. to find $\frac{du}{dv}$, the formula $\frac{du}{dv} = \frac{du/dx}{dv/dx}$ is used. Thus, to find the derivative of f(x) with respect tog(x), both are differentiated with respect to x and then the derivative of f(x) with respect to x is divided by the derivative of g(x) with respect to x. The procedure is demonstrated in illustration 29.

Illustration 29: Differentiate $\sin^2 x$ with respect to $e^{\cos x}$. **(JEE MAIN)**

Sol: Differentiate both the functions with respect to the common variable and use parametric form.

Let u(x) = $\sin^2 x$ and v(x) = $e^{\cos x}$. We want to find the value of $\frac{du}{dv} = \frac{du/dx}{dv/dx}$

Clearly, $\frac{du}{dx}$ = 2 sin x cos x and $\frac{dv}{dx} = e^{\cos x}(-\sin x) = -(\sin x)\, e^{\cos x}$

Hence, $\frac{du}{dv} = \frac{2\sin x\cos x}{(-\sin x)e^{\cos x}} = -\frac{2\cos x}{e^{\cos x}}$.

Illustration 30: Differentiate $\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ with respect to $\sqrt{1-x^4}$. **(JEE ADVANCED)**

Sol: Similar to the previous illustration.

$$\text{Let } u = \left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)^2}{(1+x^2)-(1-x^2)} = \frac{1+x^2+1-x^2+2\sqrt{1-x^4}}{2x^2}$$

$$\Rightarrow u = \frac{1+\sqrt{1-x^4}}{x^2} \Rightarrow \frac{du}{dx} = \frac{x^2\left(0+\left((-4x^3)/(2\sqrt{1-x^4})\right)\right)-\left(1+\sqrt{1-x^4}\right)2x}{x^4}$$

$$\Rightarrow \frac{du}{dx} = \frac{\left((-2x^5)/(\sqrt{1-x^4})\right)-2x\left(\sqrt{1-x^4}+1\right)}{x^4} \Rightarrow \frac{du}{dx} = \frac{-2x}{\sqrt{1-x^4}}\left(\frac{x^4+\left(\sqrt{1-x^4}\right)\left(1+\sqrt{1-x^4}\right)}{x^4}\right)$$

$$= \frac{-2x}{x^4\sqrt{1-x^4}}\left(x^4+\sqrt{1-x^4}+1-x^4\right) = \frac{-2x}{x^4\sqrt{1-x^4}}\left(\sqrt{1-x^4}+1\right) \qquad \text{....(i)}$$

Let v = $\sqrt{1-x^4}$

$$\frac{dv}{dx}=\frac{1}{2\sqrt{1-x^4}}(-4x^3)\ \Rightarrow \frac{dv}{dx}=\frac{-2x^3}{\sqrt{1-x^4}} \qquad ....(ii)$$

$$\frac{du}{dv}=\frac{du}{dx}/\frac{dv}{dx}\ \Rightarrow\ \frac{du}{dv}=\frac{-2x}{x^4\sqrt{1-x^4}}\left(\sqrt{1-x^4}+1\right)\frac{\sqrt{1-x^4}}{-2x^3}\Rightarrow \frac{du}{dv}=\frac{(1+\sqrt{1-x^4})}{x^6}$$

11. DIFFERENTIATION OF DETERMINANTS

To differentiate a determinant, one row (or column) at a time should be differentiated, keeping others unchanged, which is illustrated by the examples given below.

(i) If $F(x)=\begin{vmatrix} f(x) & g(x)\\ u(x) & v(x)\end{vmatrix}$, then $\frac{d}{dx}\{F(x)\}=\begin{vmatrix} f'(x) & g'(x)\\ u(x) & v(x)\end{vmatrix}+\begin{vmatrix} f(x) & g(x)\\ u'(x) & v'(x)\end{vmatrix}$

Also $\frac{d}{dx}\{F(x)\}=\begin{vmatrix} f'(x) & g(x)\\ u'(x) & v(x)\end{vmatrix}+\begin{vmatrix} f(x) & g'(x)\\ u(x) & v'(x)\end{vmatrix}$

(ii) If $F(x)=\begin{vmatrix} f & g & h\\ \ell & m & n\\ u & v & w\end{vmatrix}$ Where f, g, h, ℓ, m, n, u, v, w are functions of x and differentiable, then

$$F'(x)=\begin{vmatrix} f' & g' & h'\\ \ell & m & n\\ u & v & w\end{vmatrix}+\begin{vmatrix} f & g & h\\ \ell' & m' & n'\\ u & v & w\end{vmatrix}+\begin{vmatrix} f & g & h\\ \ell & m & n\\ u' & v' & w'\end{vmatrix}\ \Rightarrow F(x)=\begin{vmatrix} f' & g & h\\ \ell' & m & n\\ u' & v & w\end{vmatrix}+\begin{vmatrix} f & g' & h\\ \ell & m' & n\\ u & v' & w\end{vmatrix}+\begin{vmatrix} f & g & h'\\ \ell & m & n'\\ u & v & w'\end{vmatrix}$$

Illustration 31: If $f(x)=\begin{vmatrix} \sec\theta & \tan^2\theta & 1\\ \sec\theta & \tan x & x\\ 1 & \tan x-\tan\theta & 0\end{vmatrix}$, then find $f'(\theta)$. **(JEE MAIN)**

Sol: Above discussed method.

$$f'(x)=\begin{vmatrix} 0 & 0 & 0\\ \sec\theta & \tan x & x\\ 1 & \tan x-\tan\theta & 0\end{vmatrix}+\begin{vmatrix} \sec\theta & \tan^2\theta & 1\\ 0 & \sec^2 x & 1\\ 1 & \tan x-\tan\theta & 0\end{vmatrix}+\begin{vmatrix} \sec\theta & \tan^2\theta & 1\\ \sec\theta & \tan x & x\\ 0 & \sec^2 x & 0\end{vmatrix}$$

$$\Rightarrow\quad f'(\theta)=\begin{vmatrix} \sec\theta & \tan^2\theta & 1\\ 0 & \sec^2\theta & 1\\ 1 & 0 & 0\end{vmatrix}+\begin{vmatrix} \sec\theta & \tan^2\theta & 1\\ \sec\theta & \tan\theta & \theta\\ 0 & \sec^2\theta & 0\end{vmatrix}=(\tan^2\theta-\sec^2\theta)-\sec^2\theta\,(\theta\sec\theta-\sec\theta)=-1-(\sec^3\theta)(\theta-1)$$

12. SUCCESSIVE DIFFERENTIATION

If the first derivative $\frac{dy}{dx}$ of a function y = f(x) is also a differentiable function, then it can be further differentiated with respect to x. The derivative thus obtained is called the second derivative of y with respect to x and is denoted by $\frac{d^2y}{dx^2}$. If $\frac{d^2y}{dx^2}$ is also differentiable, then its derivative is called the third derivative of y and is denoted by $\frac{d^3y}{dx^3}$.

Similarly, $\frac{d^ny}{dx^n}$ denotes the n^{th} derivative of y. This process is known as successive differentiation and all these derivatives are called as successive derivatives of y.

The following symbols are also used to denote the successive derivatives of y = f(x):

$y_1, y_2, y_3, \ldots\ldots\ldots\ Y_n, \ldots\ldots\ldots$

$y', y'', y''', \ldots\ldots\ldots\ y^n, \ldots\ldots\ldots \Rightarrow \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots \frac{d^ny}{dx^n} \ldots\ldots, \Rightarrow Dy, D^2y, D^3y, \ldots\ldots D^ny \ldots\ldots\ldots$ (where $D \equiv \frac{d}{dx}$)

The following symbols are used to denote the value of the n^{th} derivative at x = a.

$y_n(a), y^n(a), \left(\frac{d^ny}{dx^n}\right)_{x=a}, D^ny(a)$ & $f^n(a)$

MASTERJEE CONCEPTS

Misconception: $\frac{d^ny}{dx^n} \neq \left(\frac{dy}{dx}\right)^n$

Rohit Kumar (JEE 2012 AIR 79)

13. Nth DERIVAVES OF SOME STANDARD FUNCTIONS

(a) $D^n(ax + b)^m = m(m-1)(m-2) \ldots\ldots (m-n+1)\, a^n(ax+b)^{m-n}$

(b) If $m \in N$ and $m > n$, then $D^n(ax+b)^m = \frac{m!}{(m-n)!} a^n(ax+b)^{m-n}$; $D^n(x^m) = \frac{m!}{(m-n)!} x^{m-n}$

(c) $D^n(ax+b)^n = n!\, a^n$; $D^n(x^n) = n!$

(d) $D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$; $D^n\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}$

(e) $D^n\{\log(ax+b)\} = \frac{(-1)^{n-1}(n-1)!}{(ax+b)^n} a^n$; $D^n(\log x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$

(f) $D^n(e^{ax}) = a^n e^{ax}$

(g) $D^n(a^{mx}) = m^n (\log a)^n a^{mx}$

(h) $D^n\{\sin(ax+b)\} = a^n \sin\left(ax+b+n\frac{\pi}{2}\right)$; $D^n(\sin x) = \sin\left(x+n\frac{\pi}{2}\right)$

(i) $D^n\{\cos(ax+b)\} = a^n \cos\left(ax+b+n\frac{\pi}{2}\right)$; $D^n(\cos x) = \cos\left(x+n\frac{\pi}{2}\right)$

(j) $D^n\{e^{ax}\sin(bx+c)\} = (a^2+b^2)^{n/2} e^{ax} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

(k) $D^n\{e^{ax}\cos(bx+c)\} = (a^2+b^2)^{n/2} e^{ax} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

(l) $D^n\left(\tan^{-1}\frac{x}{a}\right) = \frac{(-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta}{a^n}$, where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$

(m) $D^n(\tan^{-1}x) = (-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta$, where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

14. LEIBNITZ THEOREM

If u and v are two functions such that their n^{th} derivative exists, then the n^{th} derivative of their product can be found by the following formula:

$D^n(uv) = (D^n u)v + {}^nC_1 D^{n-1}u.\ Dv + {}^nC_2 D^{n-2}u.\ D^2v + ...+......+{}^nC_{n-1}Du.\ D^{n-1}v + ...+ u.D^nv$

The n^{th} derivative of a product of two functions can be found out by using this theorem. While using this theorem, the second function in the product is the function whose successive derivative starts to vanish (if it is possible) after some stepsand the first function is a function whose n^{th} derivative is easily known.

Illustration 32: If $y = x^3 \cos x$, find D^ny. **(JEE MAIN)**

Sol: Leibnitz theorem

Choose cos x as the first function and x^3 as the second function

$D^n(\cos x, x^3) = D^n(\cos x)(x^3) + {}^nC_1D^{n-1}(\cos x)(Dx^3) + {}^nC_2D^{n-2}(\cos x).(D^2x^3) + {}^nC_3D^{n-3}(\cos x).(D^3x^3)$

$$= x^3\cos\left(x+\frac{n\pi}{2}\right) + n.3x^2\cos\left(x+\frac{(n-1)\pi}{2}\right) + \frac{(n-1)}{1.2}6x.\cos\left(x+\frac{(n-2)\pi}{2}\right) + \frac{n(n-1)(n-2)}{1.2.3}.6.\cos\left(\frac{x+(n-3)\pi}{2}\right)$$

$$= x^3\cos\left(x+\frac{n\pi}{2}\right) + 3nx^2\sin\left(x+\frac{n\pi}{2}\right) - 3n(n-1)x\cos\left(x+\frac{n\pi}{2}\right) - n(n-1)(n-2)\sin\left(x+\frac{n\pi}{2}\right)$$

APPLICATION OF DERIVATIVES

1. THE INTERPRETATION OF THE DERIVATIVE

If y = f(x) be a given function, then the derivative/differential coefficient f′(x) or $\frac{dy}{dx}$ at the point $P(x_1, y_1)$ is called the trigonometric tangent of the angle ψ (say), which the positive direction of the tangent to the curve at P makes with the positive direction of the x-axis. Therefore, $\frac{dy}{dx}$ represents the slope of the tangent.

Thus, $f'(x) = \frac{dy}{dx}_{(x_1,y_1)} = \Psi$

Then,

(a) The inclination of the tangent with x-axis = $\tan^{-1}\frac{dy}{dx}$

(b) Slope of the tangent = $\frac{dy}{dx}$

(c) Slope of the normal = $-\frac{dx}{dy}$

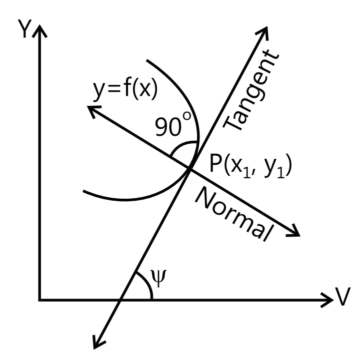

Figure 21.1

2. EQUATION OF TANGENT

(a) Equation of tangent to the curve y = f(x) at $A(x_1, y_1)$ is given by $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1)$

If the tangent at $P(x_1, y_1)$ of the curve y = f(x) is parallel to the x-axis (or perpendicular to the y-axis), then $\Psi = 0$, i.e its slop will be equal to zero.

$\Rightarrow \quad m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$

The converse also holds true. Thus, the tangent at (x_1, y_1) is parallel to the x-axis.

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$

(b) If the tangent at P (x_1, y_1) of the curve y = f(x) is parallel to the y-axis (or perpendicular to the x-axis), then $\Psi = \pi / 2$ and its slope will be infinity, i.e.

$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \infty$

The converse also holds true. Thus, the tangent at (x_1, y_1) is parallel to the y-axis.

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \infty$

(c) If at any point P (x_1, y_1) of the curve y = f(x) the tangent makes equal angles with both the axes, then at the point P, $\psi = x / 4$ or $3\pi / 4$.Therefore at P, $\tan \Psi = dy / dx = \pm 1$.

The converse of the result also holds true. Thus, at (x_1, y_1), the tangent line makes equal angles with both the axes.

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \pm 1$

(d) Concept of vertical tangent: y = f(x) has a vertical tangent at the point $x = x_0$ if

$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \infty$ or $-\infty$, but not both.

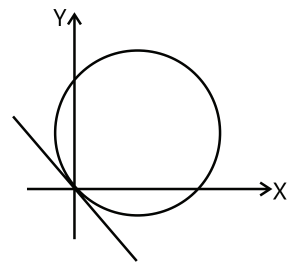

Figure 21.2

E.g. The functions $f(x) = x^{1/3}$ and f(x) = sin x both have a vertical tangent at x = 0

But $f(x) = x^{2/3}$, $f(x) = \sqrt{|x|}$ and $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ have no vertical tangents at x = 0.

(e) If a curve passes through the origin, then the equation of the tangent at the origin can be directly written by equating the lowest degree terms present in the equation of the curve to zero.

E.g.

(i) $x^2 + y^2 + 2gx + 2fy = 0$

Equation of tangent is $gx + fy = 0$

(ii) $x^3 + y^3 - 3x^2y + 3xy^2 + x^2 - y^2 = 0$

Equation of tangent at the origin is $x^2 - y^2 = 0$

(iii) $x^3 + y^3 - 3xy = 0$

Equation of tangent is $xy = 0$

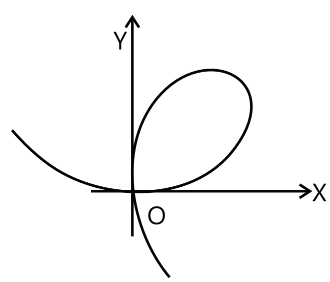

Figure 21.3

Note: This concept is valid only if the powers of x and y are natural numbers.

(f) Same line could be the tangent and normal to a given curve at a given point.

E.g. In $x^3 + y^3 - 3xy = 0$ (folium of Descartes), the line pair xy = 0 is both the tangent and normal at x = 0.

Some common parametric coordinates on a curve that are useful for differentiation

(a) For $x^{2/3} + y^{2/3} = a^{2/3}$, take parametric coordinates $x = a\cos^3$qand $y = a\sin^3\theta$.

(b) For $\sqrt{x} + \sqrt{y} = \sqrt{a}$, take $x = a\cos^4\theta$ and $y = a\sin^4\theta$.

(c) $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1$,where $x = a(\sin\theta)^{2/n}$ and $y = b(\cos\theta)^{2/n}$.

(d) For $c^2(x^2 + y^2) = x^2y^2$, take $x = c\sec\theta$ and $y = c\,\mathrm{cosec}\,\theta$.

(e) For $y^2 = x^3$, take $x = t^2$ and $y = t^3$.

Illustration 33: If the tangent to the curve $2y^3 = ax^2 + x^3$ at the point (a, a) cuts off intercepts α and β on the coordinate axes, where $a^2 + b^2 = 61$, the value of $|a|$ is ____. **(JEE MAIN)**

(A) 16 (B) 28 (C) 30 (D) 31

Sol: (C) Write the equation of the tangent and find the value of α and β in terms of a. Then use $a^2 + b^2 = 61$ to find the value of a.

The slope of the tangent is given by $\frac{dy}{dx} = \frac{2ax + 3x^2}{6y^2}$ The value of this slope at (a, a) is 5/6.

Hence, the equation of tangent is $y - a = \frac{5}{6}(x - a) \Rightarrow \frac{x}{-a/5} + \frac{y}{a/6} = 1$

Thus, the x-intercept α is $-\frac{a}{5}$, and the y-intercept β is $\frac{a}{6}$.

From $a^2 + b^2 = 61$, we get $\frac{a^2}{25} + \frac{a^2}{36} = 61 \Rightarrow a^2 = 25 \times 36 \Rightarrow |a| = 30$

3. EQUATION OF NORMAL

Equation of normal at (x_1, y_1) to the curve y = f(x) isgiven by the following formula:

$$(y - y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}(x - x_1) \quad \Rightarrow \quad (y - y_1)\left(\frac{dy}{dx}\right)_{(x_1,y_1)} + (x - x_1) = 0$$

Some facts regarding the normal

(a) Slope of the normal drawn at point P (x_1,y_1) to the curve $y = f(x) = -\left(\frac{dx}{dy}\right)_{(x_1,y_1)}$

(b) If the normal makes an angle of θ with the positive direction of the x-axis, then $-\frac{dx}{dy} = \tan\theta$ or $\frac{dy}{dx} = -\cot\theta$

(c) If the normal is parallel to the x-axis, then $\frac{dx}{dy} = 0$ or $\frac{dy}{dx} = \infty$

(d) If the normal is parallel to the y-axis, then $\left(\frac{dx}{dy}\right) = \infty$ or $\frac{dy}{dx} = 0$

(e) If the normal is equally inclined from both the axes or cuts equal intercept, then $-\left(\frac{dx}{dy}\right) = \pm 1$ or $\left(\frac{dy}{dx}\right) = \pm 1$

(f) The length of the perpendicular from the origin to the normal is $P' = \left|\frac{x_1 + y_1\left(\frac{dy}{dx}\right)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right|$

(g) The length of the intercept made by the normal on the x-axis is $x_1 + y_1\left(\frac{dy}{dx}\right)$ and the length of the intercept on the y-axis is $y_1 + x_1\left(\frac{dx}{dy}\right)$.

Illustration 34: Find out the distance between the origin and the normal to the curve $y = e^{2x} + x^2$ at the point whose abscissa is 0. **(JEE MAIN)**

(A) $\frac{1}{\sqrt{5}}$ (B) $\frac{2}{\sqrt{5}}$ (C) $\frac{3}{\sqrt{5}}$ (D) $\frac{2}{\sqrt{3}}$

Sol: (B) Write the equation of the normal and find the distance of origin from the normal.

The point on the curve corresponding to x = 0 is (0, 1)

$\frac{dy}{dx} = 2e^{2x} + 2x \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 2$

Therefore, the equation of the normal at the point (0, 1) is

$y - 1 = (-1/2)(x - 0) \Rightarrow 2y + x - 2 = 0$

Hence, the distance of the point (0, 0) from this line is $\frac{2}{\sqrt{5}}$.

4. LENGTH OF TANGENT, NORMAL, SUBTANGENT AND SUBNORMAL

4.1 Tangent

$$PT = MP \operatorname{cosec} \Psi = y\sqrt{1+\cot^2\psi} = \left|\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}\right|$$

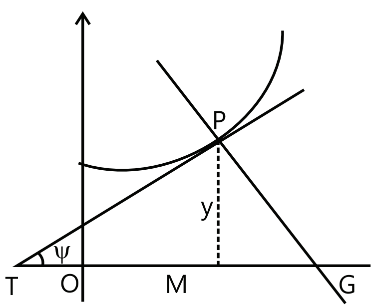

Figure 21.4

4.2 Subtangent

$$TM = MP \cot \Psi = \left|\frac{y}{(dy/dx)}\right|$$

4.3 Normal

$$GP = MP \sec \Psi = y\sqrt{1+\tan^2\psi} = \left|y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|$$

4.4 Subnormal

$$MG = MP \tan \psi = \left|y\left(\frac{dy}{dx}\right)\right|$$

Illustration 35: For the parabola $y^2 = 16x$, the ratio of the length of the subtangent to the abscissa is _____.

(A) 2 : 1 (B) 1 : 1 (C) X : Y (D) X^2 : Y **(JEE MAIN)**

Sol: (A) The length of subtangent is $\left|\frac{y}{(dy/dx)}\right|$

Differentiating, $2y\frac{dy}{dx} = 16$ Hence, $\frac{dy}{dx} = \frac{8}{y}$

Thus, the length of the subtangent is $y\frac{dx}{dy} = \frac{y^2}{8} = \frac{16x}{8} = 2x$

Therefore, the ratio of the length of the subtangent to the abscissa = 2x : x = 2 : 1.

Illustration 36: Find out the length of the normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = \pi/2$.

(JEE MAIN)

Sol: Use differentiation of the Parametric form. Length of the normal = $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2} \Rightarrow \left(\frac{dy}{dx}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

Moreover, at $\theta = \frac{\pi}{2}$, $y = a\left(1-\cos\frac{\pi}{2}\right) = a$

Therefore, the required length of the normal = $y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = a\sqrt{1+1} = \sqrt{2}a$

Illustration 37: The length of the subtangent to the ellipse $x = a\cos t$, $y = b\sin t$ at $t = \pi/4$ is _____.

(A) A (B) B (C) B/ $\sqrt{2}$ (D) A/ $\sqrt{2}$ **(JEE MAIN)**

Sol: (D) Similar to the previous illustration.

$\frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = b\cos t$; Therefore, $\left.\frac{dy}{dx}\right|_{t=\pi/4} = -\frac{b}{a}\cot(\pi/4) = -\frac{b}{a}$

Therefore, the length of the subtangent = $\left|y\frac{dx}{dy}\right|_{r=\pi/4} = \left|b\sin\frac{\pi}{4}\times-\frac{a}{b}\right| = \frac{a}{\sqrt{2}}$

5. ANGLE OF INTERSECTION OF TWO CURVES

The angle of intersection between two intersecting curves C_1 and C_2 is defined as the acute angle between their tangents (T_1 and T_2 or the normals) at the point of intersection of the two curves.

$\tan\psi = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$, where m_1 and m_2 are the slopes of the tangents T_1 and T_2 at the intersection point (x_1, y_1)

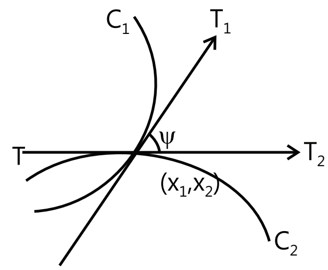

Figure 21.5

Note: If the two curves intersect orthogonally, i.e at right angle, then $\phi=\frac{\pi}{2}$. Hence, the condition will be

$$\left(\frac{dy}{dx}\right)_1 \cdot \left(\frac{dy}{dx}\right)_2 = -1$$

Illustration 38: Which of the following options represents the tangent of the angle at which the curves $y = a^x$ and $y = b^x (a \neq b > 0)$intersect? **(JEE ADVANCED)**

(A) $\frac{\log ab}{1+\log ab}$ (B) $\frac{\log a/b}{1+(\log a)(\log b)}$ (C) $\frac{\log ab}{1+(\log a)(\log b)}$ (D) None of these

Sol: (B) Differentiate the two curves and use the formula for angle between two lines.

Intersection of the two curves is given by $a^x = b^x$, which implies that x = 0. If α is the angle at which the two curves intersect, then

$$\tan\alpha = \frac{m_1 - m_2}{1+m_1m_2} = \frac{a^x \log a - b^x \log b}{1+a^xb^x(\log a)(\log b)} = \frac{(\log a/b)}{1+(\log a)(\log b)} \quad \text{(Putting } x = 0)$$

6. RATE MEASURE

Whenever a quantity y varies with another quantity x, satisfying the rule y = f(x), then $\frac{dy}{dx}$ (or f′(x)) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right|_{x=a}$ (or f′(a)) represents the rate of change of y with respect to x at x = a.

Illustration 39: The volume of a cube increases at the rate of 9 cm^3. How fast does the surface area increase when the length of an E.g. is 10 cm? **(JEE MAIN)**

Sol: Rate measurer.

Let x be the length of the side, V be the volume and S be the surface area of the cube.

Thus, $\frac{dV}{dt} = 9\ cm^3/s \Rightarrow 3x^2\frac{dx}{dt} = 9\ cm^3/s \Rightarrow \frac{dx}{dt} = \frac{3}{x^2}\ cm/s \Rightarrow \frac{dS}{dt} = \frac{d}{dt}(6x^2) = 12x\left(\frac{3}{x^2}\right) = \frac{36}{x}\ cm^2/s$

$$\Rightarrow \left.\frac{dS}{dt}\right]_{x=10cms} = 3.6\ cm^2/s$$

Illustration 40: A man of height 2 meters walks away from a 5-meter lamppost at a uniform speed of 6 meters per minute. Find the rate at which the length of his shadow increases. **(JEE MAIN)**

Sol: Use similarity to establish the relation between the rate at which length of shadow increases and speed of the man.

Let AB be the lamp-post. Let at any time t, the man CD be at a distance x metres from the lamp-post and y metres be the length of his shadow CE.

Then, $\frac{dx}{dt}$ = 6 meters / minute [given] ... (i)

Clearly, the triangles ABE and CDE are similar

$\Rightarrow \frac{AB}{CD} = \frac{AE}{CE} \Rightarrow \frac{5}{2} = \frac{x+y}{y} \Rightarrow 3y = 2x$

$\Rightarrow 3\frac{dy}{dt} = 2\frac{dx}{dt} \Rightarrow 3\frac{dy}{dt} = 2(6)$ [Using (i)]$\Rightarrow \frac{dy}{dt}$ = 4 meters / minute

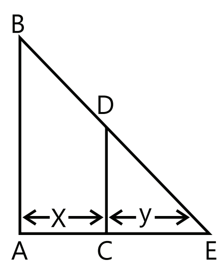

Figure 21.6

Illustration 41: An object has been moving in the clockwise direction along the unit circle $x^2 + y^2 = 1$. As it passes through the point $(1/2, \sqrt{3}/2)$, its y-coordinate decreases at the rate of 3 units per second. The rate at which the x-coordinate changes at this point is ______ units per second.

(A) 2 (B) $3\sqrt{3}$ (C) $\sqrt{3}$ (D) $2\sqrt{3}$ **(JEE MAIN)**

Sol: (B) Differentiate and proceed.

We find that $\frac{dx}{dt}$ when $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$ given that $\frac{dy}{dt} = -3$ units/s and $x^2 + y^2 = 1$.

Differentiating $x^2 + y^2 = 1$, we get $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

Putting $x = \frac{1}{2}$, $y = \sqrt{3}/2$ and $\frac{dy}{dt} = -3$, we get $\frac{1}{2}\frac{dx}{dt} + \frac{\sqrt{3}}{2}(-3) = 0 \Rightarrow \frac{dx}{dt} = 3\sqrt{3}$ (increasing)

Illustration 42: A given right circular cone has a volume p.The largest right circular cylinder that can be inscribed in the cone has a volume q. The ratio of p to q is _____. **(JEE MAIN)**

(A) 9 : 4 (B) 8 : 3 (C) 7 : 2 (D) None of these

Sol: (A) Let H be the height of the cone and α is its semi-vertical angle.

Let x be the radius of the inscribed cylinder and h be its height.

$h = QL = OL - OQ = H - x \cot a$

$p = \frac{1}{3}\pi(H \tan \alpha)^2 H$ ... (i)

V = volume of the cylinder $= \pi x^2 (H - x \cot \alpha)$

$\frac{dV}{dx} = \pi(2Hx - 3x^2 \cot \alpha)$

Hence, $\frac{dV}{dx} = 0 \Rightarrow x = 0$

$x = \frac{2}{3} H \tan \alpha$, $\left.\frac{d^2V}{dx^2}\right|_{x=\frac{2}{3}H\tan\alpha} = -2\pi H < 0$, so

V is maximum when $x = \frac{2}{3} H \tan \alpha$ and $q = V_{max} = \pi\frac{4}{9}H^2 \tan^2\alpha \frac{1}{3}H = \frac{4}{9}p$ [using (i)]

Therefore, p : q = 9 : 4

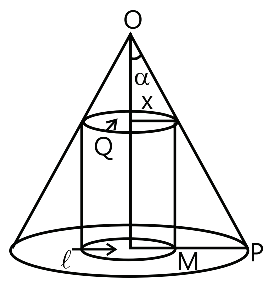

Figure 21.7

7. APPROXIMATION USING DIFFERENTIALS

To calculate the approximate value of a function, differentials may be used, wherein the differential of a function is equal to its derivative multiplied by the differential of the independent variable.

$dy = f'(x)dx$ or $df(x) = f'(x)\,dx$

MASTERJEE CONCEPTS

For the independent variable 'x', increment Δx and differential dx can be made equal, but the same cannot be applied in case of the dependent variable 'y,' i.e. $\Delta y \neq dy$.

Therefore, the approximate value of y when the increment Δx is given to the independent variable x in $y = f(x)$ is

$y + \Delta y = f(x + \Delta x) = f(x) + \frac{dy}{dx} . \Delta x$

$\Rightarrow f(x + \Delta x) = f(x) + f'(x)\,\Delta x$

Vaibhav Gupta (JEE 2009 AIR 54)

Illustration 43: Find the approximate value of the square root of 25.2. **(JEE MAIN)**

Sol: Consider a function f(x) = $\sqrt{x}$ and differentiate to get the derivative. Then replace x by x+Dx and proceed.

Let f(x) = $\sqrt{x}$, so f′(x) = $\frac{1}{2\sqrt{x}}$ We can write 25.2 as 25 + 0.2

By taking x = 25 and Δx = 0.2, now f(x + Δx) = f(x) + f′(x). Δx

$\Rightarrow \quad \sqrt{x} + \frac{1}{2\sqrt{x}}.\Delta x = \sqrt{25} + \frac{1}{2\sqrt{25}}.\ 0.2$

$= \quad 5 + \frac{0.2}{10} = 5 + 0.02 = 5.02$

Illustration 44: What is the approximate change in the volume V of a cube of side x meters caused by increasing the side by 2%? **(JEE MAIN)**

Sol: Differentiate the equation V = x^3 and use the relation $\Delta V = \frac{dV}{dx}\Delta x$.

Let Δ(x) be the change in x and ΔV be the corresponding change in V

Given that $\frac{\Delta x}{x} \times 100 = 2$

We know that $V = x^3 \ \therefore\ \frac{dV}{dx} = 3x^2$

Therefore, $\Delta V = \frac{dV}{dx}\Delta x \quad \Rightarrow \quad \Delta V = 3x^2\Delta x = 3x^2 \times \frac{2x}{100} = 0.06\ x^3 m^3$

The approximate change in volume is 0.06 x^3m^3.

Illustration 45: What is the approximate value of cos 40°? **(JEE ANDANCED)**

(A) 0.7688 (B) 0.7071 (C) 0.7117 (D) 0.7

Sol: (A) Take a function f(x) = cos x and proceed.

Let f(x) = cos x. $40° = 45° - 5° = \frac{\pi}{4} - \frac{\pi}{180} \times 5 = \frac{\pi}{4} - \frac{\pi}{36}$ radians

A differential is used to estimate the change in cos x

When x decreases from $\frac{\pi}{4}$ to $\frac{\pi}{4} - \left(\frac{\pi}{36}\right)$

f′(x) = – sin x and df (x)= f′(x) h = – h sin x

With $x = \frac{\pi}{4}$ and $h = -\frac{\pi}{36}$, df is given by

$df = -f'(x)h = -\left(-\frac{\pi}{36}\right)\sin\left(\frac{\pi}{4}\right) = \frac{\pi}{36}\cdot\frac{1}{\sqrt{2}} = \frac{\pi\sqrt{2}}{72} = 0.0617$

cos 40 ≡ cos 45 + 0.0617 ≡ 0.7071 + 0.0617 = 0.7688.

8. SHORTEST DISTANCE BETWEEN TWO CURVES

It has been found that the shortest distance between two non-intersecting curves is always along the common normal (wherever defined).

Illustration 46: Find out the shortest distance between the line y = x – 2 and the parabola $y = x^2 + 3x + 2$. **(JEE MAIN)**

Sol: The distance would be minimum at the point on the parabola where the slope of the tangent is equal to the slope of the given line.

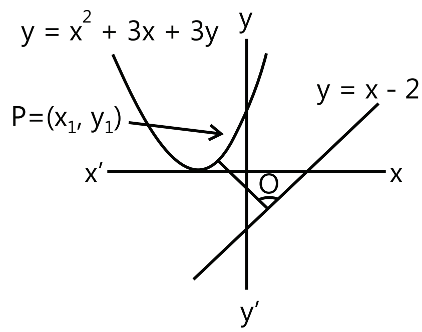

Figure 21.8

Let $P(x_1, y_1)$ is the point closest to the line y = x – 2

Then, $\left.\frac{dy}{dx}\right|_{(x_1,y_1)}$ = slope of the line

$\Rightarrow \quad 2x_1 + 3 = 1 \Rightarrow x_1 = -1$ and $y_1 = 0$

Therefore, point (–1, 0) is the closest and its perpendicular distance from the line y = x – 2 gives the shortest distance.

$\Rightarrow$ Shortest distance = $\frac{3}{\sqrt{2}}$ units

Illustration 47: Which of the following points of the curve $y = x^2$ is closest to (4, – ½)? **(JEE MAIN)**

(A) (1, 1) (B) (2, 4) (C) (2/3, 4/9) (D) (4/3, 16/9)

Sol:(A) Using distance formula find the distance of the given point from the curve and find the minima.

Let the required point be (x, y) on the curve.

Hence, d = $\sqrt{(x-4)^2 + (y+1/2)^2}$ should be minimum, which is enough to consider.

$D = (x-4)^2 + (y + 1/2)^2 = (x-4)^2 + (x^2 + 1/2)^2$

$D' = 4x^3 + 4x - 8$

Now for critical points

$D' = 0$ so $x^3 + x - 2 = 0 \Rightarrow x = 1$

Clearly D″ at x = 1 is 16 > 0.

Thus, D is minimum when x = 1. Hence the required point is (1, 1).

PROBLEM-SOLVING TACTICS

- Reduce any fractions to be as basic as possible.
- Recognise when we can use the chain rule. it enables us to differentiate functions that often seem impossible to differentiate. Whenever you see a nested function, try to assess if the chain rule is needed (it usually is).
- We always want to start a long chain of differentiation by differentiating the last part of the function to touch the input - in short, the outermost part of the function.

FORMULAE SHEET

$\frac{dc}{dx} = 0$	$\frac{d}{dx}(cu) = c\frac{du}{dx}$
$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
$\frac{d}{dx}x^n = nx^{n-1}$	$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
$\frac{d}{dx}a^x = (\ln a)\,a^x$	$\frac{d}{dx}a^u = (\ln a)\,a^u\frac{du}{dx}$
$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}e^u = e^u\frac{du}{dx}$
$\frac{d}{dx}\log_a x = \frac{1}{(\ln a)x}$	$\frac{d}{dx}\log_a u = \frac{1}{(\ln a)u}\frac{du}{dx}$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$
$\frac{d}{dx}\cot x = -\text{cosec}^2 x$	$\frac{d}{dx}\cot u = -\text{cosec}^2 u\frac{du}{dx}$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\sec u = \sec u \tan u\frac{du}{dx}$
$\frac{d}{dx}\text{cosec}\, x = -\text{cosec}\, x \cot x$	$\frac{d}{dx}\text{cosec}\, u = -\text{cosec}\, u \cot u\frac{du}{dx}$
$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$	$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$

* Equation of tangent to the curve y = f(x) at $A(x_1, y_1)$ is $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} (x - x_1)$

* Equation of normal at (x_1, y_1) to the curve y = f(x) is $(y - y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} (x - x_1)$

*** Length of Tangent, Normal, Subtangent and Subnormal**

Tangent: $PT = MP \operatorname{cosec} \Psi = y\sqrt{1+\cot^2\psi} = \left| \frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} \right|$

Subtangent: $TM = MP \cot \Psi = \left| \frac{y}{(dy/dx)} \right|$

Normal: $GP = MP \sec \Psi = y\sqrt{1+\tan^2\psi} = \left| y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \right|$

Subnormal: $MG = MP \tan \Psi = \left| y\left(\frac{dy}{dx}\right) \right|$

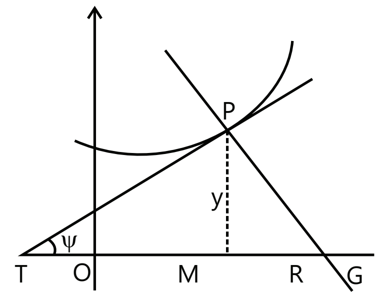

Figure 21.9

*** Angle of Intersection of Two Curves**

$\tan \Psi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$,

where m_1 and m_2 are the slopes of the tangents T_1 and T_2 at the intersection point (x_1, y_1).

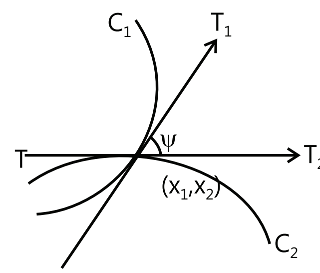

Figure 21.10

Solved Examples

JEE Main/Boards

Example 1: Show that the function f(x) = | x | is continuous at x = 0, but not differentiable at x = 0.

Sol: Evaluate $f'(0^+)$ and $f'(0^-)$.

We have $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0 = f(0)$

The function is continuous at x = 0

We also have

$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$

$f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{-(-x)-0}{-x} = -1$

Since, $f'(0^+) \neq f'(0^-)$, the function is not differentiable at x = 0

Example 2: Find the derivative of the function f(x), defined by f(x) = sin x by 1st principle.

Sol: Use the first principle to find the derivative of the given function.

Let dy be the increment in y corresponding to an increment dx in x. We have

y = sin x

y +dy = sin (x + dx)